Some Finitely Strictily Singular Operators in Analysis

Luis Rodríguez-Piazza

Universidad de Sevilla
Spain

Madrid, 25th October 2012.

Definitions

Definitions

An bounded operator $T \colon X \to Y$ between two Banach spaces is Strictly Singular (SS) if for every infinite dimensional subspace E of X, the restriction of T to E does not realize an isomorphism from E onto $T(E)$.

Equivalently,

Definitions

An bounded operator $T\colon X \to Y$ between two Banach spaces is Strictly Singular (SS) if for every infinite dimensional subspace E of X, the restriction of T to E does not realize an isomorphism from E onto $T(E)$.

Equivalently, T is (SS) if and only if: for every $\varepsilon > 0$, for every infinite dimensional subspace E of X, there exists x in the unit sphere of E such that $\|T(x)\| \leq \varepsilon$.

Definitions

An bounded operator $T\colon X \to Y$ between two Banach spaces is Strictly Singular (SS) if for every infinite dimensional subspace E of X, the restriction of T to E does not realize an isomorphism from E onto $T(E)$.

Equivalently, T is (SS) if and only if: for every $\varepsilon > 0$, for every infinite dimensional subspace E of X, there exists x in the unit sphere of E such that $\|T(x)\| \leq \varepsilon$.

This version can be quantified/localized requiring a little more:

Definitions

An bounded operator $T\colon X \to Y$ between two Banach spaces is Strictly Singular (SS) if for every infinite dimensional subspace E of X, the restriction of T to E does not realize an isomorphism from E onto $T(E)$.

Equivalently, T is (SS) if and only if: for every $\varepsilon > 0$, for every infinite dimensional subspace E of X, there exists x in the unit sphere of E such that $\|T(x)\| \leq \varepsilon$.

This version can be quantified/localized requiring a little more: The operator T is called Finitely Strictly Singular (FSS) if: for every $\varepsilon > 0$, there exists $N_\varepsilon \geq 1$ such that, for every subspace E of X with dimension greater than N_ε, there exists x in the unit sphere of E such that $\|T(x)\| \leq \varepsilon$.

Definitions

Given a bounded operator $T\colon X \to Y$ between two Banach spaces, the sequence of Bernstein numbers of T is defined by

Definitions

Given a bounded operator $T\colon X \to Y$ between two Banach spaces, the sequence of Bernstein numbers of T is defined by

$$b_n(T) = \sup_{E \subset X, dim(E) = n} \inf\{\big\|T(x)\big\| : x \in E, \|x\| = 1\}.$$

Definitions

Given a bounded operator $T\colon X \to Y$ between two Banach spaces, the sequence of Bernstein numbers of T is defined by

$$b_n(T) = \sup_{E \subset X, dim(E)=n} \inf\{\big\|T(x)\big\| : x \in E, \|x\| = 1\}.$$

T is (FSS) if and only if $\lim_{n\to\infty} b_n(T) = 0$.

Definitions

Given a bounded operator $T\colon X \to Y$ between two Banach spaces, the sequence of Bernstein numbers of T is defined by

$$b_n(T) = \sup_{E \subset X, dim(E)=n} \inf\{\|T(x)\| : x \in E, \|x\| = 1\}.$$

T is (FSS) if and only if $\lim_{n\to\infty} b_n(T) = 0$.

Observe that the Bernstein numbers are dominated by the approximation numbers

$$b_n(T) \leq a_n(T)$$

Definitions

Given a bounded operator $T\colon X \to Y$ between two Banach spaces, the sequence of Bernstein numbers of T is defined by

$$b_n(T) = \sup_{E \subset X, dim(E)=n} \inf\{\|T(x)\| : x \in E, \|x\| = 1\}.$$

T is (FSS) if and only if $\lim_{n\to\infty} b_n(T) = 0$.

Observe that the Bernstein numbers are dominated by the approximation numbers

$$b_n(T) \le a_n(T) = \inf\{\|T - R\| : R\colon X \to Y, \text{rank}(T) < n\}.$$

Definitions

Finitely strictly singular operators are also called superstrictly singular operators.

Definitions

Finitely strictly singular operators are also called superstrictly singular operators.

Theorem (Flores, Hernández, Raynaud)

An operator T is finitely strictly singular iff every ultrapower of T is strictly singular iff every operator which is locally representable in T is strictly singular.

Definitions

Finitely strictly singular operators are also called superstrictly singular operators.

Theorem (Flores, Hernández, Raynaud)

An operator T is finitely strictly singular iff every ultrapower of T is strictly singular iff every operator which is locally representable in T is strictly singular.

Moreover

$$b_n(T) = b_n(T_{\mathcal{U}}), \quad \text{for every } n \text{ and every ultrapower } T_{\mathcal{U}} \text{ of } T.$$

FSS versus Compactness

FSS versus Compactness

We always have, for an operator T,

$$\text{Compact} \quad \Rightarrow \quad \text{FSS} \quad \Rightarrow \quad \text{SS}$$

FSS versus Compactness

We always have, for an operator T,

$$\text{Compact} \quad \Rightarrow \quad \text{FSS} \quad \Rightarrow \quad \text{SS}$$

When X and Y are Hilbert spaces, these notions coincide.

FSS versus Compactness

We always have, for an operator T,

$$\text{Compact} \quad \Rightarrow \quad \text{FSS} \quad \Rightarrow \quad \text{SS}$$

When X and Y are Hilbert spaces, these notions coincide. The same is true if $X = Y = \ell^p$ $(1 \leq p < +\infty)$ or $X = Y = c_0$.

FSS versus Compactness

We always have, for an operator T,

$$\text{Compact} \quad \Rightarrow \quad \text{FSS} \quad \Rightarrow \quad \text{SS}$$

When X and Y are Hilbert spaces, these notions coincide. The same is true if $X = Y = \ell^p$ $(1 \leq p < +\infty)$ or $X = Y = c_0$. But in general

$$\text{FSS} \quad \not\Leftarrow \quad \text{SS}$$

FSS versus Compactness

We always have, for an operator T,

$$\text{Compact} \quad \Rightarrow \quad \text{FSS} \quad \Rightarrow \quad \text{SS}$$

When X and Y are Hilbert spaces, these notions coincide. The same is true if $X = Y = \ell^p$ $(1 \leq p < +\infty)$ or $X = Y = c_0$. But in general

$$\text{FSS} \quad \not\Leftarrow \quad \text{SS}$$

Ex. 1:
Take the inclusion map of $\ell^1 = \bigoplus_{\ell^1} \ell^1_n$ into the space $\bigoplus_{\ell^2} \ell^1_n$.
It is not FSS, but it is SS.

FSS versus Compactness

We always have, for an operator T,

$$\text{Compact} \quad \Rightarrow \quad \text{FSS} \quad \Rightarrow \quad \text{SS}$$

When X and Y are Hilbert spaces, these notions coincide. The same is true if $X = Y = \ell^p$ $(1 \leq p < +\infty)$ or $X = Y = c_0$. But in general

$$\text{Compact} \quad \not\Leftarrow \quad \text{FSS} \quad \not\Leftarrow \quad \text{SS}$$

Ex. 1:
Take the inclusion map of $\ell^1 = \bigoplus_{\ell^1} \ell^1_n$ into the space $\bigoplus_{\ell^2} \ell^1_n$.
It is not FSS, but it is SS.

FSS versus Compactness

We always have, for an operator T,

$$\text{Compact} \quad \Rightarrow \quad \text{FSS} \quad \Rightarrow \quad \text{SS}$$

When X and Y are Hilbert spaces, these notions coincide. The same is true if $X = Y = \ell^p$ $(1 \leq p < +\infty)$ or $X = Y = c_0$. But in general

$$\text{Compact} \quad \not\Leftarrow \quad \text{FSS} \quad \not\Leftarrow \quad \text{SS}$$

Ex. 1:
Take the inclusion map of $\ell^1 = \bigoplus_{\ell^1} \ell_n^1$ into the space $\bigoplus_{\ell^2} \ell_n^1$.
It is not FSS, but it is SS.

Ex. 2: (V. Milman '70)
For $1 \leq p < q \leq \infty$, the inclusion map $\ell^p \hookrightarrow \ell^q$ is FSS, but it is not compact.

Ex. 3:
For $1 < p < +\infty$, the inclusion map $\mathcal{C}[0,1] \hookrightarrow L^p[0,1]$ is FSS, but it is not compact. The same for $L^\infty[0,1] \hookrightarrow L^p[0,1]$.

More examples

Ex. 3:
For $1 < p < +\infty$, the inclusion map $\mathcal{C}[0,1] \hookrightarrow L^p[0,1]$ is FSS, but it is not compact. The same for $L^\infty[0,1] \hookrightarrow L^p[0,1]$.

Proposition (Mitiagin, Pełczyński '68)

Every absolutely p-summing operator is FSS.

More examples

Ex. 3:

For $1 < p < +\infty$, the inclusion map $\mathcal{C}[0,1] \hookrightarrow L^p[0,1]$ is FSS, but it is not compact. The same for $L^\infty[0,1] \hookrightarrow L^p[0,1]$.

Proposition (Mitiagin, Pełczyński '68)

Every absolutely p-summing operator is FSS.

Proposition (Flores, Hernández, Raynaud)

If $E[0,1]$ is a rearrangement invariant space and $E \neq L^\infty$; then the inclusion map $L^\infty[0,1] \hookrightarrow E[0,1]$ is FSS.

More examples

Ex. 4: (Plichko '04)

The Fourier transform $\mathcal{F}\colon L^1(\mathbb{T}) \to c_0(\mathbb{Z})$ sending

$$f \mapsto (\hat{f}(m))_{m\in\mathbb{Z}}$$

is SS, but is not FSS.

Ex. 4: (Plichko '04)

The Fourier transform $\mathcal{F}\colon L^1(\mathbb{T}) \to c_0(\mathbb{Z})$ sending

$$f \mapsto (\hat{f}(m))_{m\in\mathbb{Z}}$$

is SS, but is not FSS.

Given $n \in \mathbb{N}$ consider n real numbers $t_1, t_2, \ldots, t_n$ such that π, $t_1, t_2, \ldots, t_n$ are $\mathbb{Q}$ linearly independent. Then, by Kronecker's theorem the set $\{(e^{imt_1}, e^{imt_2}, \ldots, e^{imt_n}) : m \in \mathbb{Z}\}$ is dense in $\mathbb{T}^n$ and, putting $z_j = e^{imt_j}$, for all scalars α_j, we have

$$\Big\|\sum_j \alpha_j \widehat{\delta_{z_j}}\Big\|_{\ell^\infty(\mathbb{Z})} = \sup_m \Big|\sum_j \alpha_j e^{imt_j}\Big| = \sum_j |\alpha_j| = \Big\|\sum_j \alpha_j \delta_{z_j}\Big\|_{M(\mathbb{T})}.$$

Ex. 4: (Plichko '04)

The Fourier transform $\mathcal{F}\colon L^1(\mathbb{T}) \to c_0(\mathbb{Z})$ sending

$$f \mapsto (\hat{f}(m))_{m\in\mathbb{Z}}$$

is SS, but is not FSS.

Given $n \in \mathbb{N}$ consider n real numbers $t_1, t_2, \dots, t_n$ such that π, $t_1, t_2, \dots, t_n$ are $\mathbb{Q}$ linearly independent. Then, by Kronecker's theorem the set $\{(e^{imt_1}, e^{imt_2}, \dots, e^{imt_n}) : m \in \mathbb{Z}\}$ is dense in $\mathbb{T}^n$ and, putting $z_j = e^{imt_j}$, for all scalars α_j, we have

$$\Big\|\sum_j \alpha_j \widehat{\delta_{z_j}}\Big\|_{\ell^\infty(\mathbb{Z})} = \sup_m \Big|\sum_j \alpha_j e^{imt_j}\Big| = \sum_j |\alpha_j| = \Big\|\sum_j \alpha_j \delta_{z_j}\Big\|_{M(\mathbb{T})}.$$

Changing δ_{z_j} by $F_N * \delta_{z_j}$, where F_N is the N'th Fejér Kernel with N large enough, we see that

$$b_n(\mathcal{F}) = 1, \qquad \text{for every } n.$$

Some results of structure

Theorem (V. Milman '70)

The set $\mathcal{F}SS(X, Y)$ of all finitely strictly singular operators from X to Y is a closed linear subspace of $\mathcal{L}(X, Y)$ with the ideal property. That is, $S \circ T \circ R \in \mathcal{F}SS(X_1, Y_1)$, whenever $R \in \mathcal{L}(X_1, X)$, $T \in \mathcal{F}SS(X, Y)$, and $S \in \mathcal{L}(Y, Y_1)$.

Some results of structure

Theorem (V. Milman '70)

The set $\mathcal{F}SS(X, Y)$ of all finitely strictly singular operators from X to Y is a closed linear subspace of $\mathcal{L}(X, Y)$ with the ideal property. That is, $S \circ T \circ R \in \mathcal{F}SS(X_1, Y_1)$, whenever $R \in \mathcal{L}(X_1, X)$, $T \in \mathcal{F}SS(X, Y)$, and $S \in \mathcal{L}(Y, Y_1)$.

This result can be obtained as a corollary to Hernández, Flores, Raynaud characterization and the fact that strictly singular operators form an operator ideal.

Some results of structure

Theorem (V. Milman '70)

The set $\mathcal{F}SS(X, Y)$ of all finitely strictly singular operators from X to Y is a closed linear subspace of $\mathcal{L}(X, Y)$ with the ideal property. That is, $S \circ T \circ R \in \mathcal{F}SS(X_1, Y_1)$, whenever $R \in \mathcal{L}(X_1, X)$, $T \in \mathcal{F}SS(X, Y)$, and $S \in \mathcal{L}(Y, Y_1)$.

This result can be obtained as a corollary to Hernández, Flores, Raynaud characterization and the fact that strictly singular operators form an operator ideal.

Directly it is easy to prove: $\mathcal{F}SS(X, Y)$ is stable by multiplication by scalars, is closed in $\mathcal{L}(X, Y)$ and has the ideal property.

Some results of structure

Theorem (V. Milman '70)

The set $\mathcal{F}SS(X, Y)$ of all finitely strictly singular operators from X to Y is a closed linear subspace of $\mathcal{L}(X, Y)$ with the ideal property. That is, $S \circ T \circ R \in \mathcal{F}SS(X_1, Y_1)$, whenever $R \in \mathcal{L}(X_1, X)$, $T \in \mathcal{F}SS(X, Y)$, and $S \in \mathcal{L}(Y, Y_1)$.

This result can be obtained as a corollary to Hernández, Flores, Raynaud characterization and the fact that strictly singular operators form an operator ideal.

Directly it is easy to prove: $\mathcal{F}SS(X, Y)$ is stable by multiplication by scalars, is closed in $\mathcal{L}(X, Y)$ and has the ideal property.

It is not so obvious:

$$T, S \in \mathcal{F}SS(X, Y) \quad \Longrightarrow \quad T + S \in \mathcal{F}SS(X, Y)$$

Some results of structure

Theorem (Plichko '04)

Let $T \in \mathcal{L}(X, Y)$. Then T is FSS if and only if for every sequence $\{E_n\}$ of subspaces of X with $\dim(E_n) \to \infty$, there exist subspaces $F_n \subset E_n$ such that:

Some results of structure

Theorem (Plichko '04)

Let $T \in \mathcal{L}(X, Y)$. Then T is FSS if and only if for every sequence $\{E_n\}$ of subspaces of X with $\dim(E_n) \to \infty$, there exist subspaces $F_n \subset E_n$ such that:

- $\|T|_{F_n}\| \to 0$, and

Some results of structure

Theorem (Plichko '04)

Let $T \in \mathcal{L}(X, Y)$. Then T is FSS if and only if for every sequence $\{E_n\}$ of subspaces of X with $\dim(E_n) \to \infty$, there exist subspaces $F_n \subset E_n$ such that:

- $\|T|_{F_n}\| \to 0$, and
- $\dim(F_n) \to \infty$.

Some results of structure

Theorem (Plichko '04)

Let $T \in \mathcal{L}(X, Y)$. Then T is FSS if and only if for every sequence $\{E_n\}$ of subspaces of X with $\dim(E_n) \to \infty$, there exist subspaces $F_n \subset E_n$ such that:

- $\|T|_{F_n}\| \to 0$, and
- $\dim(F_n) \to \infty$.

Equivalent to the problem of the sum is the following fact:

$$T_1 \in \mathcal{F}SS(X_1, Y_1)\,, \quad T_2 \in \mathcal{F}SS(X_2, Y_2) \quad \Longrightarrow \quad T_1 \oplus T_2 \in \mathcal{F}SS(X_1 \times X_2, Y_1 \times Y_2),$$

where

$$T_1 \oplus T_2(x_1, x_2) = (Tx_1, Tx_2)\,, \quad (x_1, x_2) \in X_1 \times X_2.$$

The diagonal Theorem

Suppose $\{T_n\}_n$ is a uniformly bounded sequence of operators $T_n \colon X_n \to Y_n$. Let $1 \leq p < q \leq \infty$. Then the diagonal operator

$$\mathbf{X} = \bigoplus_{\ell^p} X_n, \quad \mathbf{Y} = \bigoplus_{\ell^q} Y_n, \qquad \mathbf{T} \colon \mathbf{X} \to \mathbf{Y},$$

defined by $\mathbf{T}\big((x_n)_n\big) = (T_n x_n)_n$ is bounded.

The diagonal Theorem

Suppose $\{T_n\}_n$ is a uniformly bounded sequence of operators $T_n \colon X_n \to Y_n$. Let $1 \leq p < q \leq \infty$. Then the diagonal operator

$$\mathbf{X} = \bigoplus_{\ell^p} X_n, \quad \mathbf{Y} = \bigoplus_{\ell^q} Y_n, \qquad \mathbf{T} \colon \mathbf{X} \to \mathbf{Y},$$

defined by $\mathbf{T}\big((x_n)_n\big) = (T_n x_n)_n$ is bounded.

Is it **T** an FSS operator?

The diagonal Theorem

Suppose $\{T_n\}_n$ is a uniformly bounded sequence of operators $T_n \colon X_n \to Y_n$. Let $1 \leq p < q \leq \infty$. Then the diagonal operator

$$\mathbf{X} = \bigoplus_{\ell^p} X_n, \quad \mathbf{Y} = \bigoplus_{\ell^q} Y_n, \qquad \mathbf{T} \colon \mathbf{X} \to \mathbf{Y},$$

defined by $\mathbf{T}\big((x_n)_n\big) = (T_n x_n)_n$ is bounded.

Is it $\mathbf{T}$ an FSS operator?

We say that a sequence $\{T_n\}_n$ of operators $T_n \colon X_n \to Y_n$ is uniformly finitely strictly singular if for every $\varepsilon > 0$, there exists N_ε such that for every $n \in \mathbb{N}$, and every subspace E of X_n with $\dim(E) \geq N_\varepsilon$, there exists $x \in S_E$, such that $\|T_n x\| \leq \varepsilon$.

Theorem (Lefèvre, R-P)

The diagonal operator **T** is finitely strictly singular if and only if the sequence $\{T_n\}_n$ is uniformly finitely strictly singular.

Theorem (Lefèvre, R-P)

The diagonal operator **T** is finitely strictly singular if and only if the sequence $\{T_n\}_n$ is uniformly finitely strictly singular.

About the proof:

The diagonal Theorem

Theorem (Lefèvre, R-P)

The diagonal operator **T** is finitely strictly singular if and only if the sequence $\{T_n\}_n$ is uniformly finitely strictly singular.

About the proof:

- One implication is easy. The other one is very technical.

Theorem (Lefèvre, R-P)

The diagonal operator **T** is finitely strictly singular if and only if the sequence $\{T_n\}_n$ is uniformly finitely strictly singular.

About the proof:

- One implication is easy. The other one is very technical.
- It can be reduced to the case $q = \infty$.

  use: $\|y\|_q \leq \|y\|_\infty^\theta \|y\|_p^{1-\theta}$, for $1 \leq p < q < \infty$

The diagonal Theorem

Theorem (Lefèvre, R-P)

The diagonal operator $\mathbf{T}$ is finitely strictly singular if and only if the sequence $\{T_n\}_n$ is uniformly finitely strictly singular.

About the proof:

• One implication is easy. The other one is very technical.

• It can be reduced to the case $q = \infty$.

• If $E \subset \mathbf{X}$ is finite dimensional, and $\|\mathbf{Tx}\|_\infty \geq \delta \|\mathbf{x}\|_p$, $\forall \mathbf{x} \in E$. Then there exists $N \in \mathbb{N}$ and $\varepsilon > 0$, only depending on δ, such that there is $A \subset \mathbb{N}$ with card$(A) \leq N$ and

$$\max\{\| T_n x_n\| : n \in A\} \geq \varepsilon \|\mathbf{x}\|_p \,, \quad \forall \mathbf{x} \in E.$$

Fourier Transform again

We have seen $\mathcal{F}\colon L^1(\mathbb{T}) \to c_0(\mathbb{Z})$ is SS, but not FSS.

Fourier Transform again

We have seen $\mathcal{F}\colon L^1(\mathbb{T}) \to c_0(\mathbb{Z})$ is SS, but not FSS.

Of course $\mathcal{F}\colon L^2(\mathbb{T}) \to \ell^2(\mathbb{Z})$ is not SS.

Fourier Transform again

We have seen $\mathcal{F}\colon L^1(\mathbb{T}) \to c_0(\mathbb{Z})$ is SS, but not FSS.

Of course $\mathcal{F}\colon L^2(\mathbb{T}) \to \ell^2(\mathbb{Z})$ is not SS.

Theorem (Lefèvre, R-P)

If $1 < p < 2$, and p^* is its conjugate exponent, then the Fourier Transform

$$\mathcal{F}\colon L^p(\mathbb{T}) \to \ell^{p^*}(\mathbb{Z})$$

is finitely strictly singular.

Fourier Transform again

We have seen $\mathcal{F}\colon L^1(\mathbb{T}) \to c_0(\mathbb{Z})$ is SS, but not FSS.

Of course $\mathcal{F}\colon L^2(\mathbb{T}) \to \ell^2(\mathbb{Z})$ is not SS.

Theorem (Lefèvre, R-P)

If $1 < p < 2$, and p^* is its conjugate exponent, then the Fourier Transform

$$\mathcal{F}\colon L^p(\mathbb{T}) \to \ell^{p^*}(\mathbb{Z})$$

is finitely strictly singular.

In fact this result is valid for the Fourier transform in every locally compact abelian group; in particular, for the Fourier transform in $\mathbb{R}^d$. Moreover this is a direct consequence of a more general result.

An interpolation result

Suppose (Ω, μ) and (Δ, ν) are two measure spaces and T is an operator such that

$$T\colon L^2(\mu) \to L^2(\nu), \quad \text{and}$$

An interpolation result

Suppose (Ω, μ) and (Δ, ν) are two measure spaces and T is an operator such that

$$T\colon L^2(\mu) \to L^2(\nu), \quad \text{and} \quad T\colon L^1(\mu) \to L^\infty(\nu)\,.$$

An interpolation result

Suppose (Ω, μ) and (Δ, ν) are two measure spaces and T is an operator such that

$$T\colon L^2(\mu) \to L^2(\nu), \quad \text{and} \quad T\colon L^1(\mu) \to L^\infty(\nu)\,.$$

Then, for $1 \le p \le 2$, we have

$$T_p = T\colon L^p(\mu) \to L^{p^*}(\nu) \quad \text{is bounded}$$

An interpolation result

Suppose (Ω, μ) and (Δ, ν) are two measure spaces and T is an operator such that

$$T\colon L^2(\mu) \to L^2(\nu), \quad \text{and} \quad T\colon L^1(\mu) \to L^\infty(\nu)\,.$$

Then, for $1 \leq p \leq 2$, we have

$$T_p = T\colon L^p(\mu) \to L^{p^*}(\nu) \quad \text{is bounded}$$

Theorem

For $1 < p < 2$, the operator T_p is FSS.

An interpolation result

Suppose (Ω, μ) and (Δ, ν) are two measure spaces and T is an operator such that

$$T\colon L^2(\mu) \to L^2(\nu), \quad \text{and} \quad T\colon L^1(\mu) \to L^\infty(\nu)\,.$$

Then, for $1 \leq p \leq 2$, we have

$$T_p = T\colon L^p(\mu) \to L^{p^*}(\nu) \quad \text{is bounded}$$

Theorem

For $1 < p < 2$, the operator T_p is FSS.
Moreover, for every $p \in (1, 2)$, there exists $K_p > 0$ such that, if $\|T_1\| \leq 1$ and $\|T_2\| \leq 1$, then

$$b_n(T_p) \leq k_p n^{-1/r}, \quad \text{for every } n,$$

where $\frac{1}{r} = \frac{1}{p} - \frac{1}{2}$.

A lemma

Suppose T is like in the theorem, and $\frac{1}{r} = \frac{1}{p} - \frac{1}{2} = \frac{1}{2} - \frac{1}{p^*}$.

A lemma

Suppose T is like in the theorem, and $\frac{1}{r} = \frac{1}{p} - \frac{1}{2} = \frac{1}{2} - \frac{1}{p^*}$.

Take $g \in L^r(\mu)$, $u \in L^r(\nu)$ and define

$$T_{g,u} \colon L^2(\mu) \to L^2(\nu) \,, \qquad T_{g,u} f = u \cdot T(g \cdot f) \,.$$

A lemma

Suppose T is like in the theorem, and $\frac{1}{r} = \frac{1}{p} - \frac{1}{2} = \frac{1}{2} - \frac{1}{p^*}$.

Take $g \in L^r(\mu)$, $u \in L^r(\nu)$ and define

$$T_{g,u} \colon L^2(\mu) \to L^2(\nu)\,, \qquad T_{g,u}f = u \cdot T(g \cdot f)\,.$$

That is $T_{g,u} = M_u \circ T \circ M_g$

A lemma

Suppose T is like in the theorem, and $\frac{1}{r} = \frac{1}{p} - \frac{1}{2} = \frac{1}{2} - \frac{1}{p^*}$.

Take $g \in L^r(\mu)$, $u \in L^r(\nu)$ and define

$$T_{g,u}\colon L^2(\mu) \to L^2(\nu)\,, \qquad T_{g,u}f = u \cdot T(g \cdot f)\,.$$

That is $T_{g,u} = M_u \circ T \circ M_g$
We have

Lemma

The operator $T_{g,h}$ is in the Schatten class $\mathcal{S}_r\big(L^2(\mu), L^2(\nu)\big)$. Moreover, if $\|g\|_r \le 1$, and $\|u\|_r \le 1$, we have

$$\sum_{k=1}^{\infty} a_k(T_{g,h})^r \le 1 \quad \text{and} \quad a_n(T_{g,h}) \le n^{-1/r}\,, \forall n \in \mathbb{N}$$

Proof of the lemma. Assume $\|T_1\|, \|T_2\| \leq 1$. For $r \in [2, +\infty]$, consider the following bilinear operator:

A lemma

Proof of the lemma. Assume $\|T_1\|, \|T_2\| \leq 1$. For $r \in [2, +\infty]$, consider the following bilinear operator:

$$\begin{aligned} \Phi\colon L^r(\mu) \times L^r(\nu) &\longrightarrow \mathcal{L}\big(L^2(\mu), L^2(\nu)\big) \\ (g, u) &\longmapsto T_{g,u} \end{aligned}$$

A lemma

Proof of the lemma. Assume $\|T_1\|, \|T_2\| \le 1$. For $r \in [2, +\infty]$, consider the following bilinear operator:

$$\begin{array}{rcl} \Phi\colon L^r(\mu) \times L^r(\nu) & \longrightarrow & \mathcal{L}\big(L^2(\mu), L^2(\nu)\big) \\ (g, u) & \longmapsto & T_{g,u} \end{array}$$

- For $r = 2$, $T_{g,u}$ is an order bounded operator. Indeed

$$|T_{g,u}f| = |u| \cdot |T(gf)| \le \|T(gf)\|_{L^\infty(\nu)}|u| \le \|g\|_2\|f\|_2|u|,$$

Proof of the lemma. Assume $\|T_1\|, \|T_2\| \leq 1$. For $r \in [2, +\infty]$, consider the following bilinear operator:

$$\begin{aligned} \Phi\colon L^r(\mu) \times L^r(\nu) &\longrightarrow \mathcal{L}\big(L^2(\mu), L^2(\nu)\big) \\ (g, u) &\longmapsto T_{g,u} \end{aligned}$$

- For $r = 2$, $T_{g,u}$ is an order bounded operator. Indeed

$$|T_{g,u}f| = |u| \cdot |T(gf)| \leq \|T(gf)\|_{L^\infty(\nu)}|u| \leq \|g\|_2\|f\|_2|u|,$$

Then $T_{g,u}$ is in the Schatten class $\mathcal{S}_2$, and $\|T_{g,u}\|_{\mathcal{S}_2} \leq \|g\|_2\|u\|_2$.

Proof of the lemma. Assume $\|T_1\|, \|T_2\| \leq 1$. For $r \in [2, +\infty]$, consider the following bilinear operator:

$$\begin{aligned} \Phi \colon L^r(\mu) \times L^r(\nu) &\longrightarrow \mathcal{L}\big(L^2(\mu), L^2(\nu)\big) \\ (g, u) &\longmapsto T_{g,u} \end{aligned}$$

- For $r = 2$, $T_{g,u}$ is an order bounded operator. Indeed

$$|T_{g,u}f| = |u| \cdot |T(gf)| \leq \|T(gf)\|_{L^\infty(\nu)}|u| \leq \|g\|_2\|f\|_2|u|,$$

Then $T_{g,u}$ is in the Schatten class $\mathcal{S}_2$, and $\|T_{g,u}\|_{\mathcal{S}_2} \leq \|g\|_2\|u\|_2$.

- For $r = \infty$, $\|T_{g,u}\| \leq \|g\|_\infty\|u\|_\infty$

A lemma

Proof of the lemma. Assume $\|T_1\|, \|T_2\| \leq 1$. For $r \in [2, +\infty]$, consider the following bilinear operator:

$$\begin{aligned} \Phi\colon L^r(\mu) \times L^r(\nu) &\longrightarrow \mathcal{L}\big(L^2(\mu), L^2(\nu)\big) \\ (g, u) \quad &\longmapsto \quad T_{g,u} \end{aligned}$$

- For $r = 2$, $T_{g,u}$ is an order bounded operator. Indeed

$$|T_{g,u}f| = |u| \cdot |T(gf)| \leq \|T(gf)\|_{L^\infty(\nu)}|u| \leq \|g\|_2\|f\|_2|u|,$$

Then $T_{g,u}$ is in the Schatten class $\mathcal{S}_2$, and $\|T_{g,u}\|_{\mathcal{S}_2} \leq \|g\|_2\|u\|_2$.

- For $r = \infty$, $\|T_{g,u}\| \leq \|g\|_\infty\|u\|_\infty$

From an interpolation argument the lemma follows: Φ sends $L^r(\mu) \times L^r(\nu)$ into $\mathcal{S}_r$.

Proof of the theorem

Proof of the theorem

Suppose $\dim E = n$, $\delta > 0$, and $\|Tf\|_{p^*} \geq \delta \|f\|_p$, for every $f \in E$

Proof of the theorem

Suppose $\dim E = n$, $\delta > 0$, and $\|Tf\|_{p^*} \geq \delta \|f\|_p$, for every $f \in E$

$$L^p(\mu) \supseteq E \underset{T^{-1}}{\overset{T}{\rightleftarrows}} T(E) \subseteq L^{p^*}(\nu)$$

$\|T^{-1}\| \leq 1/\delta$,

Proof of the theorem

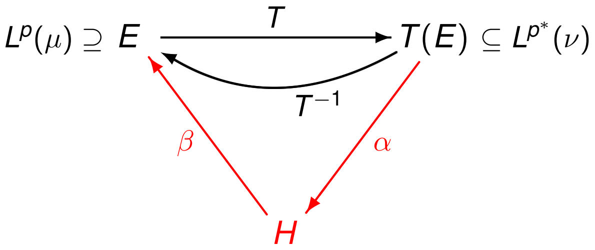
Suppose $\dim E = n$, $\delta > 0$, and $\|Tf\|_{p^*} \geq \delta \|f\|_p$, for every $f \in E$

$$L^p(\mu) \supseteq E \underset{T^{-1}}{\overset{T}{\rightleftarrows}} T(E) \subseteq L^{p^*}(\nu)$$

$\|T^{-1}\| \leq 1/\delta$, and we can apply Kwapien's Theorem

Proof of the theorem

Suppose $\dim E = n$, $\delta > 0$, and $\|Tf\|_{p^*} \geq \delta \|f\|_p$, for every $f \in E$

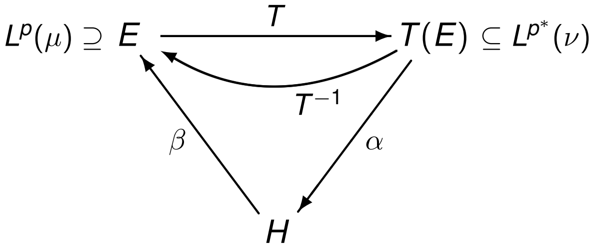


$\|T^{-1}\| \leq 1/\delta$, and we can apply Kwapien's Theorem

$\dim H = n, \quad \|\beta\| = 1, \quad \|\alpha\| \lessapprox 1/\delta$

Proof of the theorem

Suppose $\dim E = n$, $\delta > 0$, and $\|Tf\|_{p^*} \geq \delta \|f\|_p$, for every $f \in E$



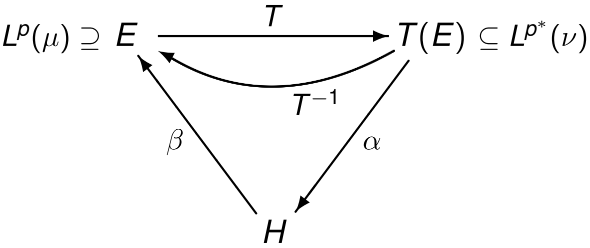
$\|T^{-1}\| \leq 1/\delta$, and we can apply Kwapien's Theorem

$$\dim H = n, \quad \|\beta\| = 1, \quad \|\alpha\| \lessapprox 1/\delta$$

Observe that $Id_H = \alpha \circ T \circ \beta$

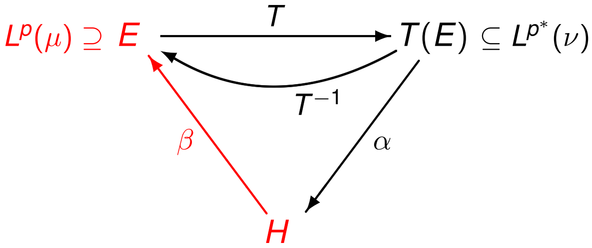
Proof of the theorem

We have $\dim H = n, \quad \|\beta\| = 1, \quad \|\alpha\| \lesssim 1/\delta, \quad Id_H = \alpha \circ T \circ \beta.$



Proof of the theorem

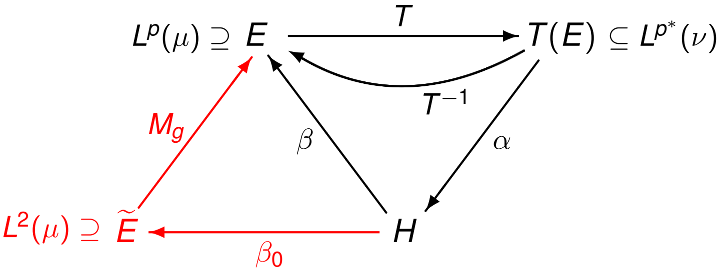
We have $\dim H = n, \quad \|\beta\| = 1, \quad \|\alpha\| \lesssim 1/\delta, \quad Id_H = \alpha \circ T \circ \beta.$



We apply Maurey's Factorization Theorem to β

Proof of the theorem

We have $\dim H = n$, $\|\beta\| = 1$, $\|\alpha\| \lesssim 1/\delta$, $Id_H = \alpha \circ T \circ \beta$.



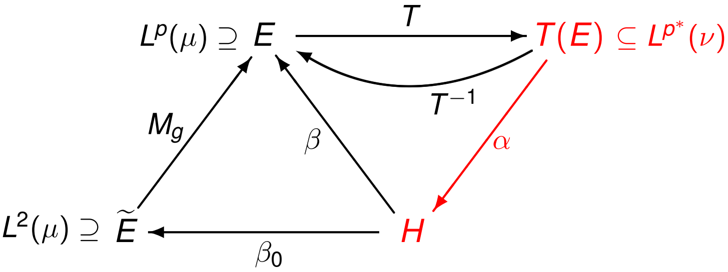
We apply Maurey's Factorization Theorem to β

$\exists g \in L^r(\mu)$, $\|g\|_r = 1$, $\widetilde{E} = \{f/g : f \in E\}$, $\|\beta_0\| \lesssim \|\beta\| = 1$

$\frac{1}{r} = \frac{1}{p} - \frac{1}{2}$

Proof of the theorem

We have dim $H = n$, $\|\beta\| = 1$, $\|\alpha\| \lesssim 1/\delta$, $Id_H = \alpha \circ T \circ \beta$.



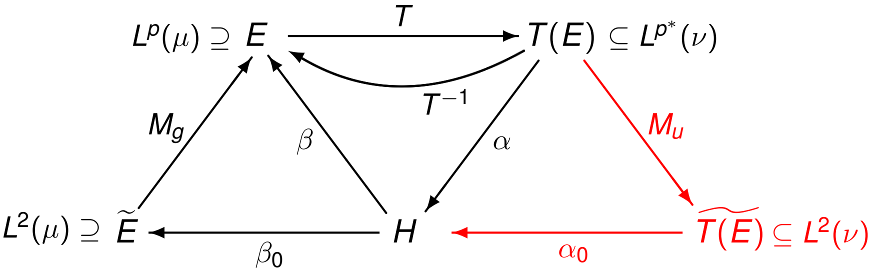
We apply Maurey's Factorization Theorem to β and to α:

$\exists g \in L^r(\mu)$, $\|g\|_r = 1$, $\widetilde{E} = \{f/g : f \in E\}$, $\|\beta_0\| \lesssim \|\beta\| = 1$

$\frac{1}{r} = \frac{1}{p} - \frac{1}{2}$

Proof of the theorem

We have $\dim H = n, \quad \|\beta\| = 1, \quad \|\alpha\| \lesssim 1/\delta, \quad Id_H = \alpha \circ T \circ \beta.$



We apply Maurey's Factorization Theorem to β and to α:

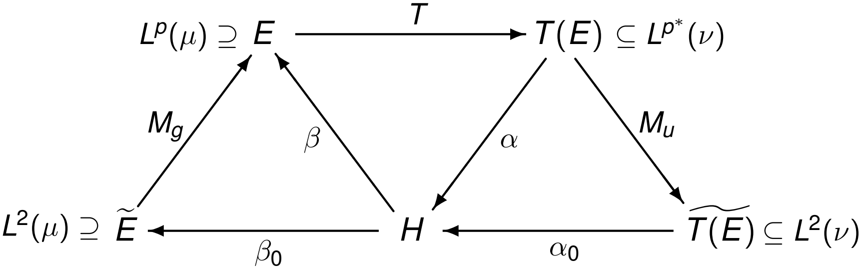
$\exists g \in L^r(\mu), \quad \|g\|_r = 1, \quad \widetilde{E} = \{f/g : f \in E\}, \quad \|\beta_0\| \lesssim \|\beta\| = 1$

$\frac{1}{r} = \frac{1}{p} - \frac{1}{2} = \frac{1}{2} - \frac{1}{p^*}$ $\qquad \exists u \in L^r(\nu), \quad \|u\|_r = 1,$

$\widetilde{T(E)} = \{h \cdot u : h \in T(E)\}, \qquad \|\alpha_0\| \lesssim \|\alpha\| \lesssim 1/\delta.$

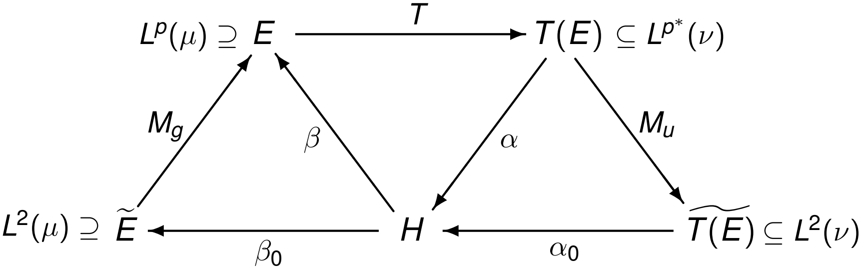
Proof of the theorem

Then $\dim H = n, \quad \|\beta_0\| \lesssim 1, \quad \|\alpha_0\| \lesssim 1/\delta, \quad Id_H = \alpha \circ T \circ \beta.$



Proof of the theorem

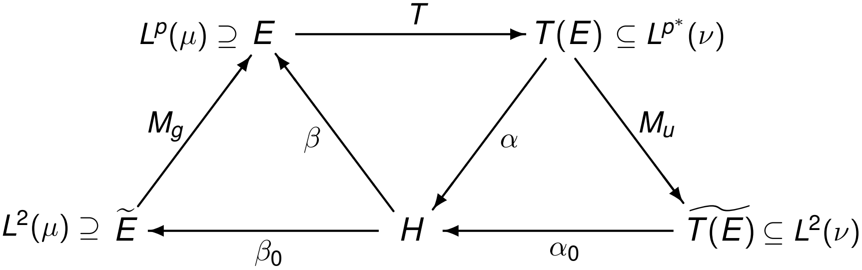
Then $\dim H = n$, $\|\beta_0\| \lesssim 1$, $\|\alpha_0\| \lesssim 1/\delta$, $Id_H = \alpha \circ T \circ \beta$.



We have $Id_H = \alpha \circ T \circ \beta = \alpha_0 \circ M_u \circ T \circ M_g \circ \beta_0$, and then

Proof of the theorem

Then $\dim H = n, \quad \|\beta_0\| \lesssim 1, \quad \|\alpha_0\| \lesssim 1/\delta, \quad Id_H = \alpha \circ T \circ \beta.$

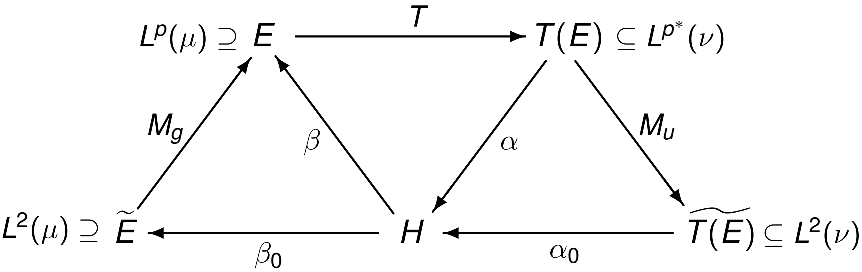


We have $Id_H = \alpha \circ T \circ \beta = \alpha_0 \circ M_u \circ T \circ M_g \circ \beta_0$, and then

$$1 = a_n(Id_H) \leq \|\beta_0\| a_n\big(M_u \circ T \circ M_g\big) \|\alpha_0\|$$

Proof of the theorem

Then $\dim H = n, \quad \|\beta_0\| \lesssim 1, \quad \|\alpha_0\| \lesssim 1/\delta, \quad Id_H = \alpha \circ T \circ \beta.$

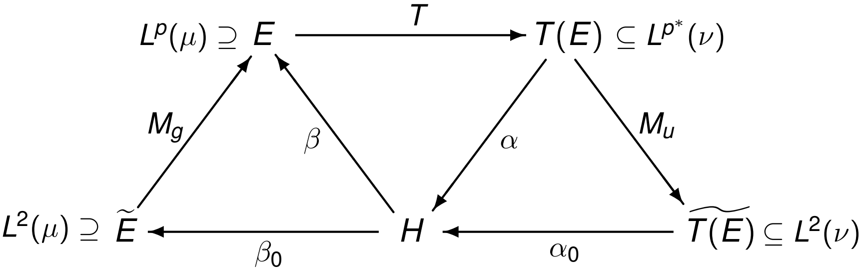


We have $Id_H = \alpha \circ T \circ \beta = \alpha_0 \circ M_u \circ T \circ M_g \circ \beta_0$, and then

$$1 = a_n(Id_H) \leq \|\beta_0\| a_n\big(M_u \circ T \circ M_g\big) \|\alpha_0\| \leq \frac{K_p}{\delta}\, n^{-1/r}\,.$$

Proof of the theorem

Then $\dim H = n, \quad \|\beta_0\| \lessapprox 1, \quad \|\alpha_0\| \lessapprox 1/\delta, \quad Id_H = \alpha \circ T \circ \beta.$



We have $Id_H = \alpha \circ T \circ \beta = \alpha_0 \circ M_u \circ T \circ M_g \circ \beta_0$, and then

$$1 = a_n(Id_H) \leq \|\beta_0\| a_n\big(M_u \circ T \circ M_g\big) \|\alpha_0\| \leq \frac{K_p}{\delta}\, n^{-1/r}\,.$$

Therefore $\delta \leq K_p n^{-1/r}$, and we have, for the Bernstein numbers,

$$b_n(T_p) \leq K_p n^{-1/r}\,, \qquad \text{for every } n.$$

Fourier Transform again

Remark: The estimate $b_n(T_p) \lessapprox n^{-1/r}$ is sharp. This can be shown with the Fourier transform $\mathcal{F}_p \colon L^p(\mathbb{T}) \to \ell^{p^*}(\mathbb{Z})$.

Fourier Transform again

Remark: The estimate $b_n(T_p) \lesssim n^{-1/r}$ is sharp. This can be shown with the Fourier transform $\mathcal{F}_p \colon L^p(\mathbb{T}) \to \ell^{p^*}(\mathbb{Z})$.

Let us denote, for $m \in \mathbb{Z}$, by e_m the exponential

$$e_m(t) = e^{imt} .$$

and consider the subspace E of $L^p(\mathbb{T})$, generated by $\{e_m : 1 \leq m \leq n\}$.

Fourier Transform again

Remark: The estimate $b_n(T_p) \lessapprox n^{-1/r}$ is sharp. This can be shown with the Fourier transform $\mathcal{F}_p \colon L^p(\mathbb{T}) \to \ell^{p^*}(\mathbb{Z})$.

Let us denote, for $m \in \mathbb{Z}$, by e_m the exponential

$$e_m(t) = e^{imt} .$$

and consider the subspace E of $L^p(\mathbb{T})$, generated by $\{e_m : 1 \leq m \leq n\}$. Then $\dim E = n$, and for every $f \in E$ we have $\hat{f}$ is supported by $\{1, 2, \ldots, n\}$ and

$$n^{1/r} \|\hat{f}\|_{p^*} \geq \|\hat{f}\|_2 = \|f\|_2 \geq \|f\|_p .$$

Fourier Transform again

Remark: The estimate $b_n(T_p) \lesssim n^{-1/r}$ is sharp. This can be shown with the Fourier transform $\mathcal{F}_p \colon L^p(\mathbb{T}) \to \ell^{p^*}(\mathbb{Z})$.

Let us denote, for $m \in \mathbb{Z}$, by e_m the exponential

$$e_m(t) = e^{imt} .$$

and consider the subspace E of $L^p(\mathbb{T})$, generated by $\{e_m : 1 \leq m \leq n\}$. Then $\dim E = n$, and for every $f \in E$ we have $\hat{f}$ is supported by $\{1, 2, \dots, n\}$ and

$$n^{1/r} \|\hat{f}\|_{p^*} \geq \|\hat{f}\|_2 = \|f\|_2 \geq \|f\|_p .$$

This yields $b_n(\mathcal{F}_p) \geq n^{-1/r}$.

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ the open unit disk, and $1 \leq p < +\infty$.

Hardy spaces

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ the open unit disk, and $1 \leq p < +\infty$. The Hardy space $H^p = H^p(\mathbb{D})$ is formed by the holomorphic functions $f : \mathbb{D} \to \mathbb{C}$ such that

$$\|f\|_{H^p} = \sup_{0 \leq r < 1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p \, dt \right)^{1/p} < +\infty \, .$$

Hardy spaces

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ the open unit disk, and $1 \leq p < +\infty$. The Hardy space $H^p = H^p(\mathbb{D})$ is formed by the holomorphic functions $f\colon \mathbb{D} \to \mathbb{C}$ such that

$$\|f\|_{H^p} = \sup_{0\leq r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p \, dt \right)^{1/p} < +\infty \,.$$

Let $\mathbb{T} = \partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$. On the torus $\mathbb{T}$ we consider the normalized arc–length measure m. Every $f \in H^p(\mathbb{D})$ has almost everywhere radial limit f^*

$$f^*(e^{it}) = \lim_{r\to 1^-} f(re^{it}) \,.$$

It is known that $f^* \in L^p(\mathbb{T}) = L^p(m)$ and $\|f\|_{H^p} = \|f^*\|_{L^p}$.

Bergman spaces

Let us denote by $\mathcal{A}$ the normalized area measure on the unit disk $\mathbb{D}$; that is $d\mathcal{A} = \frac{dx\,dy}{\pi}$. Let $1 \leq q < \infty$.

Bergman spaces

Let us denote by $\mathcal{A}$ the normalized area measure on the unit disk $\mathbb{D}$; that is $d\mathcal{A} = \frac{dx\,dy}{\pi}$. Let $1 \leq q < \infty$.

The Bergman space $B^q(\mathbb{D})$ is $B^q(\mathbb{D}) = L^p(\mathcal{A}) \cap \mathcal{H}(\mathbb{D})$; that is $B^q(\mathbb{D})$ is formed by the holomorphic functions $f\colon \mathbb{D} \to \mathbb{C}$ such that

$$\|f\|_{B^q} = \left(\int_{\mathbb{D}} |f(z)|^q \, d\mathcal{A}(z)\right)^{1/q} < +\infty\,.$$

Bergman spaces

Let us denote by $\mathcal{A}$ the normalized area measure on the unit disk $\mathbb{D}$; that is $d\mathcal{A} = \frac{dx\,dy}{\pi}$. Let $1 \le q < \infty$.

The Bergman space $B^q(\mathbb{D})$ is $B^q(\mathbb{D}) = L^p(\mathcal{A}) \cap \mathcal{H}(\mathbb{D})$; that is $B^q(\mathbb{D})$ is formed by the holomorphic functions $f\colon \mathbb{D} \to \mathbb{C}$ such that

$$\|f\|_{B^q} = \Big(\int_{\mathbb{D}} |f(z)|^q \, d\mathcal{A}(z)\Big)^{1/q} < +\infty \,.$$

Observe that, putting $f_r(z) = f(rz)$. We have

$$\|f\|_{B^q}^q = \int_0^1 \|f_r\|_{H^q}^q 2r\,dr \,.$$

Hardy and Bergman spaces

It is known that $H^p \subset B^q$ if and only if $q \leq 2p$.

It is known that $H^p \subset B^q$ if and only if $q \leq 2p$.
For $q > 2p$, if $f(z) = (1 - z)^{-2/q}$, $f \notin B^q$ and $f \in H^p$.

It is known that $H^p \subset B^q$ if and only if $q \leq 2p$.
For $q > 2p$, if $f(z) = (1-z)^{-2/q}$, $f \notin B^q$ and $f \in H^p$.

For $q < 2p$, the inclusion $H^p \hookrightarrow B^q$ is a compact operator.

Hardy and Bergman spaces

It is known that $H^p \subset B^q$ if and only if $q \leq 2p$.
For $q > 2p$, if $f(z) = (1 - z)^{-2/q}$, $f \notin B^q$ and $f \in H^p$.

For $q < 2p$, the inclusion $H^p \hookrightarrow B^q$ is a compact operator.

The inclusion $H^p \hookrightarrow B^{2p}$ is not compact.

Hardy and Bergman spaces

It is known that $H^p \subset B^q$ if and only if $q \leq 2p$.
For $q > 2p$, if $f(z) = (1 - z)^{-2/q}$, $f \notin B^q$ and $f \in H^p$.

For $q < 2p$, the inclusion $H^p \hookrightarrow B^q$ is a compact operator.

The inclusion $H^p \hookrightarrow B^{2p}$ is not compact.

Theorem (Lefèvre, R-P)

The natural inclusion $H^p \hookrightarrow B^{2p}$ is FSS, for every $p \in [1, +\infty)$.

Proof for $p = 1$

For $p = 1$ we can use Hardy inequality. We denote by $\hat{f}(n)$ the n'th Taylor coefficient in 0 of $f \in \mathcal{H}(\mathbb{D})$. Then

$$f(z) = \sum_{n \geq 0} \hat{f}(n) z^n , \quad \text{for all } z \in \mathbb{D} .$$

Proof for $p = 1$

For $p = 1$ we can use Hardy inequality. We denote by $\hat{f}(n)$ the n'th Taylor coefficient in 0 of $f \in \mathcal{H}(\mathbb{D})$. Then

$$f(z) = \sum_{n \geq 0} \hat{f}(n) z^n, \quad \text{for all } z \in \mathbb{D}.$$

Hardy inequality

For every $f \in H^1(\mathbb{D})$, we have $\displaystyle\sum_{n \geq 0} \frac{|\hat{f}(n)|}{n+1} \leq \pi \|f\|_{H^1}$.

Proof for $p = 1$

For $p = 1$ we can use Hardy inequality. We denote by $\hat{f}(n)$ the n'th Taylor coefficient in 0 of $f \in \mathcal{H}(\mathbb{D})$. Then

$$f(z) = \sum_{n \geq 0} \hat{f}(n) z^n, \quad \text{for all } z \in \mathbb{D}.$$

Hardy inequality

For every $f \in H^1(\mathbb{D})$, we have $\displaystyle\sum_{n \geq 0} \frac{|\hat{f}(n)|}{n+1} \leq \pi \|f\|_{H^1}$.

It is not difficult to see that

$$\|f\|_{B^2}^2 = \sum_{n \geq 0} \frac{|\hat{f}(n)|^2}{n+1}.$$

Proof for $p = 1$

This allows us to factorize the inclusion $H^1 \hookrightarrow B^2$ through the inclusion $L^1(\mu) \cap L^\infty(\mu) \hookrightarrow L^2(\mu)$, for μ the measure defined on $\mathbb{N}$ by

$$\mu(B) = \sum_{n \in B} \frac{1}{n+1}, \qquad B \subset \mathbb{N}.$$

Proof for $p = 1$

This allows us to factorize the inclusion $H^1 \hookrightarrow B^2$ through the inclusion $L^1(\mu) \cap L^\infty(\mu) \hookrightarrow L^2(\mu)$, for μ the measure defined on $\mathbb{N}$ by

$$\mu(B) = \sum_{n \in B} \frac{1}{n+1}, \qquad B \subset \mathbb{N}.$$

We conclude thanks to

Proposition (Lefèvre, R-P)

For every positive measure μ, the natural inclusion $L^1(\mu) \cap L^\infty(\mu) \hookrightarrow L^2(\mu)$ is FSS.

Proof for $p = 1$

Proof of the Proposition.

Proof for $p = 1$

Proof of the Proposition. Suppose E is an n-dimensional subspace such that $\|f\|_2 \geq \delta \max\{\|f\|_1, \|f\|_\infty\}$, for every $f \in E$,

Proof for $p = 1$

Proof of the Proposition. Suppose E is an n-dimensional subspace such that $\|f\|_2 \geq \delta \max\{\|f\|_1, \|f\|_\infty\}$, for every $f \in E$,

$\{f_j\}_{j=1}^n$ an orthonormal basis of E, and $\quad S = \left(\sum_{j=1}^n |f_j|^2\right)^{1/2}$.

Proof for $p = 1$

Proof of the Proposition. Suppose E is an n-dimensional subspace such that $\|f\|_2 \geq \delta \max\{\|f\|_1, \|f\|_\infty\}$, for every $f \in E$,

$\{f_j\}_{j=1}^n$ an orthonormal basis of E, and $\quad S = \left(\sum_{j=1}^n |f_j|^2\right)^{1/2}$.

From $\|\cdot\|_\infty \leq (1/\delta)\|\cdot\|_2$, we deduce $\|S\|_\infty \leq 1/\delta$.

Proof for $p = 1$

Proof of the Proposition. Suppose E is an n-dimensional subspace such that $\|f\|_2 \geq \delta \max\{\|f\|_1, \|f\|_\infty\}$, for every $f \in E$,

$\{f_j\}_{j=1}^n$ an orthonormal basis of E, and $S = \left(\sum_{j=1}^n |f_j|^2\right)^{1/2}$.

From $\|\cdot\|_\infty \leq (1/\delta)\|\cdot\|_2$, we deduce $\|S\|_\infty \leq 1/\delta$.

If $\{r_j\}$ are Rademacher functions, we have

$$\|S\|_1 \leq K_1 \int \int_0^1 \left|\sum_{j=1}^n r_j(t) f_j\right| dt\, d\mu$$

Proof for $p = 1$

Proof of the Proposition. Suppose E is an n-dimensional subspace such that $\|f\|_2 \geq \delta \max\{\|f\|_1, \|f\|_\infty\}$, for every $f \in E$,

$\{f_j\}_{j=1}^n$ an orthonormal basis of E, and $S = \left(\sum_{j=1}^n |f_j|^2\right)^{1/2}$.

From $\|\cdot\|_\infty \leq (1/\delta)\|\cdot\|_2$, we deduce $\|S\|_\infty \leq 1/\delta$.

If $\{r_j\}$ are Rademacher functions, we have

$$\|S\|_1 \leq K_1 \int\int_0^1 \left|\sum_{j=1}^n r_j(t) f_j\right| dt\, d\mu = K_1 \int_0^1 \left\|\sum_{j=1}^n r_j(t) f_j\right\|_{L^1} dt$$

Proof for $p = 1$

Proof of the Proposition. Suppose E is an n-dimensional subspace such that $\|f\|_2 \geq \delta \max\{\|f\|_1, \|f\|_\infty\}$, for every $f \in E$,

$\{f_j\}_{j=1}^n$ an orthonormal basis of E, and $S = \left(\sum_{j=1}^n |f_j|^2\right)^{1/2}$.

From $\|\cdot\|_\infty \leq (1/\delta)\|\cdot\|_2$, we deduce $\|S\|_\infty \leq 1/\delta$.

If $\{r_j\}$ are Rademacher functions, we have

$$\|S\|_1 \leq K_1 \int\int_0^1 \left|\sum_{j=1}^n r_j(t) f_j\right| dt\, d\mu = K_1 \int_0^1 \left\|\sum_{j=1}^n r_j(t) f_j\right\|_{L^1} dt$$
$$\leq \frac{K_1}{\delta} \left(\int_0^1 \left\|\sum_{j=1}^n r_j(t) f_j\right\|_{L^2}^2 dt\right)^{1/2} = \frac{K_1 \sqrt{n}}{\delta}.$$

Proof for $p = 1$

Proof of the Proposition. Suppose E is an n-dimensional subspace such that $\|f\|_2 \geq \delta \max\{\|f\|_1, \|f\|_\infty\}$, for every $f \in E$,

$\{f_j\}_{j=1}^n$ an orthonormal basis of E, and $S = \left(\sum_{j=1}^n |f_j|^2\right)^{1/2}$.

From $\|\cdot\|_\infty \leq (1/\delta)\|\cdot\|_2$, we deduce $\|S\|_\infty \leq 1/\delta$.

If $\{r_j\}$ are Rademacher functions, we have

$$\|S\|_1 \leq K_1 \int\int_0^1 \left|\sum_{j=1}^n r_j(t) f_j\right| dt\, d\mu = K_1 \int_0^1 \left\|\sum_{j=1}^n r_j(t) f_j\right\|_{L^1} dt$$

$$\leq \frac{K_1}{\delta}\left(\int_0^1 \left\|\sum_{j=1}^n r_j(t) f_j\right\|_{L^2}^2 dt\right)^{1/2} = \frac{K_1\sqrt{n}}{\delta}.$$

In consequence $n = \int S^2\, d\mu \leq \|S\|_1 \|S\|_\infty \leq \frac{K_1\sqrt{n}}{\delta^2}$,

Proof for $p = 1$

Proof of the Proposition. Suppose E is an n-dimensional subspace such that $\|f\|_2 \geq \delta \max\{\|f\|_1, \|f\|_\infty\}$, for every $f \in E$,

$\{f_j\}_{j=1}^n$ an orthonormal basis of E, and $S = \left(\sum_{j=1}^n |f_j|^2\right)^{1/2}$.

From $\|\cdot\|_\infty \leq (1/\delta)\|\cdot\|_2$, we deduce $\|S\|_\infty \leq 1/\delta$.

If $\{r_j\}$ are Rademacher functions, we have

$$\|S\|_1 \leq K_1 \int\int_0^1 \left|\sum_{j=1}^n r_j(t) f_j\right| dt\, d\mu = K_1 \int_0^1 \left\|\sum_{j=1}^n r_j(t) f_j\right\|_{L^1} dt$$

$$\leq \frac{K_1}{\delta}\left(\int_0^1 \left\|\sum_{j=1}^n r_j(t) f_j\right\|_{L^2}^2 dt\right)^{1/2} = \frac{K_1\sqrt{n}}{\delta}.$$

In consequence $n = \int S^2\, d\mu \leq \|S\|_1 \|S\|_\infty \leq \frac{K_1\sqrt{n}}{\delta^2}$, and $n \leq K_1^2/\delta^4$.

Somes ingredients in the proof for $p > 1$

We use here Littlewood-Paley decomposiiton. For $j \geq 0$,

$$\Lambda_j = (2^{j-1} - 1, 2^j) \cap \mathbb{Z}.$$

Somes ingredients in the proof for $p > 1$

We use here Littlewood-Paley decomposiiton. For $j \geq 0$,

$$\Lambda_j = (2^{j-1} - 1, 2^j) \cap \mathbb{Z}.$$

For $f \in H^p$ let us define $P_j f$ by

$$P_j f(z) = \sum_{m \in \Lambda_j} \hat{f}(m) z^m , \quad z \in \mathbb{D} .$$

Littewood-Paley decomposition

For $1 < p < +\infty$, we have:

Somes ingredients in the proof for $p > 1$

We use here Littlewood-Paley decomposiiton. For $j \geq 0$,

$$\Lambda_j = (2^{j-1} - 1, 2^j) \cap \mathbb{Z}.$$

For $f \in H^p$ let us define $P_j f$ by

$$P_j f(z) = \sum_{m \in \Lambda_j} \hat{f}(m) z^m , \quad z \in \mathbb{D} .$$

Littewood-Paley decomposition

For $1 < p < +\infty$, we have:

- $\|f\|_{H^p} \approx \left\| \left(\sum_{j \geq o} |P_j f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{T})}$.

Somes ingredients in the proof for $p > 1$

We use here Littlewood-Paley decomposiiton. For $j \geq 0$,

$$\Lambda_j = (2^{j-1} - 1, 2^j) \cap \mathbb{Z}.$$

For $f \in H^p$ let us define $P_j f$ by

$$P_j f(z) = \sum_{m \in \Lambda_j} \hat{f}(m) z^m \,, \quad z \in \mathbb{D}\,.$$

Littewood-Paley decomposition

For $1 < p < +\infty$, we have:

- $\|f\|_{H^p} \approx \left\| \left(\sum_{j \geq o} |P_j f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{T})}.$
- $\|f\|_{B^p} \approx \left\| \left(\sum_{j \geq o} |P_j f|^2 \right)^{1/2} \right\|_{L^p(\mathcal{A})}.$

Somes ingredients in the proof for $p > 1$

As a consequence we have, for $1 < p \leq 2$,

$$\Big(\sum_{j\geq 0} \|P_j f\|^2_{H^p}\Big)^{1/2} \lessapprox \|f\|_{H^p} \lessapprox \Big(\sum_{j\geq 0} \|P_j f\|^p_{H^p}\Big)^{1/p};$$

Somes ingredients in the proof for $p > 1$

As a consequence we have, for $1 < p \leq 2$,

$$\Big(\sum_{j\geq 0} \|P_j f\|_{H^p}^2\Big)^{1/2} \lesssim \|f\|_{H^p} \lesssim \Big(\sum_{j\geq 0} \|P_j f\|_{H^p}^p\Big)^{1/p};$$

and, for $p \geq 2$,

$$\Big(\sum_{j\geq 0} \|P_j f\|_{H^p}^p\Big)^{1/p} \lesssim \|f\|_{H^p} \lesssim \Big(\sum_{j\geq 0} \|P_j f\|_{H^p}^2\Big)^{1/2};$$

Lemma 1

If $f \in H^p$, and $\hat{f}(m) = 0$, for $m < N$; Then $\|f\|_{B^p}^p \leq \frac{2}{N}\|f\|_{H^p}^p$

Somes ingredients in the proof for $p > 1$

Lemma 1

If $f \in H^p$, and $\hat{f}(m) = 0$, for $m < N$; Then $\|f\|_{B^p}^p \leq \frac{2}{N}\|f\|_{H^p}^p$

Define $H_k^p = \{f \in H^p : \hat{f}(m) = 0, \text{ for all } m \notin \Lambda_k\}$

Somes ingredients in the proof for $p > 1$

Lemma 1

If $f \in H^p$, and $\hat{f}(m) = 0$, for $m < N$; Then $\|f\|_{B^p}^p \leq \frac{2}{N}\|f\|_{H^p}^p$

Define $H_k^p = \{f \in H^p : \hat{f}(m) = 0, \text{ for all } m \notin \Lambda_k\}$

Lemma 2

For $f \in H_k^p$, we have $\|f\|_{B^p}^p \approx 2^{-k}\|f\|_{H^p}^p$

Somes ingredients in the proof for $p > 1$

Lemma 1

If $f \in H^p$, and $\hat{f}(m) = 0$, for $m < N$; Then $\|f\|_{B^p}^p \leq \frac{2}{N}\|f\|_{H^p}^p$

Define $H_k^p = \{f \in H^p : \hat{f}(m) = 0, \text{ for all } m \notin \Lambda_k\}$

Lemma 2

For $f \in H_k^p$, we have $\|f\|_{B^p}^p \approx 2^{-k}\|f\|_{H^p}^p$

Lemma 3

Let J_k be the inclusion $J_k : H_k^p \hookrightarrow B^{2p}$. Then the sequence $\{J_k\}_{k\geq 0}$ is uniformly finitely strictly singular.

Somes ingredients in the proof for $p > 1$

Proposition

For $1 < p < +\infty$, there exists $M_p > 0$ such that:

Somes ingredients in the proof for $p > 1$

Proposition

For $1 < p < +\infty$, there exists $M_p > 0$ such that:

(1) $\|f\|_{B^{2p}} \leq M_p \Big(\sum_k \|P_k f\|_{H^p}^{2p} \Big)^{1/2p}$, when $p \leq 2$, and

Somes ingredients in the proof for $p > 1$

Proposition

For $1 < p < +\infty$, there exists $M_p > 0$ such that:

(1) $\|f\|_{B^{2p}} \leq M_p \left(\sum_k \|P_k f\|_{H^p}^{2p} \right)^{1/2p}$, when $p \leq 2$, and

(2) $\|f\|_{B^{2p}} \leq M_p (\sup_k \|P_k f\|_{H^p})^{1/2} \left(\sum_k \|P_k f\|_{H^p}^{p} \right)^{1/2p}$, when $p \geq 2$.

Somes ingredients in the proof for $p > 1$

Proposition

For $1 < p < +\infty$, there exists $M_p > 0$ such that:

(1) $\|f\|_{B^{2p}} \leq M_p \left(\sum_k \|P_k f\|_{H^p}^{2p}\right)^{1/2p}$, when $p \leq 2$, and

(2) $\|f\|_{B^{2p}} \leq M_p (\sup_k \|P_k f\|_{H^p})^{1/2} \left(\sum_k \|P_k f\|_{H^p}^{p}\right)^{1/2p}$, when $p \geq 2$.

Proof of (1) for $p = 2$.

Somes ingredients in the proof for $p > 1$

Proposition

For $1 < p < +\infty$, there exists $M_p > 0$ such that:

(1) $\|f\|_{B^{2p}} \leq M_p\Big(\sum_k \|P_k f\|_{H^p}^{2p}\Big)^{1/2p}$, when $p \leq 2$, and

(2) $\|f\|_{B^{2p}} \leq M_p(\sup_k \|P_k f\|_{H^p})^{1/2}\Big(\sum_k \|P_k f\|_{H^p}^{p}\Big)^{1/2p}$, when $p \geq 2$.

Proof of (1) for $p = 2$. Put $f_k = P_k f$. By Littlewood–Paley

$$\|f\|_{B^4}^4 \lessapprox \int_{\mathbb{D}} \Big(\sum_k |f_k|^2\Big)^2 \, d\mathcal{A} = \sum_{k,l} \int_{\mathbb{D}} |f_k|^2 |f_l|^2 \, d\mathcal{A} = \sum_{k,l} \|f_k f_l\|_{B^2}^2$$

Somes ingredients in the proof for $p > 1$

Proposition

For $1 < p < +\infty$, there exists $M_p > 0$ such that:

(1) $\|f\|_{B^{2p}} \leq M_p \left(\sum_k \|P_k f\|_{H^p}^{2p}\right)^{1/2p}$, when $p \leq 2$, and

(2) $\|f\|_{B^{2p}} \leq M_p (\sup_k \|P_k f\|_{H^p})^{1/2} \left(\sum_k \|P_k f\|_{H^p}^{p}\right)^{1/2p}$, when $p \geq 2$.

Proof of (1) for $p = 2$. Put $f_k = P_k f$. By Littlewood–Paley

$$\|f\|_{B^4}^4 \lessapprox \int_{\mathbb{D}} \Big(\sum_k |f_k|^2\Big)^2 \, d\mathcal{A} = \sum_{k,l} \int_{\mathbb{D}} |f_k|^2 |f_l|^2 \, d\mathcal{A} = \sum_{k,l} \|f_k f_l\|_{B^2}^2$$

If $k \leq l$, we have $\|f_k f_l\|_{B^2}^2 \lessapprox 2^{-l} \|f_k f_l\|_{H^2}^2 \leq 2^{k-l} \|f_k\|_{H^2}^2 \|f_l\|_{H^2}^2$.

Somes ingredients in the proof for $p > 1$

Therefore

$$\|f\|_{B^4}^4 \lesssim 2 \sum_{k \leq l} \|f_k f_l\|_{B^2}^2 \lesssim 2 \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} 2^{-j} \|f_k\|_{H^2}^2 \|f_{k+j}\|_{H^2}^2$$

Somes ingredients in the proof for $p > 1$

Therefore

$$\|f\|_{B^4}^4 \lesssim 2\sum_{k\le l}\|f_k f_l\|_{B^2}^2 \lesssim 2\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}2^{-j}\|f_k\|_{H^2}^2\|f_{k+j}\|_{H^2}^2$$

by Cauchy-Schwartz

$$\|f\|_{B^4}^4 \lesssim \sum_j 2^{-j}\Big(\sum_k \|f_k\|_{H^2}^4\Big)^{1/2}\Big(\sum_k \|f_{k+j}\|_{H^2}^4\Big)^{1/2} \lesssim \sum_k \|f_k\|_{H^2}^4 .$$