Daugavet-like properties and numerical indices in some function spaces

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Universidad de Granada

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The talk is based on the papers

- Vladimir Kadets, Miguel Martín, Javier Merí and Dirk Werner,
  Lushness, numerical index one and the Daugavet property in rearrangement invariant spaces.
  Canadian J. Math. (to appear).

- Han-Ju Lee and Miguel Martín,
  Polynomial numerical indices of Banach spaces with 1-unconditional bases.
  Linear Algebra Appl. (2012).

- Han-Ju Lee, Miguel Martín and Javier Merí,
  Polynomial numerical indices of Banach spaces with absolute norm.
  Linear Algebra Appl. (2011).

Sketch of the talk

1. Introduction and preliminaries
   - Notation
   - The two main properties we are dealing with
2. Sequence spaces
   - Definitions
   - Numerical index one
   - Polynomial numerical index one
3. Function spaces
   - Definitions
   - Lush spaces
   - Daugavet property
4. Open problems

Introduction and preliminaries

Basic notation

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X real or complex Banach space.

- S_X unit sphere
- B_X closed unit ball
- $\mathbb{T}$ modulus-one scalars
- X^* dual space
- $L(X)$ bounded linear operators from X to X.
- $\mathrm{aconv}(\cdot)$ absolutely convex hull.

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The Daugavet property (Kadets-Shvidkoy-Sirotkin-Werner, 1997 - 2000)

X has the Daugavet property if

$$\|\mathrm{Id} + T\| = 1 + \|T\| \qquad \text{(DE)}$$

for rank-one operators $T \in L(X)$.

- Then every $T \in L(X)$ not fixing copies of ℓ_1 also satisfies (DE).

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Banach spaces with numerical index one (Lumer, 1968)

X has numerical index one if

$$\max_{\theta \in \mathbb{T}} \|\mathrm{Id} + \theta T\| = 1 + \|T\| \qquad \text{(aDE)}$$

for EVERY operator T on X.

- Equivalently,

$$\|T\| = \sup\{|x^*(Tx)| : x \in S_X, x^* \in S_{X^*}, x^*(x) = 1\}$$

  for EVERY $T \in L(X)$.

On the Daugavet property

On the Daugavet property

Examples

1. $C(K,E)$ when K is perfect
2. $L_1(\mu,E)$ and $L_\infty(\mu,E)$ when μ is atomless
3. the disk algebra $A(\mathbb{D})$ and H^∞
4. function algebras with perfect Choquet boundary
5. $\mathrm{Lip}(K)$ when K is a compact convex subset of ℓ_p
6. non-atomic C^*-algebras and preduals of non-atomic von Neumann algebras
7. some "big" subspaces of $C[0,1]$

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Characterization

X has the Daugavet property iff

$$B_X = \overline{\mathrm{co}}\big(\{y\in B_X \ : \ \|x-y\|\geqslant 2-\varepsilon\}\big)$$

for every $x\in S_X$ and every $\varepsilon>0$

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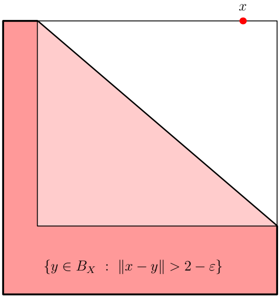

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Some results

X with the Daugavet property. Then:

- Every weakly-open subset of B_X has diameter 2.
- X contains a copy of ℓ_1.
- Actually, given $x_0 \in S_X$ and slices $\{S_n : n \geqslant 1\}$, one may take $x_n \in S_n$ $\forall n \geqslant 1$ such that $\{x_n : n \geqslant 0\}$ is equivalent to the ℓ_1-basis.
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Characterization

X has the Daugavet property iff for every $x \in S_X$, $x^* \in S_{X^*}$ and $\varepsilon > 0$, there exists $y \in B_X$ such that

$$\|x + y\| \geqslant 2 - \varepsilon \qquad \text{and} \qquad \operatorname{Re} x^*(y) > 1 - \varepsilon.$$

On the numerical index one

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Examples

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Characterization

We do not know of any operator-free characterization!!

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Some results

X with numerical index one, $\dim(X) = \infty$. Then:

- X^* is not smooth and X^* is not strictly convex.
- In some particular cases, it is possible to prove that X is not smooth and that X is not strictly convex.
- Nevertheless, there is a strictly convex **non-complete** X such that $X^* \equiv L_1(\mu)$ (and so X has numerical index one).
- In the real case, $X^* \supseteq \ell_1$.
- The norm of X cannot be Fréchet smooth.
- There are no LUR points in S_X.

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The alternative Daugavet property (M.–Oikhberg, 2007)

A Banach space X has the alternative Daugavet property (ADP) if the norm equality

$$\max_{\theta\in\mathbb{T}} \|\mathrm{Id} + \theta T\| = 1 + \|T\| \qquad \text{(aDE)}$$

holds for every for every RANK-ONE operator $T \in L(X)$.

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- There are some sufficient geometrical conditions.
- The weakest property of this kind is the following:

Lushness (Boyko–Kadets–M.–Werner, 2007)

X is lush if given $x, y \in S_X$, $\varepsilon > 0$, there is $x^* \in S_{X^*}$ such that

$$x \in S := \{z \in B_X : \operatorname{Re} x^*(z) > 1 - \varepsilon\} \quad \text{and} \quad \operatorname{dist}(y, \operatorname{aconv}(S)) < \varepsilon.$$

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Relationship between the properties

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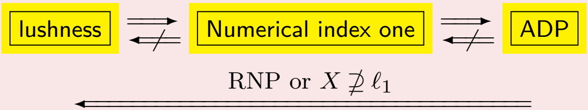

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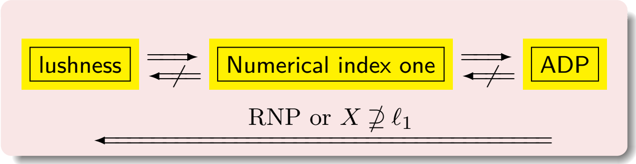

Examples

- $C\big([0,1],\ell_2\big)$ has ADP but not numerical index one
- there exists $\mathcal{X}$ with numerical index one which is not lush

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Determine which spaces have the Daugavet property or have numerical index one among Köthe sequence or function spaces.

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Determine which spaces have the Daugavet property or have numerical index one among Köthe sequence or function spaces.

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- For sequence spaces: we show which r.i. spaces have numerical index one and we show a results about spaces with polynomial numerical index one.
- For function spaces: we characterize separable r.i. spaces with the Daugavet property or which are lush.

Sequence spaces

2. Sequence spaces
   - Definitions
   - Numerical index one
   - Polynomial numerical index one

Definitions and remarks

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Definitions

1. A sequence space with absolute norm is a Banach subspace X of $\mathbb{K}^{\mathbb{N}}$ with
   - if $x, y \in \mathbb{K}^{\mathbb{N}}$ with $|x| \leqslant |y|$ and $y \in X$, then $x \in X$ with $\|x\| \leqslant \|y\|$,
   - for every $n \in \mathbb{N}$, $e_n := \mathbf{1}_{\{n\}} \in X$ with $\|e_n\| = 1$.

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2. A sequence space with absolute norm X is a rearrangement invariant (r.i.) space if, in addition,
   - for every bijection $\tau : \mathbb{N} \to \mathbb{N}$ and every $x \in X$, $\|x \circ \tau\| = \|x\|$.
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- A separable sequence space with absolute norm is nothing than a Banach space with 1-unconditional basis.
- A separable r.i. sequence space is nothing than a Banach space with 1-symmetric basis.

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Polynomial numerical index of order 2 equal to one and the 2-ADP (Choi–Garcia–Kim–Maestre, 2006; Choi–Garcia–Maestre–M., 2007)

X has polynomial numerical index of order 2 equal to one if the norm equality

$$\max_{\theta\in\mathbb{T}} \|\mathrm{Id} + \theta\, P\| = 1 + \|P\| \qquad \text{(aDE)}$$

holds for every 2-homogeneous polynomial from X to X
(the norm in of the space of all polynomials).

- If every **rank-one** 2-homogeneous polynomial from X to X satisfies (aDE), we say that X has the 2-ADP.

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Examples

- complex $C_0(L)$ has polynomial numerical index of order 2 equal to one,
- complex $C_0(L,E)$ has the 2-ADP if L is perfect,
- no real space of dimension greater than 1 is known to have the 2-ADP,
- the real or complex $L_1(\mu)$ spaces do not have the 2-ADP.

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- c_0 and ℓ_∞^m are the only complex Banach spaces with 1-unconditional basis which have polynomial numerical index of order 2 equal to one.
- Apart of $\mathbb{R}$, there is no real Banach space with 1-unconditional basis which has polynomial numerical index of order 2 equal to one.

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1. X with 1-unconditional basis and polynomial numerical index of order 2 equal to one: this implies that X has numerical index one and so, it is lush.

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Conversely

If $c_0 \subseteq X \subseteq \ell_\infty$ isometrically, then X has polynomial numerical index of order 2 equal to one.

Function spaces

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A (separable) rearrangement invariant space on $[0,1]$ is a separable Banach space X consisting on equivalence classes of locally integrable scalar functions on $[0,1]$ satisfying

(a) if $|f| \leqslant |g|$ a.e. with f measurable and $g \in X \implies f \in X$ and $\|f\| \leqslant \|g\|$.

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Examples

1. $L_p[0,1]$ spaces for $1 \leqslant p < \infty$
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3. separable Orlitz spaces

Lush spaces

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Let us give the proof of this result:

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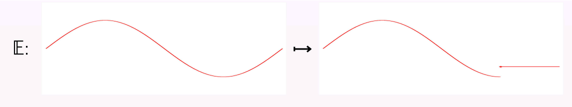

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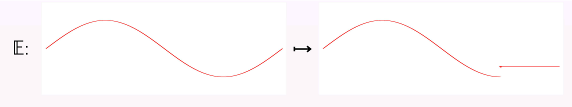

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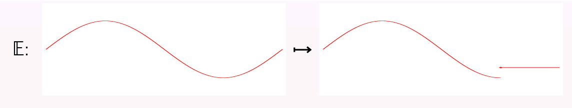

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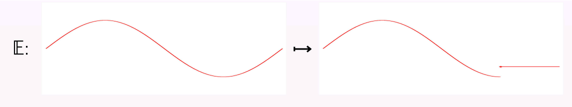

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So it remains to find $g \in X$ with small support and $\|g\|_X \approx \|g\|_{L_1} \approx 1$ in order to prove that $X = L_1$!

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Here choose $f_0 = \mathbf{1}$ and $\ell_0 = -\int$; hence there exists $f \in X$ with

- $\|f\|_X \leqslant 1$,
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- $\int_0^1 f(t)\,dt \leqslant -1 + \varepsilon$.

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Consequently, for $t = \mu(A)$ and $g = f\mathbf{1}_A$:

$$1 \approx \|g\|_{L_1} \leqslant \frac{\phi(t)}{t}\|g\|_{L_1} \leqslant \|g\|_X \leqslant \|f\|_X \leqslant 1,$$

which implies that

$$\lim_{t\to 0} \frac{\phi(t)}{t} = 1,$$

and $X = L_1$.

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Problem 4

- Are the ADP, numerical index one and lushness equivalent for Köthe spaces?
- Are the ADP and the Daugavet property equivalent for Köthe spaces on $[0,1]$?