# Boyd Indices on Banach Function Spaces

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## Rearrangement invariant Banach function space

Let $X$ be a rearrangement invariant Banach function space (r.i.); that is

$$||f||_X = ||f^*||_{\bar{X}}$$

where $f^*$ is the decreasing rearrangement of $f$ respect to the Lebesgue measure:

$$f^*(t) = \inf\{s > 0; \lambda_f(s) \leq t\}$$

with

$$\lambda_f(s) = |\{x; |f(x)| > s\}|.$$

**Examples:**

(i) $X = L^p$, $1 \leq p \leq \infty$.

(ii) Orlicz spaces (Ex.: $L \log L$).

(iii) Lorentz spaces $L^{p,q}$.

**Definition 1** *The upper Boyd index $\alpha_X$ is defined as follows:*

$$\alpha_X := \lim_{t\to\infty} \frac{\log ||D_t||_X}{\log t},$$

*with $||D_t||_X$ the norm of the dilation operator*

$$||D_t||_X = \sup_{||f||_X\leq 1} ||D_t f||_X, \qquad D_t f(s) = f(s/t).$$

*And the lower Boyd index $\beta_X$ is defined by*

$$\beta_X := \lim_{t\to 0} \frac{\log ||D_t||_X}{\log t}.$$

We will be interested in the following classical result concerning the following two important operators in Harmonic Analysis: the Hardy-Littlewood maximal operator

$$Mf(x) = \sup_{x\in I} \frac{1}{|I|} \int_I |f(y)| dy,$$

where the supremum is taken over all intervals $I$ of the real line, and the Hilbert transform

$$Hf(x) = \frac{1}{\pi} \lim_{\varepsilon \to 0^+} \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y}\, dy,$$

whenever this limits exists almost everywhere.

## Classical Results

**Theorem 1 (Lorentz-Shimogaki, 1967)**

*Given a r.i. Banach Function Space $X$ on $\mathbb{R}$,*

$$M : X \longrightarrow X \qquad \textit{is bounded} \qquad \Longleftrightarrow \qquad \alpha_X < 1.$$

**Theorem 2 (Boyd, 1967)**

$$H : X \longrightarrow X \qquad \textit{is bounded} \qquad \Longleftrightarrow \qquad \alpha_X < 1, \quad \textit{and} \quad \beta_X > 0.$$

**Example 1:** $X = L^p$, then $||D_t||_{L^p} = t^{1/p}$ and therefore

$$\alpha_X = \beta_X = \frac{1}{p}.$$

Hence Lorentz-Shimogaki Theorem says that

$$M : L^p \longrightarrow L^p \iff p > 1,$$

and Boyd's Theorem

$$H : L^p \longrightarrow L^p \iff 1 < p < \infty.$$

That is, we recover the Riesz-Kolmogorov theorem (1920's).

## Idea of the Proof

## Lorentz-Shimogaki's

It is known that

$$(Mf)^*(t) \approx \frac{1}{t}\int_0^t f^*(s)ds = \int_0^1 f^*(st)ds,$$

and hence

$$||Mf||_X \leq \int ||f(\cdot s)||_X ds \leq ||f||_X \int_0^1 ||D_{1/s}||ds.$$

Everything follows now, from the fact that

$$\int_0^1 ||D_{1/s}||ds < \infty \iff ||D_{1/s}|| \leq \frac{1}{s^\alpha}, \alpha < 1 \iff \alpha_X < 1.$$

and the hidden reason because all these equivalences are true is the fact that the function $h(s) = ||D_{1/s}||$ is submultiplicative; that is $h(st) \leq h(s)h(t)$.

**Boyd's theorem:**

$$(Hf)^*(t) \lesssim \frac{1}{t}\int_0^t f^*(s)ds + \int_t^\infty f^*(s)\frac{ds}{s} = \int_0^1 f^*(st)ds + \int_0^1 f^*\Big(\frac{t}{s}\Big)\frac{ds}{s},$$

and, as before,

$$||Hf||_X \lesssim ||f||_X \bigg(\int_0^1 ||D_{1/s}||ds + \int_0^1 ||D_s||\frac{ds}{s}\bigg)$$

The first term is controlled by the condition $\alpha_X < 1$ and the second term by $\beta_X > 0$ and again, the main property is the submultiplicity property.

**Example 2:**

## Classical Lorentz spaces

Let us recall that the Lorentz spaces $\Lambda^p(w)$ were introduced by Lorentz in 1951 and are defined by the condition

$$\|f\|_{\Lambda^p(w)} = \bigg(\int_0^\infty f^*(t)^p w(t)dt\bigg)^{1/p} < \infty.$$

**Examples:**

(i) If $w = 1$, $\Lambda^p(w) = L^p$.

(ii) If $w(t) = t^{p/q-1}$, $\Lambda^p(w) = L^{q,p}$.

(iii) If $w(t) = 1 + \log^+ \frac{1}{t}$, and $p = 1$, $\Lambda^1(w) = L \log L$.

(iv) If $p = 1$, $\Lambda^1(w) = \Lambda_W$ is a minimal Lorentz space with fundamental function $W$.

**Question 1:**

Since $\Lambda^p(w)$ are r.i., what does the Lorentz-Shimogaki and Boyd's theorems say?

**Proposition 1**

*For every* $0 < p < \infty$,

$$\alpha_{\Lambda^p(w)} := \lim_{t\to\infty} \frac{\log \overline{W}^{1/p}(t)}{\log t}, \qquad \beta_{\Lambda^p(w)} := \lim_{t\to 0} \frac{\log \overline{W}^{1/p}(t)}{\log t},$$

*where*

$$\overline{W}(t) := \sup_{s\in[0,+\infty)} \frac{W(st)}{W(s)}.$$

Then, the Lorentz-Shimogaki's Theorem applied to $\Lambda^p(w)$ says that

**Corollary 3**

$$M : \Lambda^p(w) \longrightarrow \Lambda^p(w) \quad \Longleftrightarrow \quad \lim_{t\to\infty} \frac{\log \overline{W}^{1/p}(t)}{\log t} < 1.$$

**Corollary 4**

$$H : \Lambda^p(w) \longrightarrow \Lambda^p(w) \iff \lim_{t\to\infty} \frac{\log \overline{W}^{1/p}(t)}{\log t} < 1,\ \lim_{t\to 0} \frac{\log \overline{W}^{1/p}(t)}{\log t} > 0.$$

**Theorem 5 (Ariño-Muckenhoupt, 1990)**

*For every* $p > 0$,

$$M : \Lambda^p(w) \longrightarrow \Lambda^p(w)$$

*if and only if* $w \in B_p$, *that is*

$$r^p \int_r^\infty \frac{w(t)}{t^p} dt \lesssim \int_0^r w(t) dt.$$

**Theorem 6 (E. Sawyer, 1991)**

*For every* $p > 0$,

$$H : \Lambda^p(w) \longrightarrow \Lambda^p(w)$$

*if and only if* $w \in B_p \cap B_\infty^*$, *where* $w \in B_\infty^*$ *if*

$$\int_0^r \frac{1}{t}\int_0^t w(s)dsdt \lesssim \int_0^r w(s)ds.$$

As a consequence:

**Corollary 7**

$$\lim_{t\to\infty} \frac{\log \overline{W}^{1/p}(t)}{\log t} < 1 \qquad \textit{if and only if} \qquad w \in B_p.$$

**Corollary 8**

$$\lim_{t\to 0} \frac{\log \overline{W}^{1/p}(t)}{\log t} > 0 \qquad \textit{if and only if} \qquad w \in B_\infty^*.$$

**Remark 1**

$$w \in B_p \qquad \textit{if and only if} \qquad \overline{W}^{1/p}(t) \lesssim t^\alpha, \qquad \alpha < 1.$$

**Lemma 1**

*For every submutiplicative increasing function $\varphi$ defined in $[1,\infty)$,*

$$\lim_{t\to\infty} \frac{\log \varphi(t)}{\log t} < 1 \qquad \textit{if and only if} \qquad \varphi(x) \lesssim x^{\alpha}$$

*for some $\alpha < 1$ and every $x > 1$.*

**Question 2:** How can we define Boyd's indices in spaces which are Banach Function Spaces non necessarily rearrangement invariant?

# Weighted Lebesgue spaces

In the 70's the following theorem was proved.

**Theorem 9 (Muckenhoupt, 1972)**

*If* $p > 1$,

$$M : L^p(u) \to L^p(u)$$

*if and only if* $u \in A_p$*:*

$$\sup_Q \left( \frac{1}{|Q|} \int_Q u(x)dx \right) \left( \frac{1}{|Q|} \int_Q u(x)^{-1/(p-1)} dx \right)^{p-1} < \infty.$$

**Theorem 10 (Hunt, Muckenhoupt, Wheeden, 1973)**

$$H : L^p(u) \to L^p(u) \iff u \in A_p$$

**Question 3:**

Is there an analogue to the Lorentz-Shimogaki and Boyd's theorem for the spaces $L^p(u)$?

In 2007, Lerner and Pérez defined the upper Boyd's index for more general spaces than r.i. and proved the analogue to Lorentz-Shimogaki's theorem.

*To pursue this direction we introduce a generalized definition of the upper Boyd index. In this new approach the main role is played by the so-called local maximal operator* $m_\lambda f$ *defined by*

$$m_\lambda f(x) = \sup_{x\in Q}(f\chi_Q)^*(\lambda|Q|), \qquad 0 < \lambda < 1.$$

*We give the following generalization of the upper Boyd index.*

**Definition 2** *For any quasi-Banach function space* $X$ *over* $\mathbb{R}^n$ *we define the non-increasing function* $\Phi_X$ *on* $(0,1)$ *as the operator norm* *of* $m_\lambda$ *on* $X$*, namely,*

$$\Phi_X(\lambda) = ||m_\lambda||_X = \sup_{||f||_X \leq 1} ||m_\lambda f||_X.$$

*We define the generalized upper Boyd index as*

$$\alpha_X = \lim_{\lambda \to 0} \frac{\log \Phi_X(\lambda)}{\log \frac{1}{\lambda}}.$$

**Theorem 11 (Lerner-Pérez, 2007)**

$$M : X \longrightarrow X$$

*if and only if* $\alpha_X < 1$.

**Examples:**

1)

$$\alpha_{L^p(u)} = \lim_{t \to \infty} \frac{\log \nu_u^{1/p}(t)}{\log t} < 1,$$

where

$$\nu_u(t) = \sup \left\{ \frac{u(I)}{u(E)};\ E \subset I,\ \frac{|I|}{|E|} = t \right\}$$

**Theorem 12 (Lerner-Pérez, 2007)**

$$M : L^p(u) \longrightarrow L^p(u)$$

*if and only if*

$$\alpha_{L^p(u)} = \lim_{t\to\infty} \frac{\log \nu_u^{1/p}(t)}{\log t} < 1,$$

*where*

$$\nu_u(t) = \sup \left\{ \frac{u(I)}{u(E)};\ E \subset I,\ \frac{|I|}{|E|} = t \right\}$$

**Question 4:** Where the functions $m_\lambda$ and $\Phi_X$ appear?

$$\frac{1}{|Q|}\int_Q |f(x)|dx = \int_0^1 (f\chi_Q)^*(\lambda|Q|)d\lambda,$$

and hence

$$Mf(x) \leq \int_0^1 m_\lambda f(x)d\lambda$$

$$||Mf||_X \leq \int_0^1 ||m_\lambda f||_X d\lambda \leq ||f||_X \int_0^1 \Phi_X(\lambda)dx.$$

**Question 5:** how can we define the lower Boyd index for a general BFS?

## Weighted Lorentz Spaces, 1951

The weighted Lorentz spaces $\Lambda_u^p(w)$ are defined by the condition

$$\|f\|_{\Lambda_u^p(w)} = \left(\int_0^\infty f_u^*(t)^p w(t)dt\right)^{1/p} < \infty,$$

where $f_u^*$ is the decreasing rearrangement of $f$ respect to $u$,

$$f_u^*(t) = \inf\{s > 0; \lambda_f^u(s) \leq t\}$$

with

$$\lambda_f^u(s) = u(\{x; |f(x)| > s\}).$$

**Examples:**

1) If $w = 1$, then $\Lambda_u^p(w) = L^p(u)$

2) If $u = 1$, then $\Lambda_u^p(w) = \Lambda^p(w)$

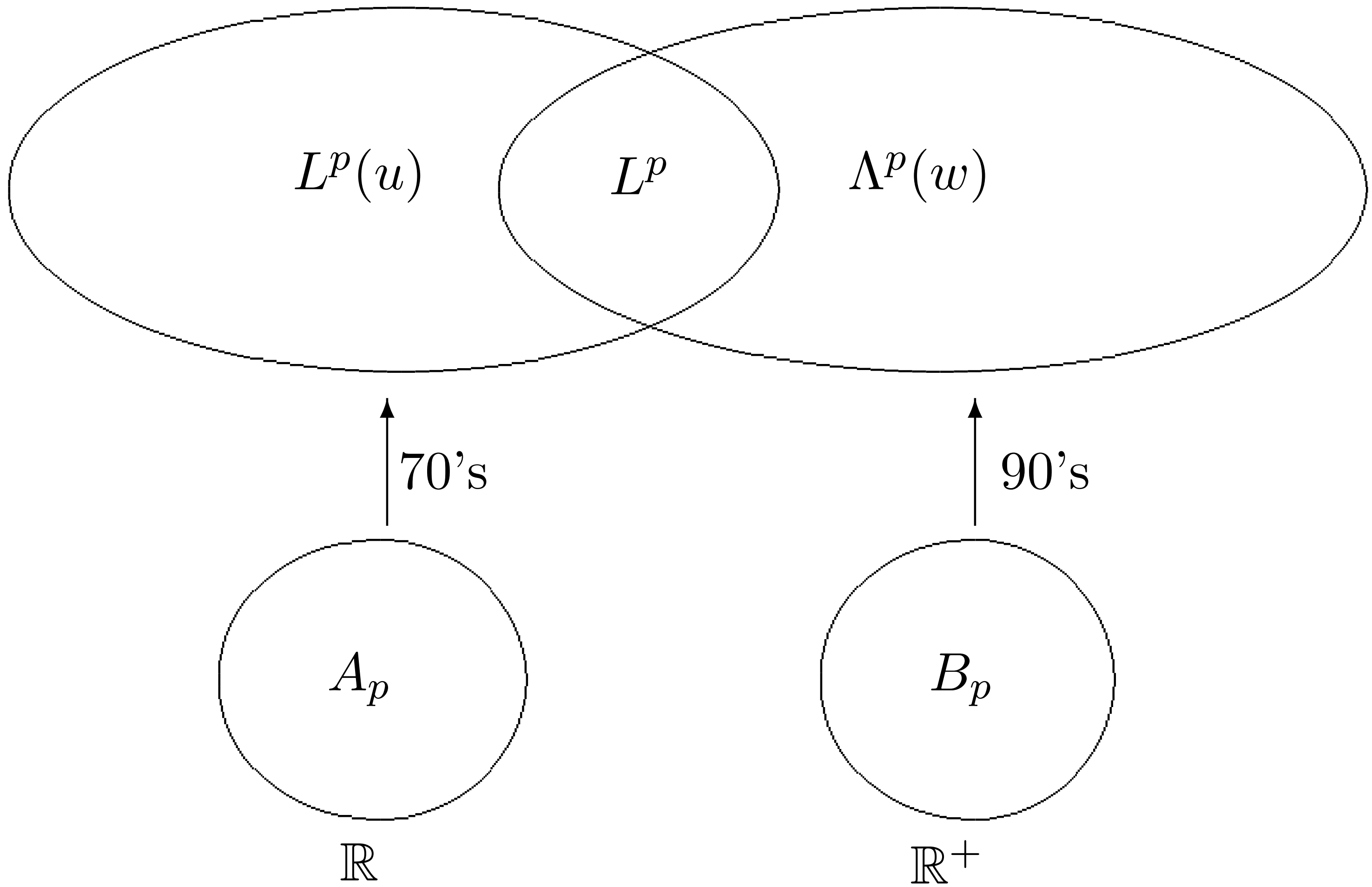
3) If $u = 1$ and $w(t) = t^{p/q-1}$ then $\Lambda_u^p(w) = L^{q,p}$.

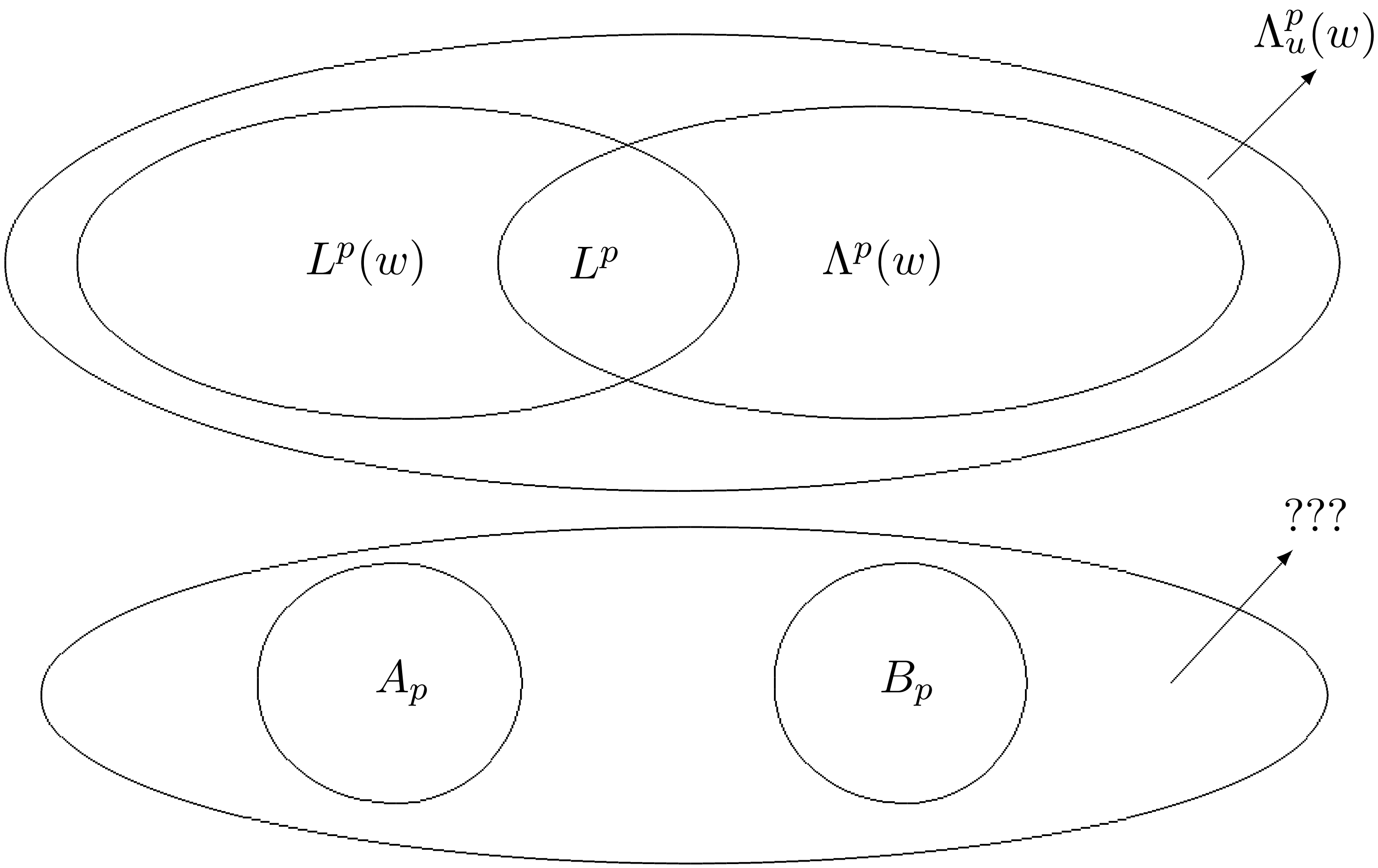
4) If $w(t) = t^{p/q-1}$ then $\Lambda_u^p(w) = L^{q,p}(u)$.

**Question 6:** (J. A. Raposo's thesis)

Which is the characterization of the weights $u$ and $w$ such that

$$M : \Lambda_u^p(w) \longrightarrow \Lambda_u^p(w)?$$


$L^p(u)$
$L^p$
$\Lambda^p(w)$
70's
90's
$A_p$
$B_p$
$\mathbb{R}$
$\mathbb{R}^+$


$\Lambda_u^p(w)$
$L^p(w)$
$L^p$
$\Lambda^p(w)$
???
$A_p$
$B_p$

**Theorem 13 (C-Raposo-Soria, 2007)**

*If* $0 < p < \infty$,

$$M : \Lambda^p_u(w) \longrightarrow \Lambda^p_u(w)$$

*is bounded if and only if there exists* $\alpha < 1$ *such that for every* $t > 1$,

$$\overline{W}_u^{1/p}(t) \lesssim t^{\alpha}, \tag{1}$$

*where, for every* $t > 1$,

$$\overline{W}_u(t) := \sup \left\{ \frac{W\left(u\left(\bigcup_j I_j\right)\right)}{W\left(u\left(\bigcup_j S_j\right)\right)} : \; S_j \subseteq I_j, \frac{|I_j|}{|S_j|} = t \right\},$$

*with* $I_j$ *disjoint intervals and all unions are finite.*

**Remark 2**

*In the paper of Lerner-Pérez, they compute*

$$\alpha_{\Lambda^p_u(w)} = \lim_{t\to\infty} \frac{\log \overline{W}_u^{\frac{1}{p}}(t)}{\log t}, \tag{2}$$

and proved that

$$M : \Lambda^p_u(w) \longrightarrow \Lambda^p_u(w) \iff \lim_{t\to\infty} \frac{\log \overline{W}_u^{\frac{1}{p}}(t)}{\log t} < 1.$$

Since we also have that

$$M : \Lambda^p_u(w) \longrightarrow \Lambda^p_u(w) \iff \overline{W}_u^{1/p}(t) \lesssim t^{\alpha}, \quad \alpha < 1$$

**Proposition 2**

*The function* $\overline{W}_u$ *is submultiplicative on* $[1, \infty)$.

**Lemma 2**

*Let* $I$ *be an interval and let* $S = \cup_{k=1}^{N}(a_k, b_k)$ *be union of disjoint intervals such that* $S \subset I$. *Then, for every* $t \in [1, |I|/|S|]$ *there exists a collection of disjoint subintervals* $\{I_n\}_{n=1}^{M}$ *satisfying that* $S \subset \cup_n I_n$ *such that, for every* $n$,

$$t|S \cap I_n| = |I_n|. \tag{3}$$

**Question 6:** (E. Agora's thesis)

Which is the characterization of the weights $u$ and $w$ such that

$$H : \Lambda^p_u(w) \longrightarrow \Lambda^p_u(w)?$$

**Theorem 14**

*If* $p > 1$ *then*

$$H : \Lambda_u^p(w) \longrightarrow \Lambda_u^p(w)$$

*if and only if the three following condition holds:*

*(i)* $u \in A_\infty = \cup_p A_p$.

*(ii)* $w \in B_\infty^*$.

*(iii)*

$$M : \Lambda_u^p(w) \longrightarrow \Lambda_u^p(w).$$

**Question:** Is there some relation between the conditions (i) and (ii) in the previous theorem and some lower Boyd's index?

## The Boyd Theorem for $\Lambda_u^p(w)$

### Definition 3

*If $\lambda \in (0,1]$, we define*

$$\underline{W_u}(\lambda) := \sup \left\{ \frac{W\left(u\left(\bigcup_j S_j\right)\right)}{W\left(u\left(\bigcup_j I_j\right)\right)} : \ S_j \subseteq I_j, \ \frac{|S_j|}{|I_j|} = \lambda \right\},$$

*where $I_j$ are disjoint open intervals and all unions are finite.*

### Proposition 3

*The function $\underline{W_u}$ is submultiplicative in $[0,1]$.*

By analogy with the case of the upper index, we give the following definition.

**Definition 4**

*We define the generalized lower Boyd index associated to* $\Lambda_u^p(w)$ *as*

$$\beta_{\Lambda_u^p(w)} := \lim_{t\to 0} \frac{\log \underline{W_u}^{1/p}(t)}{\log t}.$$

**Proposition 4**

*A couple of weights* $u$ *and* $w$ *satisfy that* $u \in A_\infty$ *and* $w \in B_\infty^*$ *if and only if*

$$\beta_{\Lambda_u^p(w)} > 0.$$

**Theorem 15**

*If* $p > 1$ *then*

$$H : \Lambda_u^p(w) \longrightarrow \Lambda_u^p(w)$$

*if and only if*

$$\alpha_{\Lambda_u^p(w)} < 1 \qquad \textit{and} \qquad \beta_{\Lambda_u^p(w)} > 0.$$

**Theorem 16**

*Let* $0 < p < \infty$. *If*

$$H : \Lambda_u^p(w) \longrightarrow \Lambda_u^p(w)$$

*is bounded then*

$$\beta_{\Lambda_u^p(w)} > 0.$$

**Theorem 17**

*Let* $0 < p < \infty$. *If*

$$\alpha_{\Lambda_u^p(w)} < 1 \qquad \text{and} \qquad \beta_{\Lambda_u^p(w)} > 0$$

*then*

$$H : \Lambda_u^p(w) \longrightarrow \Lambda_u^p(w)$$

*is bounded.*

So it remains to prove that, for every $0 < p \leq 1$,

$$H : \Lambda_u^p(w) \longrightarrow \Lambda_u^p(w) \implies \alpha_{\Lambda_u^p(w)} < 1.$$

# Indices for Banach Function Spaces

## Definition 5

*Let $X$ be a r.i. space with fundamental function $\varphi_X$ and let*

$$\overline{\varphi}_X(t) = \sup_s \frac{\varphi_X(st)}{\varphi_X(s)}.$$

*Then the lower and upper fundamental indices are defined by*

$$\underline{\beta}_X = \lim_{t\to 0} \frac{\log \overline{\varphi}_X(t)}{\log t} \quad \textit{and} \quad \overline{\beta}_X = \lim_{t\to\infty} \frac{\log \overline{\varphi}_X(t)}{\log t}.$$

**Remark 3** *If we rewrite our function* $\overline{W_u}^{1/p}$ *we see that, if* $X = \Lambda_u^p(w)$,

$$\overline{W_u}^{1/p}(\lambda) = \sup\left\{\frac{W^{1/p}\left(u\left(\bigcup_j I_j\right)\right)}{W^{1/p}\left(u\left(\bigcup_j S_j\right)\right)} : S_j \subseteq I_j, \frac{|I_j|}{|S_j|} = \lambda\right\}$$

$$= \sup\left\{\frac{||\chi_{\bigcup_j I_j}||_X}{||\chi_{\bigcup_j S_j}||_X} : S_j \subseteq I_j, \frac{|I_j|}{|S_j|} = \lambda\right\}$$

*Now, if we take the last expression and we think that* $X$ *is r.i., we obtain that*

$$\sup\left\{\frac{||\chi_{\bigcup_j I_j}||_X}{||\chi_{\bigcup_j S_j}||_X} : S_j \subseteq I_j, \frac{|I_j|}{|S_j|} = \lambda\right\} = \sup\left\{\frac{||\chi_{(0,r)}||_X}{||\chi_{(0,s)}||_X} : \frac{r}{s} = \lambda\right\}$$

$$= \sup\left\{\frac{\varphi_X(r)}{\varphi_X(s)}||_X : r = s\lambda\right\} = \overline{\varphi}_X(\lambda)$$

*And, if we rewrite our function* $\underline{W_u}^{1/p}$ *we see that, if* $X = \Lambda^p_u(w)$,

$$\underline{W_u}^{1/p}(\lambda) = \sup\left\{\frac{W^{1/p}\left(u\left(\bigcup_j S_j\right)\right)}{W^{1/p}\left(u\left(\bigcup_j I_j\right)\right)} : S_j \subseteq I_j, \frac{|S_j|}{|I_j|} = \lambda\right\}$$
$$= \sup\left\{\frac{||\chi_{\bigcup_j S_j}||_X}{||\chi_{\bigcup_j I_j}||_X} : S_j \subseteq I_j, \frac{|S_j|}{|I_j|} = \lambda\right\}$$

*and, if we take the last expression and we think that* $X$ *is r.i., we obtain that*

$$\sup\left\{\frac{||\chi_{\bigcup_j S_j}||_X}{||\chi_{\bigcup_j I_j}||_X} : S_j \subseteq I_j, \frac{|S_j|}{|I_j|} = \lambda\right\} = \sup\left\{\frac{||\chi_{(0,r)}||_X}{||\chi_{(0,s)}||_X} : \frac{r}{s} = \lambda\right\}$$
$$= \sup\left\{\frac{\varphi_X(r)}{\varphi_X(s)}||_X : r = s\lambda\right\} = \overline{\varphi}_X(\lambda).$$

**Definition 6**

*Given a Banach function space X, we define*

$$\overline{\varphi}_X(\lambda) = \sup\left\{ \frac{||\chi_{\bigcup_j I_j}||_X}{||\chi_{\bigcup_j S_j}||_X} : \ S_j \subseteq I_j, \frac{|I_j|}{|S_j|} = \lambda \right\}, \qquad \lambda \geq 1$$

*and*

$$\overline{\varphi}_X(\lambda) = \sup\left\{ \frac{||\chi_{\bigcup_j S_j}||_X}{||\chi_{\bigcup_j I_j}||_X} : \ S_j \subseteq I_j, \frac{|S_j|}{|I_j|} = \lambda \right\}, \qquad \lambda < 1.$$

*Then, we define the lower and upper fundamental indices as follows:*

$$\underline{\beta}_X = \lim_{t\to 0} \frac{\log \overline{\varphi}_X(t)}{\log t} \quad \textit{and} \quad \overline{\beta}_X = \lim_{t\to\infty} \frac{\log \overline{\varphi}_X(t)}{\log t}.$$

## Questions

(i) Which is the relation between this new upper index and the Boyd index of Pérez-Lerner?

Open question.

(ii) Is the Lorentz-Shimogaki theorem true with this new upper index?

No. In 1970, Shimogaki gave an example of a r.i. space $X$ such that the fundamental function is the same than $L^2$ but the maximal operator is not bounded in $X$.

(iii) Is the Boyd theorem true with these new upper and lower index?

No.

... etc ...

## Some progress and recent results

**Proposition 5**

*$\bar{\varphi}_X$ is submultiplicative in $[0, \infty)$.*

**Theorem 18**

$$M : X \to X \quad \Longrightarrow \quad \overline{\beta}_X < 1.$$

**Theorem 19**

*If $\overline{\beta}_X < 1$, then*

$$||M\chi_E||_X \leq ||\chi_E||_X$$

*for every measurable set $E$.*

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