Perturbation classes for semi-Fredholm operators in some Banach lattices

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Workshop Operators and Banach lattices
Madrid, 25-26 October 2012

Joint work with Javier Pello, Universidad Rey Juan Carlos, Móstoles.

Semi-Fredholm operators

X, Y Banach spaces;
$T : X \longrightarrow Y$ bounded operator: $T \in \mathcal{L}(X, Y)$.

T *upper semi-Fredholm:* $\dim \ker(T) < \infty$ and $R(T)$ closed.

$\Phi_+(X, Y)$: upper semi-Fredholm operators from X into Y.

$$T \in \Phi_+(X, Y) \Rightarrow \begin{array}{l} X = \ker(T) \oplus M, \text{ with } M \text{ closed subspace,} \\ T \text{ isomorphism from } M \text{ into } Y. \end{array}$$

$T \in \Phi_+$ means T isomorphism into, up to a finite dim. subspace.

T *lower semi-Fredholm:* dim $Y/R(T) < \infty$ and $R(T)$ closed.

$\Phi_-(X, Y)$: lower semi-Fredholm operators from X into Y.

$T \in \Phi_-(X, Y)$ means T surjective, up to a finite dim. subspace.

Duality relations:

- $T \in \Phi_-(X, Y) \Leftrightarrow T^* \in \Phi_+(Y^*, X^*)$;
- $T \in \Phi_+(X, Y) \Leftrightarrow T^* \in \Phi_-(Y^*, X^*)$.

Let $\mathcal{S}$ be Φ_+ or Φ_-, and suppose $\mathcal{S}(X, Y) \neq \emptyset$

$$P\mathcal{S}(X, Y) := \{K \in \mathcal{L}(X, Y) : \forall T \in \mathcal{S}(X, Y), T + K \in \mathcal{S}\}.$$

Perturbation class of $\mathcal{S}$.

Questions: for concrete pairs of spaces X and Y,

- determine $P\mathcal{S}(X, Y)$;
- find intrinsic characterizations $P\mathcal{S}(X, Y)$.

Strictly singular and strictly cosingular operators

Given a closed subspace M of X,
$J_M : M \to X$ is the inclusion and $Q_M : X \to X/M$ the quotient map.

$K \in \mathcal{L}(X, Y)$ *strictly singular*
$K \in \mathcal{SS}(X, Y)$: $(KJ_M)^{-1}$ continuous $\Rightarrow \dim M < \infty$.

Kato (1954): $\mathcal{SS}(X, Y) \subset P\Phi_+(X, Y)$ when $\Phi_+(X, Y) \neq \emptyset$.

$K \in \mathcal{L}(X, Y)$ *strictly cosingular*
$K \in \mathcal{SC}(X, Y)$: $Q_N K$ surjective $\Rightarrow \dim Y/N < \infty$.

Vladimirskii (1967): $\mathcal{SC}(X, Y) \subset P\Phi_-(X, Y)$ when $\Phi_-(X, Y) \neq \emptyset$.

Perturbation classes problem for Φ_+ and Φ_-

Gohberg, Markus, Feldman (1960):

- $\mathcal{SS}(X, Y) = P\Phi_+(X, Y)$? (when $\Phi_+(X, Y) \neq \emptyset$)

Caradus, Pfaffenberger, Yood (1974):

- $\mathcal{SC}(X, Y) \subset P\Phi_-(X, Y)$? (when $\Phi_-(X, Y) \neq \emptyset$)

Positive answers for some pairs of spaces X, Y provide intrinsic characterizations of the perturbation classes.

Some positive solutions (before 2002)

We have $P\Phi_+(X, Y) = \mathcal{SS}(X, Y)$ in the following cases:

- Y subprojective. Whitley (1964);
- $X = Y = L_p(\mu)$, $1 \leq p \leq \infty$. Weis (1977);
- X hereditarily indecomposable. Aiena, G. (2001);
- X separable and $Y \supset C[0, 1]$ complemented.
  Aiena, G., Martinón (2002).

We have $P\Phi_-(X, Y) = \mathcal{SC}(X, Y)$ in the following cases:

- X superprojective. Aiena, G. (2001);
- $X = Y = L_p(\mu)$, $1 \leq p \leq \infty$. Weis (1977);
- Y quotient indecomposable. Aiena, G. (2001);
- $X \supset \ell_1$ complemented and Y separable.
  Aiena, G., Martinón (2002).

Some counterexamples

1. There exists a reflexive space X for which $\mathcal{SS}(X) \neq P\Phi_+(X)$ and $\mathcal{SC}(X^*) \neq P\Phi_-(X^*)$ G. (2003).
2. There exists a space Z with $Z^* \simeq \ell_1$ for which $\mathcal{SS}(Z) = P\Phi_+(Z)$ and $\mathcal{SC}(Z) \neq P\Phi_-(Z)$ G. (2011).
3. For each $1 < p < \infty$ there exists a reflexive X_p such that $\mathcal{SC}(X_p \times \ell_p) \neq P\Phi_-(X_p \times \ell_p)$ and $\mathcal{SS}(X_p^* \times \ell_p^*) \neq P\Phi_+(X_p^* \times \ell_p^*)$ Giménez, G., Martínez-Abejón (2012).

Question 1. Find counterexamples with classical Banach spaces.

Further positive solutions

Theorem (G., Salas-Brown (2010))

Suppose that Y contains L_p and one of the following conditions holds:

1. $p = 1$ *and Y is weakly sequentially complete;*
2. $1 < p < 2$ *and Y satisfies the Orlicz property;*
3. $2 \leq p \leq \infty$.

Then $P\Phi_+(L_p, Y) = \mathcal{SS}(L_p, Y)$.

Orlicz property: every weakly null (x_n) with $\inf_n \|x_n\| > 0$ has a subsequence satisfying a lower 2-estimate:

$$\|\sum_{k=1}^{\infty} a_k x_{n_k}\| \geq C\Big(\sum_{k=1}^{\infty} |a_k|^2\Big)^{1/2} \quad \text{for each } (a_k) \subset \mathbb{K}.$$

Corollary

For $1 \leq q \leq p < 2$, $P\Phi_+(L_p, L_q) = \mathcal{SS}(L_p, L_q)$.

Proposition

Suppose that X has a quotient isomorphic to L_q and one of the following conditions holds:

1. $2 < q < \infty$ *and* X^* *satisfies the Orlicz property;*
2. $1 \leq q \leq 2$.

Then $P\Phi_-(X, L_q) = \mathcal{SC}(X, L_q)$.

Corollary

For $2 \leq q \leq p \leq \infty$, $P\Phi_-(L_p, L_q) = \mathcal{SC}(L_p, L_q)$.

Further positive solutions revisited

We saw that $P\Phi_+(L_p, Y) = \mathcal{SS}(L_p, Y)$ when Y contains L_p,

1. $p = 1$ and Y is weakly sequentially complete;
2. $1 < p < 2$ and Y satisfies the Orlicz property;
3. $2 \leq p \leq \infty$.

Case 3: For $2 \leq p < \infty$ the space L_p is strongly subprojective: every inf. dim. closed subspace contains an inf. dim. subspace complemented in L_p with complement isomorphic to L_p.

Cases 1 and 2: We need additional conditions on Y.

Questions.

- Are these additional conditions necessary?
- Is it possible to replace L_p by other Banach lattices?

Strictly singular operators on Banach lattices

Let X be a Banach lattice, Y a Banach space, and $T \in Lc(X, Y)$.

T disjointly strictly singular: M subspace of X generated by a disjoint sequence $\Rightarrow$ TJ_M is not an isomorphism into.

Theorem (FHKT, 2009)

Under certain conditions, $T \in Lc(X, Y)$ is strictly singular if and only if it is disjointly strictly singular and ℓ_2-singular.

[FHKT, 2009] J. Flores, F.L. Hernández, N.J. Kalton and P. Tradacete. *J. London Math. Soc. 79 (2009), 612–630.*

Perturbation classes on Banach lattices (1)

Theorem

Let $p \in (1, \infty)$ and let X be a Banach lattice with finite cotype such that
(a) every copy of ℓ_2 in X contains a complemented copy;
(b) every subspace of X spanned by a disjoint sequence contains a further subspace complemented in X and isomorphic to ℓ_p;
(c) for every subspace M of X isomorphic to ℓ_p, there exist subspaces N of M and H of X with $H \simeq X$, $N \cap H = 0$ and $N + H$ is closed.

Let Y be a Banach space containing an isomorphic copy of X and such that $\mathcal{SS}(\ell_2, Y) = \mathcal{K}(\ell_2, Y)$.

Then $P\Phi_+(X, Y) = \mathcal{SS}(X, Y)$.

Perturbation classes on Banach lattices (2)

$P\Phi_+(X, Y) = \mathcal{SS}(X, Y)$ for

$X = L_{p,q}(0, 1)$, $L_{p,q}(0, \infty)$, or $\Lambda(W, p)$ $(1 < p < 2,\ 1 \leq q < \infty)$
and Y containing a copy of X and satisfying $\mathcal{SS}(\ell_2, Y) = \mathcal{K}(\ell_2, Y)$.

Perturbation classes on Banach lattices (3)

Theorem

Let $p \in (1, \infty)$ and let Y be a reflexive Banach lattice with finite type such that
(a) every copy of ℓ_2 in Y^* contains a complemented copy;
(b) every subspace of Y^* spanned by a disjoint sequence contains a further subspace complemented in Y^* and isomorphic to ℓ_p;
(c) for every subspace M of Y^* isomorphic to ℓ_p, there exist subspaces $N \subseteq M$ and $H \subseteq Y^*$ such that H is isomorphic to Y^*, $N \cap H = 0$ and $N + H$ is closed.

Let X be a Banach space admitting a quotient isomorphic to X and such that $\mathcal{SS}(\ell_2, X^*) = \mathcal{K}(\ell_2, X^*)$.

Then $P\Phi_-(X, Y) = \mathcal{SC}(X, Y)$.

Perturbation classes on Banach lattices (4)

$P\Phi_-(X, Y) = \mathcal{SC}(X, Y)$ for

$Y = L_{p,q}(0, 1)$, $L_{p,q}(0, \infty)$, or $\Lambda(W, p)$ $(2 < p < \infty, 1 < q < \infty)$

and X admitting a quotient isomorphic to Y and satisfying $\mathcal{SS}(\ell_2, X^*) = \mathcal{K}(\ell_2, X^*)$.

Thank you for your attention.