On weak compactness in Lebesgue-Bochner spaces

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Operators and Banach lattices
Madrid – October 25th, 2012

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6. Hence $K \subset nT(B_{L^2(\mu)}) + \varepsilon B_{L^1(\mu)}$. □

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Theorem (Diestel-Ruess-Schachermayer 1993)

A set $C \subset L^1(\mu, X)$ is **relatively weakly compact** if and only if

1. C is equi-integrable and bounded;
2. for every sequence $(f_n) \subset C$ there exist $g_n \in \mathrm{co}\{f_k : k \geq n\}$ such that $(g_n(\omega))$ is weakly (resp. norm) convergent in X for μ-a.e. $\omega \in \Omega$.

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Theorem (Talagrand 1984)

$L^1(\mu, X)$ is **weakly sequentially complete** if X is.

A partial answer

Theorem (Lajara-R. 2012)

If X is SWCG, then there is a weakly compact set $G \subset L^1(\mu, X)$ such that:

for every decomposable weakly compact set $K \subset L^1(\mu, X)$ and $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that $K \subset nG + \varepsilon B_{L^1(\mu,X)}$.

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Decomposable means: $f\mathbb{1}_A + g\mathbb{1}_{\Omega \setminus A} \in K$ for every $f, g \in K$ and $A \in \Sigma$.

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$$L(W) = \left\{ f \in L^1(\mu, X) : f(\omega) \in W \text{ for } \mu\text{-a.e. } \omega \in \Omega \right\}, \quad \text{where } W \subset X.$$

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1. W is relatively weakly compact $\Longrightarrow$ $L(W)$ is relatively weakly compact.
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- *Decomposable sets arise as collections of selectors of set-valued functions.*

Decomposability and set-valued functions

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Let $D \subset L^1(\mu, X)$ be a **decomposable** closed nonempty set. Then there is a measurable $F : \Omega \to cl(X)$ such that

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Let $D \subset L^1(\mu, X)$ be a **decomposable** closed nonempty set. Then there is a measurable $F : \Omega \to cl(X)$ such that

$$D = S_F^1$$

Measurable means: $\{\omega \in \Omega : F(\omega) \cap U \neq \emptyset\} \in \Sigma$ for every open set $U \subset X$.

Theorem (Klei 1988) – assuming that X is separable

Let $F : \Omega \to cl(X)$ be measurable. If S_F^1 is relatively weakly compact, then

$F(\omega)$ is **relatively weakly compact** for μ-a.e. $\omega \in \Omega$.