Selected topics in positive and regular operators acting between Banach lattices

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$E = (E, \|\cdot\|), F = (F, \|\cdot\|)$ Banach lattices

Arnoud C. M. van Rooij, *When do regular operators between two Riesz spaces form a Riesz space ?* (Report 8410, Department of Mathematics, Catholic University Nijmegen, 1984, 97 pp.)

positive operators $L_+(E, F)$ ($T(E_+) \subset F_+$)

regular operators $L_r(E, F)$ ($T = T_1 - T_2$, $T_i \in L_+(E, F)$, $L_r(E, F) = \operatorname{span} L_+(E, F)$)

order bounded operators $L_b(E, F)$ ($A \subset E$ order bounded $\Rightarrow$ $T(A)$ order bounded in F)

continuous operators $L(E, F)$

$L_r(E, F)$, $L_b(E, F)$, $L(E, F)$ are ordered vector space with respect to the order generated by $L_+(E, F)$:
$T \leqslant S \;\Leftrightarrow\; S - T \in L_+(E, F)$

$T : E \to F$ order continuous ($x_\alpha \xrightarrow{(o)} x \;\Rightarrow\; T(x_\alpha) \xrightarrow{(o)} T(x)$)
$\Rightarrow\; T \in L_b(E, F)$

$L_b(E, F) \subset L(E, F)$

$\|x_n\| \to 0 \;\Rightarrow_{\text{passing to a subsequence if necessary}}\; x = \sum_{n=1}^\infty n|x_n| \;\Rightarrow$
$\{nx_n : n \in \mathbb{N}\} \subset [-x, x] \;\Rightarrow\; \{nT(x_n) : n \in \mathbb{N}\}$ order bounded
$\|nT(x_n)\| < M \;\Rightarrow\; \|T(x_n)\| \to 0$

$L_+(E, F) \subsetneq L_r(E, F) \subsetneq L_b(E, F) \subsetneq L(E, F)$

for $E = C[0, 1]$, $F = C[0, 1] \times \ell^p$ ($p < \infty$)

$L_r(\ell^1, F) = L_b(\ell^1, F) = L(\ell^1, F)$ for an arbitrary F

$T \in L(\ell^1, F) \;\Rightarrow\; \sum_{n=1}^\infty a_n|T(e_n)| = S((a_n))$
$S, (S - T) \in L_+(\ell^1, F)$ and $T = S - (S - T)$

$L_r(E,F) = L_b(E,F) = L(E,F)$ for an arbitrary $F \Rightarrow E$ is order isomorphic to $\ell^1(\Gamma)$

$L_b(E,F) = L_r(E,F^\delta) \cap L(E,F)$ because F is full (= cofinal) in its Dedekind completion F^δ (i.e., $\forall_{y \in F^\delta} \exists_{x \in F}\ y \leqslant x$)

finite rank operators $\mathcal{F}(E,F) \subset L_r(E,F)$: $T = \sum_{k=1}^n f_k \otimes x_k$, $x_k \in E, f_k \in F^*$, then
$T = \sum_{k=1}^n |f_k| \otimes |x_k| - (\sum_{k=1}^n |f_k| \otimes |x_k| - T)$

$T_1 = \bigvee_{k=1}^n |f_k| \otimes \bigvee_{k=1}^n |x_k| \Rightarrow T = T_1 - (T_1 - T)$

Question: Is an operator of rank k a difference of two positive operators of the same rank k ?

$f \otimes x$ is a difference of two positive rank-one operators iff x or f is comparable with zero.

Suppose f and x are not comparable with zero but
$T = f \otimes x = T_1 - T_2$ where $T_i = f_i \otimes x_i \geqslant 0$.
$T_i \geqslant 0 \quad \Rightarrow \quad$ we can assume without loss of generality $x_i, f_i \geqslant 0$.
T is not comparable with zero $\quad \Rightarrow \quad T_i \neq 0$.
$f \otimes x = f_1 \otimes x_1 - f_2 \otimes x_2 \quad \Rightarrow \quad \operatorname{Ker} f_1 \cap \operatorname{Ker} f_2 \subset \operatorname{Ker} f$. Hence
$f = \alpha f_1 + \beta f_2$
Suppose f_1, f_2 are linearly independent. Therefore
$\operatorname{Ker} f_2 \not\subseteq \operatorname{Ker} f_1$. Hence $f(y)x = \alpha f_1(y)x_1 \neq 0$ for some $y \in E$ which is impossible because x is not comparable with zero.
Assume now that f_1, f_2 are linearly dependent. But now $f = \gamma f_1$ for some nonzero γ which is impossible again because f is not comparable with zero.

$L_r(E, F)$ contains nuclear operators, i.e., operators of the form $\sum_{k=1}^{\infty} f_k \otimes x_k$ where $\sum_{k=1}^{\infty} \|f_k\| < \infty$ and $\sum_{k=1}^{\infty} \|x_k\| < \infty$.

Topological properties of $L_r(E, F)$ and $L_b(E, F)$.

Let K be a metrizable compact space.
c_0 is Dedekind complete $\Rightarrow$ $L_r(C(K), c_0) = L_b(C(K), c_0)$
order bounded sets in c_0 = relatively compact sets in c_0 $\Rightarrow$ $L_r(C(K), c_0) = \mathcal{K}(C(K), c_0)$ but $\mathcal{K}(C(K), c_0) \neq L(C(K), c_0)$ because $C(K)$ is separable and and so c_0 is complemented in $C(K)$ but a projection is not compact.

Let $p, q \in (1, \infty)$. There exists $T \in \mathcal{K}(\ell^p, \ell^q) \setminus L_r(\ell^p, \ell^q)$.
ℓ^q has the approximation property $\Rightarrow$ $T = \lim_{n\to\infty} T_n$ where $T_n \in \mathcal{F}(\ell^p, \ell^q) \subset L_r(\ell^p, \ell^q)$, i.e., $L_r(\ell^p, \ell^q) = L_b(\ell^p, \ell^q)$ is not closed in $L(\ell^p, \ell^q)$. If $q < p$, then $L_r(\ell^p, \ell^q)$ is a proper dense subset in $L(\ell^p, \ell^q)$ because $L(\ell^p, \ell^q) = \mathcal{K}(\ell^p, \ell^q)$ by Pitt's theorem.

$L_r(E, F)$ is a Banach space with respect to a norm called the regular norm $\|\cdot\|_r$
$\|T\|_r = \inf\{\|S\| : \pm T \leqslant S\} = \inf\{\|T_1 + T_2\| : T = T_1 - T_2, T_i \in L_+(E, F)\}$.

$\|T\| \leqslant \|T\|_r$ (there exists $T : (\mathbb{R}^{2^n}, \|\cdot\|_2) \to (\mathbb{R}^{2^n}, \|\cdot\|_2)$ with $\|T\| = 1$ and $\|T\|_r = \sqrt{2^n}$; for $n = 1$ we can choose $T = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$)

When do spaces $L_r(E, F)$ form a Riesz space ?

F is Dedekind complete $\Rightarrow$ $L_r(E, F)$ is a Dedekind complete Riesz space for every E and the modulus $|T| = \sup\{T, -T\}$ of T is given by the Kantorovich's formula $|T|(x) = \sup_{|y| \leqslant x} |T(y)|$ for every positive $x \in E$

On the other hand if F is not necessary Dedekind complete but $T : E \to F$ is such that the above formula makes sense for each $x \in E_+$ then it defines a positive operator, and it is precisely the modulus of T. When this is the case we say that the *modulus exists properly*, or that T has a proper modulus. Also, when the modulus $|T|$ exists we shall say that it exists properly at x if $|T|(x)$ is given by the Kantorovich's equality.

old open question: does there exist a regular operator with non-proper modulus?

Assume that an operator $T : E \to F$ possesses the modulus and let $E_{p|T|} = \{x \in E_+ : T \text{ exists properly at } x\}$

Proposition

The set $E_{p|T|}$ possesses the following properties.

- $E_{p|T|}$ *is a cone.*
- $0 \leqslant y \leqslant x \in E_{p|T|} \quad \Rightarrow \quad y \in E_{p|T|}$.
- $E_{p|T|}$ *is closed under finite infima and suprema* $(x, y \in E_{p|T|} \quad \Rightarrow \quad x \wedge y, x \vee y \in E_{p|T|})$.

If $T = f \otimes x$, then $|T| = |f| \otimes |x|$ – but there exists a finite rank operator whose modulus is not a finite rank operator.

Theorem

If E, F are Banach lattices, then every continuous finite rank operator $T : E \to F$ has a proper modulus $|T|$ in $L_r(E, F)$ and the modulus $|T|$ is compact.

F is Dedekind complete $\Rightarrow$ $L_r(E, F)$ is a Riesz space – what about $\Leftarrow$?

Y.A. Abramovich, V.A. Gejler, A.C.M. van Rooij, A.W. Wickstead:

Theorem

For a Banach lattice F the following statements are equivalent.

(a) $L_r(E, F)$ *is a Riesz space for all Banach lattices* E.

(b) $L_r(L^1(\mu), F)$ *is a Riesz space for all measures* μ.

(c) $L_r(C(K), F)$ *is a Riesz space for every compact set* K.

(d) $L_r(c(S), F)$ *is a Riesz space for every set* S, *where* $c(S) = \{f \in \mathbb{R}^S : \exists_{r>0} \forall_{\varepsilon>0} |f(s) - r| \geqslant \varepsilon$ *for at most finitely many* $s\}$, *and* $\|f\| = \sup_{s \in S} |f(s)|$.

(e) F *is Dedekind complete.*

A. van Rooij's characterization of σ-Dedekind complete Banach lattices.

Theorem

For a Banach lattice F the following statements are equivalent.

(a) $L_r(L^1[0,1], F)$ *is a Riesz space.*

(b) $L_r(c, F)$ *is a Riesz space, where* $c = c(\mathbb{N})$.

(c) $L_r(C(K), F)$ *is a Riesz space for every infinite metrizable compact space* K.

(d) F *is* σ*-Dedekind complete.*

Y.A. Abramovich, A.W. Wickstead:

Theorem

(d) $\Leftrightarrow$ (e): $L_r(E, F)$ *are* σ*-Dedekind complete Riesz spaces for all separable Banach lattices* E.

Proposition

If $L_r(c, F)$ is a Riesz space, then every $T \in L_r(c, F)$ has the proper modulus.

Theorem

For a Banach lattice E the following statements are equivalent.

(a) *$L_r(E, F)$ is a Riesz space for all Banach lattices F and every $T \in L_r(E, F)$ has the proper modulus.*

(b) *$L_r(E, F)$ is a Riesz space for all Banach lattices F.*

(c) *$L_r(E, C(K))$ is a Riesz space for every compact space K.*

(d) *E is discrete and its norm is order continuous.*

(e) *E is σ-Dedekind complete and $L(E, c_0) = L_r(E, c_0)$.*

(f) *Every $x \in E$ lies in an ideal of E that is order isomorphic to a quotient Riesz space of c_0.*

$0 < x \in E$ is discrete iff $|y| \leqslant x \quad \Rightarrow \quad y = tx$ for some scalar t (unit vectors are discrete in classical sequence Banach lattices).

E is discrete if $\forall_{0<x\in E}\ \exists_{\text{discrete } e\in E}\ e \leqslant x$.

E is discrete $\quad \Rightarrow \quad \exists_\Gamma\ \exists_{\text{sublattice } F\subset\mathbb{R}^\Gamma}$ such that $\text{span}\{1_{\{\gamma\}} : \gamma \in \Gamma\} \subset F \sim E$.

Examples of discrete spaces: all classical sequence Banach lattices.

A Banach lattice is continuous when it contains none discrete elements ($C[0,1]$, $L^p(\mu)$ for atomless measures μ, ℓ^∞/c_0).

A Banach lattice $E = (E, \|\cdot\|)$ has order continuous norm if $x_\alpha \downarrow 0 \quad \Rightarrow \quad \|x_\alpha\| \to 0$.

When is $L_r(E, F)$ discrete or continuous ?

Theorem

Let E, F be two Banach lattices and let F be Dedekind complete.

(a) $L_r(E, F)$ *is discrete iff* E^* *and* F *are discrete.*

(b) $L_r(E, F)$ *is continuous iff* E^* *or* F *is continuous.*

Moreover, $T \in L_r(E, F)$ *is discrete iff* $T = f \otimes e$ *where* e *is discrete in* E *and* $f \in F^*$ *is a homomorphism* (*i.e.,* f *is discrete in the dual space*).

Z.L. Chen

Theorem

The regular norm is order continuous on $L_r(E, F)$ *iff positive operators between* E *and* F *are simultaneously L-weakly and M-weakly compact (i.e.,* $\|y_n\| \to 0$ *whenever* $y_n \in \text{sol}\, T(B_E)$ *are disjoint and* $\|Tx_n\| \to 0$ *for each norm bounded disjoint sequence* $(x_n) \subset E$ *).*

Corollary

If a Banach lattice F has order continuous norm, then the regular norm on $L_r(C(K), F)$ is order continuous too.

norm on F is order continuous $\Leftrightarrow$ regular operators from $C(K)$ into F are weakly compact, but weakly compact operators on $C(K)$ spaces coincide with M-weakly compact operators. The dual of $C(K)$ has order continuous norm and now we can use the Dodds-Fremlin theorem: M-weakly compact operators mapping $T : E \to F$ are L-weakly compact and vice versa whenever E^* and F have order continuous norms.

Corollary

The regular norm is order continuous on $L_r(E, E)$ iff E is finite dimensional.

E is a KB space $\Leftrightarrow$ $(0 \leqslant x_n \uparrow$ and $\sup_n \|x_n\| < \infty) \Rightarrow (x_n)$ is convergent $\Leftrightarrow$ E does not contain any subspace isomorphic to c_0

Z.L. Chen

Theorem

The following statements are equivalent.

(a) $(L_r(E,F), \|\cdot\|_r)$ *is a KB-space.*

(b) $\|\cdot\|_r$ *is order continuous and* F *is a KB-space.*

(c) F *is a KB-space and every positive* $T : E \to F$ *is M-weakly compact.*

Corollary

If $L^p(\mu), L^q(\nu)$ *are infinite dimensional, then* $(L_r(L^p(\mu), L^q(\nu)), \|\cdot\|_r)$ *is a KB-space iff* $q < p$.

E has the positive Schur property ($E \in (\mathrm{PSP})$) whenever
$0 \leqslant x_n \xrightarrow{\sigma(E,E^*)} 0 \ \Rightarrow \ \|x_n\| \to 0$.

$L^1(\mu) \in (\mathrm{PSP})$

D. Leung: μ finite, φ an Orlicz function such that $\lim_{s\to\infty} \frac{\varphi^*(2s)}{\varphi^*(s)} = \infty$ (where $\varphi^*(t) = \sup_{t>0}(st - \varphi(s))$) $\Rightarrow$ $L^\varphi(\mu) \in (\mathrm{PSP})$; moreover $(L^\varphi(\mu))^{(2n)} \in (\mathrm{PSP})$ for all n.

Corollary

Suppose that E^* (respectively F) possesses the positive Schur property. Then $L_r(E, F)$ with the regular norm is a KB-space iff F (respectively E^*) is a KB-space.

A. van Rooij

Proposition

The space $L_r(\ell^\infty, F)$ is a Riesz space iff F is $\mathfrak{c}$-complete, i.e., every order bounded from above subset X with $\operatorname{card} X \leqslant \mathfrak{c}$ has a supremum.

When $L(E,F) = L_r(E,F)$?

? L.V Kantorovich, B.Z. Vulikh

Theorem

If E, F are such that F is order isomorphic to a Dedekind complete space $C(K)$ or E is order isomorphic to $L^1(\mu)$ and simultaneously there exists norm one positive projection $P : F^{**} \to F$, then every continuous operator $T : E \to F$ is regular (and so $L(E,F)$ is a Riesz space). Moreover the operator and regular norms are equal.

conjecture: $L(E,F) = L_r(E,F) \quad \Rightarrow \quad E$ is order isomorphic to $L^1(\mu)$ or F is order isomorphic to a closed Riesz subspace in some $C(K)$ space.

D. Cartwright and H.P. Lotz

Theorem

Let E, F be Banach lattices such that F (resp. E^*) contains a closed Riesz subspace order isomorphic to ℓ^p for a finite p. If every compact operator $T : E \to F$ belongs to $L_r(E, F^{**})$, then E is order isomorphic to $L^1(\mu)$ (resp. F is order isomorphic to a closed Riesz subspace of some $C(K)$).

Corollary

$L(E, L^1(\mu)) = L_r(E, L^1(\mu))$ for infinite dimensional $L^1(\mu)$ iff E is order isomorphic to $L^1(\nu)$.

Y.A. Abramovich and A.W. Wickstead: arguments “supporting” the conjecture.

Theorem

The following conditions on a Banach lattice F are equivalent.

(a) *F is order isomorphic to a Dedekind complete $C(K)$ space.*

(b) *For every Banach lattice E the space $L(E, F)$ is a Riesz space.*

(c) *For every Banach lattice E every continuous $T : E \to F$ is regular and $L_r(E, F)$ forms a Riesz space.*

Y.A. Abramovich disproved the conjecture – there exits E and F such that E is not order isomorphic to any $L^1(\mu)$, F is not order isomorphic to any AM-space but every $T \in L(E, F)$ has the modulus (in particular $L(E, F) = L_r(E, F)$).

We have already mentioned that $L(L^1(\mu), F) = L_r(L^1(\mu), F)$ whenever there exists a contractive positive projection $P : F^{**} \to F$. The assumption about F can be slightly weakened – it is enough to require that F has the Levi property, i.e., increasing norm bounded nets of positive elements have a supremum in F.

Abramovich and Wickstead noticed that this modified version of the theorem can be reversed.

Theorem

The following conditions on a Banach lattice F are equivalent.

(a) *F has the Levi property.*

(b) *$L(L^1(\mu), F)$ is a Riesz space for every measure μ.*

(c) *$L(L^1(\mu), F) = L_r(L^1(\mu), F)$ for every μ and F is Dedekind complete.*

$\|\cdot\|$ is order continuous on E ($E \in \mathrm{(o.c.)}$) iff $x_\alpha \downarrow 0 \quad \Rightarrow \|x_\alpha\| \to 0$

$\|\cdot\|$ is σ-order continuous on E ($E \in (\sigma\text{-o.c.})$) iff $x_n \downarrow 0 \quad \Rightarrow \|x_n\| \to 0$

order continuity = σ-order continuity when E is σ-Dedekind complete

$\|\cdot\|_\infty$ is (σ-o.c.) on spaces $c(S)$ for every uncountable sets S but $\|\cdot\|_\infty \notin \mathrm{(o.c.)}$
the same holds for the quotient norm on E/F whenever E consists of sequences, $F = \overline{\mathrm{span}}\{e_n : n \in \mathbb{N}\}$ and $F \neq E$

$E_A = \{x \in E : |x| \geqslant x_\alpha \downarrow 0 \;\Rightarrow\; \|x_\alpha\| \to 0\}$
E_A is always a norm closed ideal (but it may happen $E_A = \{0\}$)

Theorem

If E_A is a proper order dense ideal in a σ-Dedekind complete Banach lattice E, then the quotient norm on E/E_A is σ-order continuous and the norm is not order continuous. Additionally E/E_A is continuous and none nonzero ideal in the quotient is σ-Dedekind complete.

Characterizations of order continuity:
G.Ja. Lozanovskii – A σ-Dedekind complete Banach lattice has order continuous norm iff E does not contain any closed subspace isomorphic to ℓ^∞ (equivalently: E does not contain any closed Riesz subspace order isomorphic to ℓ^∞)
D. Fremlin and P. Meyer-Nieberg: E has order continuous norm iff $x_n \wedge x_m = 0$ and $x_n \leqslant x \quad \Rightarrow \quad \|x_n\| \to 0$.
Characterizations of σ-order continuity:

Theorem

For a Banach lattice E the following statements are equivalent.

(a) $E \in (\sigma\text{-o.c.})$.

(b) *E is order complete and E does not contain any closed σ-regular Riesz subspace order isomorphic to ℓ^∞.*

(c) *E is order complete and if elements $x_n \in E$, $n \in \mathbb{N}$ are such that $x_n \wedge x_m = 0$ and $\sup_n x_n$ exists, then $\|x_n\| \to 0$.*

Explanations:
E is order complete means that every sequence $(x_n) \subset E$ satisfying the order Cauchy condition:

$$\exists_{v_n \downarrow 0} \, \forall_{n,k} \, |x_{n+k} - x_n| \leqslant v_n$$

is order convergent.
Examples: σ-Dedekind complete Banach lattices, ℓ^∞ / c_0 (it is not σ-Dedekind complete); the spaces $C[0,1]$ and c are not order complete.

A Riesz subspace $F \subset E$ is σ-regular if every countable subset of F having an infimum (or a supremum) in F has the same infimum (supremum) in E.
Examples: ideals, order dense Riesz subspace; but $\{(x_n) \in c_0 : x_{2n} = 0\} \oplus \mathbb{R}1_{\mathbb{N}}$ is not σ-regular in ℓ^∞.

Let us note that ℓ^∞ / c_0 contains many Riesz subspaces order isomorphic to ℓ^∞ but none copy is σ-regular.

Problem: does every Banach space possess an unconditional basic sequence ? (No – W.T. Gowers and B. Maurey)

T. Figiel, J. Lindenstaruss, L. Tzafriri

Theorem

A Banach lattice E has an order continuous norm iff it is σ-Dedekind complete and every closed subspace of E has an unconditional basic sequence.

Operator characterizations of the order continuity.

Theorem

For a Banach lattice E the following statements are equivalent.

(a) *E has order continuous norm.*

(b) *If K is an arbitrary compact space and $T : C(K) \to E$ is positive, then T is weakly compact.*

(c) *E is σ-Dedekind complete and every positive operator $T : \ell^\infty \to E$ is weakly compact.*

(d) *E is σ-Dedekind complete and every Dunford-Pettis operator $T : E \to c_0$ is order bounded.*

Theorem

For a Banach lattice E the following statements are equivalent.

(a) $E^* \in (\text{o.c.})$.

(b) *Every Dunford-Pettis operator on E is weakly compact.*

(c) *Every continuous operator T from E into a Banach space without any subspace isomorphic to c_0 is weakly compact.*

(d) *Every continuous operator $T : E \to L^1(\mu)$ is weakly compact.*

(e) *Every positive operator $T : E \to L^1[0,1]$ is weakly compact.*

(f) *Every positive operator $T : E \to \ell^1$ is compact.*

(g) *Every continuous operator $T : E \to E^*$ is weakly compact.*

Operator characterizations of the σ-order continuity.
C. Aliprantis, O. Burkinshaw, P. Kranz

Theorem

For a Banach lattice E the following statements are equivalent.

(a) $E \in (\sigma\text{-o.c.})$.

(b) *If* $0 \leqslant T_n, T : E \to E$ *satisfy* $T_n(x) \uparrow T(x)$ *for each* $x \geqslant 0$, *then also* $T_n^2(x) \uparrow T^2(x)$.

J. Schur (1920): $(x_n) \subset \ell^1$, $x_n \xrightarrow{\sigma(\ell^1,\ell^\infty)} 0 \;\Rightarrow\; \|x_n\| \to 0$

Proof: gliding hump technique or the theory of basis argument

– if $x_n \xrightarrow{\sigma(\ell^1,\ell^\infty)} 0$ and $\|x_n\| \geqslant \varepsilon > 0 \;\;\Rightarrow\;\; \exists_{(n_k)}\, (x_{n_k}) \sim (e_k)_{\ell^1}$

X has the Schur property ($X \in (\mathrm{SP})$) if

$x_n \xrightarrow{\sigma(X,X^*)} 0 \;\Rightarrow\; \|x_n\| \to 0$

Examples.

1. $\ell^1(\Gamma) \in (\mathrm{SP})$ for every set Γ.
2. $X_n \in (\mathrm{SP}) \ \Rightarrow \ (\oplus \mathrm{X_n})_{\ell^1} \in (\mathrm{SP})$,

in particular $X = (\oplus \ell^2_n)_{\ell^1} \in (\mathrm{SP})$, but $X \nsim \ell^1$ because X^* contains a complemented copy of ℓ^2.

3. Consider a weighted Orlicz sequence space $\ell^\varphi(a_n)$ generated by a convex function φ satisfying two conditions: $\lim_{u\to 0} \frac{\varphi(u)}{u} = 0$, $\lim_{u\to\infty} \frac{\varphi(u)}{u} = \infty$ and let $(a_n) \in \ell^1_{++} = \{(c_n) \in \ell^1 : \forall_n \ c_n > 0\}$. If $\lim_{u\to\infty} \frac{\varphi^*(2u)}{\varphi^*(u)} = \infty$, then $\ell^\varphi(a_n) \in (\mathrm{SP})$.

$W_\varphi = \{(b_n) \in \ell^1_{++} : \ell^\varphi(b_n) \sim \ell^1\}$ is of the first category in ℓ^1_+ and ℓ^1_{++} is a dense G_δ set in ℓ^1_+. Hence $\ell^1_{++} \setminus W_\varphi \neq \emptyset$.
Conclusion: for every φ there exists a lot (a_n) such that $\ell^\varphi(a_n) \in (\mathrm{SP})$ and $\ell^\varphi(a_n) \nsim \ell^1$.

4. Nakano sequence space $\ell^{(p_n)}$, $p_n \in [1, \infty)$.
I. Halperin and H. Nakano (1953): $\ell^{(p_n)} \in (\mathrm{SP})$ iff $p_n \to 1$
If $(1 - \frac{1}{p_n}) \log n \to \infty$, then $\ell^{(p_n)} \not\approx \ell^1$.
5. R. Ryan (1987) – $L(X, Y) \in (\mathrm{SP})$ iff $X^*, Y \in (\mathrm{SP})$.
$(\oplus \ell_n^\infty)_{\ell^1})$ is not isomorphic to any subspace of ℓ^1 $\Rightarrow$
$L(c_0, (\oplus \ell_n^\infty)_{\ell^1}) \not\approx \ell^1(\Gamma)$ for every Γ because $L(X, Y)$ contains Y isometrically.

Theorem

(a) $E \in (\mathrm{SP})$.

(b) *If μ is an arbitrary measure, then every weakly compact operator $T : L^1(\mu) \to E$ is compact.*

(c) *Every positive weakly compact operator $T : \ell^1 \to E$ is compact.*

(d) *E has order continuous norm and every continuous linear operator $T : E \to c_0$ is Dunford-Pettis $($= T maps weak null sequences into norm null$)$.*

Every Banach lattice possessing the Schur property is a dual space.

Theorem

For a Banach lattice E the following statements are equivalent.

(a) $E^* \in (\mathrm{SP})$.

(b) *Every weakly compact operator on E is compact.*

(c) *Every weakly compact operator $T : E \to c_0$ is compact.*

(d) *If F is a Banach lattice with order continuous norm and $T : E \to F$ is weakly compact, then T is L-weakly compact.*

E has the positive Schur property ($E \in (\mathrm{PSP})$) whenever

$$0 \leqslant x_n \xrightarrow{\sigma(E,E^*)} 0 \ \Rightarrow \ \|x_n\| \to 0$$

Useful characterization:

$$E \in (\mathrm{PSP}) \ \Leftrightarrow \ (\mathrm{x_k} \wedge \mathrm{x_m} = 0,\ \mathrm{x_n} \xrightarrow{\sigma(\mathrm{E},\mathrm{E}^*)} 0 \ \Rightarrow \ \|\mathrm{x_n}\| \to 0)$$

If E is **discrete**, then $E \in (\mathrm{PSP}) \ \Leftrightarrow \ \mathrm{E} \in (\mathrm{SP})$.

Theorem

For a Banach lattice E the following statements are equivalent.

(a) $E \in (\mathrm{PSP})$.

(b) *Every normalized sequence of pairwise disjoint positive elements contains a subsequence equivalent to the unit vector basis in ℓ^1 (and so E is saturated by order copies of ℓ^1).*

(c) *E is σ-Dedekind complete and an operator $T : E \to c_0$ is a Dunford-Pettis operator iff T is regular*

Josefson-Nissenzweig theorem –
$\dim X = \infty \ \Rightarrow \ \exists_{(f_n)\in X^*} \ \|f_n\| = 1$ and $f_n \xrightarrow{\sigma(X^*,X)} 0$.

But if $0 \leqslant f_n \xrightarrow{\sigma((C(K))^*,C(K))} 0$ and μ_n are regular Borel measures representing f_n, then $\|f_n\| = \int_K 1 \ d\mu_n = f_n(1) \to 0$.

B. Aqzzouz, A. Elbour, A. Wickstead (2010) – E has the dual positive Schur property ($E \in \mathrm{(DPSP)}$) if
$0 \leqslant f_n \xrightarrow{\sigma(E^*,E)} 0 \ \Rightarrow \ \|f_n\| \to 0$.

$E \in \mathrm{(DPSP)} \ \Rightarrow$ E has the positive Grothendieck property ($E \in \mathrm{(PGP)}$), i.e., $0 \leqslant f_n \xrightarrow{\sigma(E^*,E)} 0 \ \Rightarrow \ f_n \xrightarrow{\sigma(E^*,E^{**})} 0$

$E \in \mathrm{(DPSP)} \ \Leftrightarrow \ \mathrm{E} \in \mathrm{(PGP)}$ and $\mathrm{E}^* \in \mathrm{(PSP)} \ \Leftrightarrow$ every $0 \leqslant T : E \to c_0$ is compact

Examples.

1.

Theorem

For an AM-space E $(= E$ is order isomorphic and isometric to a closed sublattice in some $C(K))$ the following statements are equivalent.

- $E \in (\mathrm{DPSP})$.
- $E \in (\mathrm{PGP})$.
- *E does not contain any positively complemented order copy of c_0.*

2. If $L^{\varphi}(\mu)$ is an Orlicz space, then
$L^{\varphi}(\mu) \in (\mathrm{DPSP}) \Leftrightarrow \mathrm{L}^{\varphi^*}(\mu) \in (\mathrm{PSP})$. Moreover
$L^{\varphi}(\mu) \in (\mathrm{DPSP}) \Rightarrow (\mathrm{L}^{\varphi}(\mu))^{**} \in (\mathrm{DPSP})$, $(L^{\varphi}(\mu))^{****} \in (\mathrm{DPSP})$, $\ldots$, $(L^{\varphi}(\mu))^{(2n)} \in (\mathrm{DPSP})$

For a finite measure μ and an Orlicz function φ satisfying $\lim_{u\to\infty} \frac{\varphi(2u)}{\varphi(u)} = \infty$ we obtain $L^{\varphi}(\mu) \in (\mathrm{DPSP})$.

3. Let $1 \leqslant q_n \to \infty$ and let $\varphi_n(u) = \frac{1}{q_n} u^{q_n}$. Then
$\ell^{(\varphi_n)} \in (\mathrm{DPSP}), (\ell^{(\varphi_n)})^{**} \in (\mathrm{DPSP}), (\ell^{(\varphi_n)})^{****} \in (\mathrm{DPSP}), \ldots,$
$(\ell^{(\varphi_n)})^{(2n)} \in (\mathrm{DPSP})$ and

$(\oplus \ell_n^{q_n})_{\ell^\infty} \in (\mathrm{DPSP}), ((\oplus \ell_n^{q_n})_{\ell^\infty})^{**} \in (\mathrm{DPSP}),$
$((\oplus \ell_n^{q_n})_{\ell^\infty})^{****} \in (\mathrm{DPSP}), \ldots, ((\oplus \ell_n^{q_n})_{\ell^\infty})^{(2n)} \in (\mathrm{DPSP})$

Theorem

For a Banach lattice E the following statements are equivalent.

(a) $E \in (\mathrm{DPSP})$.

(b) *If* $f_m \wedge f_k = 0$ *and* $f_n \xrightarrow{\sigma(E^*,E)} 0$, *then* $\|f_n\| \to 0$.

(c) *Every order weakly compact operator on E is* M*-weakly compact.*

(d) *Every positive weakly compact operator* $T : E \to F$ *is semi-compact* $(i.e., \forall_{\varepsilon>0} \exists_{0 \leqslant y \in F} \; T(B_E(1)) \subset [-y, y] + \varepsilon B_F(1))$.

(e) *If F is a discrete Banach lattice with order continuous norm, then every positive operator* $T : E \to F$ *is compact.*

The word “discrete”can not be rejected in the last statement.

The last statement formulated in the theorem motivates considerations of the following property of a Banach lattice E.

$(*)$ *If F is a Banach lattice with order continuous norm and $T : E \to F$ is positive, then T is compact.*

It is a surprise that σ-Dedekind complete Banach lattices E satisfying $(*)$ are finite dimensional. On the other hand there exist spaces $C(K)$ satisfying $(*)$ and we can characterize them.

Theorem

For a space $E = C(K)$ the following statements are equivalent.

(a) *E^* is order isomorphic to $\ell^1(\Gamma)$ for some set Γ.*

(b) *E does not contain any closed subspace isomorphic (i.e., linearly homeomorphic) to ℓ^1.*

(c) *E satisfies $(*)$.*

(d) *Every positive operator $T : E \to (\ell^\infty)^*$ is compact.*

If K is a countable compact space then $C(K)$ satisfies $(*)$ because $(C(K))^*$ is isomorphic to ℓ^1.