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Binz-Butzmann duality versus Pontryagin duality

By

M. J. Chasco and E. Martín-Peinador *)

Introduction. A topological group G is said to be reflexive if the natural embedding α_G from G into its bidual $G^{\wedge\wedge}$ is a topological isomorphism, assuming that both $G^\wedge$ and $G^{\wedge\wedge}$ are endowed with the compact open topology. The Pontryagin duality theorem says that every locally compact abelian group is reflexive. This theorem has been extended in order to establish other sorts of groups which are also reflexive. For instance every Banach space considered in its additive structure is a reflexive group; locally convex reflexive spaces are reflexive as groups; arbitrary products of reflexive groups are reflexive, etc.

On the other hand, Fischer defined the convergence groups, as groups endowed with a convergence structure compatible with the addition. Every topological group defines a convergence structure, the one given by its convergent filters or nets. However, not every compatible convergence structure of a group comes from a topology on the supporting set. Binz and Butzmann have defined reflexivity in convergence groups and also in convergence vector spaces. In the present paper we deal with the relationship between reflexivity in the sense of Pontryagin and reflexivity in the sense of Binz-Butzmann (*BB*-reflexivity), for a topological group G.

A *convergence structure* Ξ on a set X is a set of pairs $(\mathscr{F}, x)$, where $\mathscr{F}$ denotes a filter on X, and x an element of X, such that:

1) For every $x \in X$, $(\mathscr{F}_x, x) \in \Xi$, where $\mathscr{F}_x$ is the filter generated by $\{x\}$.
2) If $(\mathscr{F}, x)$, $(\mathscr{G}, x)$ are in Ξ, $(\mathscr{F} \cap \mathscr{G}, x)$ is also in Ξ.
3) If $(\mathscr{F}, x)$ is in Ξ and $\mathscr{G}$ is a filter finer than $\mathscr{F}$ (i.e. $\mathscr{F} \subseteqq \mathscr{G}$), then $(\mathscr{G}, x)$ is also in Ξ.

If $(\mathscr{F}, x) \in \Xi$, we say that $\mathscr{F}$ converges to x, and write $\mathscr{F} \to x$. The pair (X, Ξ), denoted also briefly by X, will be called a *convergence space*. In particular we shall deal with the continuous convergence structure on the set of characters of a topological group, to be defined below.

Many topological properties can be stated in terms of convergence, therefore they have corresponding definitions for convergence spaces. For a detailed discussion on them see [5], where products and quotients of convergence spaces are also described.

We recall, for a later use, the following definitions: In a convergence space (X, Ξ) a subset $K \subseteqq X$ is *compact* if every ultrafilter in K converges to some x in K. The space X is *locally compact* if every convergent filter in X has a member which is compact.

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A *convergence group* (G, Ξ) is defined as a group endowed with a convergence structure Ξ such that the mapping $\mu: G \times G \to G$ given by $\mu((x, y)) = xy^{-1}$ is continuous in the sense of convergence.

Denote by T the group $\mathbb{R}/\mathbb{Z}$, endowed with the natural quotient topology; we shall identify the elements of T with points in $(-1, 1]$. For an arbitrary abelian topological group G, the symbol ΓG denotes the set of characters (i.e. continuous homomorphisms from G into T). A topology on ΓG is called admissible if the evaluation mapping $w: \Gamma G \times G \to T$ defined by $w((\xi, x)) = \xi(x)$ is continuous. If G is locally compact and ΓG carries the compact open topology, then w is continuous, not being so without the former assumptions.

The *continuous convergence structure* Λ, in ΓG, is defined in the following way: a filter $\mathscr{F}$ in ΓG converges in Λ to an element $\xi \in \Gamma G$ if for every $x \in G$ and every filter $\mathscr{H}$ in G, convergent to x, $w(\mathscr{F} \times \mathscr{H})$ converges to $\xi(x)$ in T. ($\mathscr{F} \times \mathscr{H}$ denotes the filter generated by the products $F \times H$, $F \in \mathscr{F}$ and $H \in \mathscr{H}$, and $w(F \times H) := \{f(x); f \in F \text{ and } x \in H\}$). If ΓG is endowed with Λ, w takes convergent filters into convergent filters, therefore it is continuous and in fact Λ is the coarsest convergence structure making the evaluation map continuous.

Summarizing: For an abelian topological group G, ΓG is also an abelian group. If it is endowed with the continuous convergence structure it becomes a convergence group, from now on denoted by $\Gamma_c G$. The same set endowed with the compact open topology is a topological group, called $G^\wedge$. A neighborhood basis of zero in $G^\wedge$ is given by the sets $(S, V) := \{f \in \Gamma G; f(S) \subseteq V\}$, where $S \subseteq G$ is compact and V a zero neighborhood in T. If G is locally compact then $\Gamma_c G$ may be identified with $G^\wedge$. We shall denote by $\Gamma_c \Gamma_c G := \Gamma_c(\Gamma_c G)$ and $G^{\wedge\wedge} := (G^\wedge)^\wedge$ the Binz-Butzmann bidual (*BB*-bidual) and the Pontryagin bidual respectively, and by $\kappa_G: G \longrightarrow \Gamma_c \Gamma_c G$, $\alpha_G: G \longrightarrow G^{\wedge\wedge}$ the corresponding canonical embeddings.

Relationship between Pontryagin reflexivity and Binz-Butzmann reflexivity. The following simple fact will be used repeatedly.

Lemma 1. *If a filter $\mathscr{F}$ converges to ϕ in $\Gamma_c G$, it also converges to ϕ in $G^\wedge$.*

Proposition 1. *Let G be a topological group such that $\alpha_G: G \longrightarrow G^{\wedge\wedge}$ is continuous. Then, the convergence group $\Gamma_c G$ is locally compact.*

Proof. Suppose $\mathscr{F}$ is a filter in $\Gamma_c G$ convergent to the zero homomorphism. Let $E_G(0)$ be the filter of all the neighborhoods of zero in G; we have $w(\mathscr{F} \times E_G(0)) \to 0$, and therefore we can find $F \in \mathscr{F}$, $N \in E_G(0)$ such that $f(N) \subseteq [-1/4, 1/4]$, for every f in F.

The set $N^0 := \{\tau \in \Gamma G, \tau(N) \subseteq [-1/4, 1/4]\}$ is compact in the compact open topology τ, being the annihilator of a zero neighborhood, [1] (1.5). Let us prove that N^0 is compact in the convergence structure Λ. For this purpose take an ultrafilter $\mathscr{U}$ in N^0. Since N^0 is compact in the compact open topology τ of ΓG, $\mathscr{U}$ is τ-convergent, say $\mathscr{U} \xrightarrow{\tau} \varphi$. We must see that $\mathscr{U} \xrightarrow{\Lambda} \varphi$. Take a neighbourhood of zero in T, say $(-\varepsilon, \varepsilon)$; due to continuity of α_G, N^0 is equicontinuous; therefore, we can find $V \in E_G(0)$ such that $\phi(V) \subseteq (-\varepsilon, \varepsilon)$, $\forall \phi \in N^0$. Thus $w(U \times V) \subseteq (-\varepsilon, \varepsilon)$ for every U in $\mathscr{U}$, which shows that

$w(\mathscr{U} \times E_G(0)) \longrightarrow \varphi(0)$ and consequently $\mathscr{U} \xrightarrow{\Lambda} \varphi$. On the other hand, N^0 contains F, therefore $N^0 \in \mathscr{F}$.

Remark 1. Observe that for a topological group G with α_G continuous, the above argument together with Lemma 1 imply that $\Gamma_c G$ and $G^\wedge$ have the same compact subsets.

Corollary 1. *If G is a topological group such that $\alpha_G: G \to G^{\wedge\wedge}$ is continuous, then the continuous convergence structure on $\Gamma(\Gamma_c G)$ has exactly the same convergent filters as the compact open topology on the same set. Thus we can say that $\Gamma_c \Gamma_c G$ is topological.*

Proof. Apply to $\Gamma_c G$ the following result of [2] (Th. 32, pg. 52): "A convergence c-embedded space X is locally compact if and only if $C_c(X)$ is topological. Furthermore, whenever $C_c(X)$ carries a topology, it must be the compact open topology". Here $C_c(X)$ denotes the set of continuous, real valued functions defined on X, endowed with the structure of continuous convergence. In our case, $\Gamma_c G$ is c-embedded [3], and locally compact by Proposition 1. Therefore, by the quoted result, $C_c \Gamma_c G$ is topological, carrying the compact open topology. Since $\Gamma_c \Gamma_c G$ can be identified with a subspace of $C_c \Gamma_c G$, it is also topological.

Now we give the following:

Theorem 1. *Let G be a topological group such that $\alpha_G: G \longrightarrow G^{\wedge\wedge}$ is continuous. The Pontryagin bidual $G^{\wedge\wedge}$ is a topological subgroup of the Binz-Butzmann bidual $\Gamma_c \Gamma_c G$.*

Proof. Let κ be a continuous character defined on $G^\wedge$. We must prove that κ is also continuous on $\Gamma_c G$. To this end, take a convergent filter $\mathscr{F}$ in $\Gamma_c G$, say $\mathscr{F} \longrightarrow \xi$. By Lemma 1 $\mathscr{F}$ is also τ-convergent to ξ. Since κ is τ-continuous, the filter generated by $\kappa(\mathscr{F})$ converges to $\kappa(\xi)$. Thus κ is in $\Gamma_c \Gamma_c G$, and this proves $G^{\wedge\wedge} \subseteqq \Gamma_c \Gamma_c G$. On the other hand, $\Gamma_c \Gamma_c G$ carries the compact open topology, τ, and by Remark 1, $\Gamma_c G$ and $G^\wedge$ have the same compact subsets. So $G^{\wedge\wedge}$ is a topological subgroup.

A few comments now will make clear that in general $G^{\wedge\wedge} \neq \Gamma_c \Gamma_c G$, the equality holding, for instance, when G is locally compact, and in some other cases as well.

Comments and examples. For a topological vector space X, the reflexivity in the sense of Binz-Butzmann is defined as follows: let $\mathscr{L}_c(X)$ be the set of all continuous linear functionals on X endowed with the structure of continuous convergence. Then X is said to be BB-reflexive if the natural embedding from X into $\mathscr{L}_c \mathscr{L}_c(X)$ is a bicontinuous isomorphism. In [3] it is proved that X is BB-reflexive if and only if it is locally convex and complete.

On the other hand, we can consider the topological group underlying the topological vector space X. As defined above, X is a BB-reflexive group if the natural embedding from X into $\Gamma_c \Gamma_c X$ is a bicontinuous isomorphism. In [4] it is proved that X is BB-reflexive as a vector space iff it is BB-reflexive as a group. This is far from being so in the ordinary sense. A reflexive topological vector space is reflexive regarded as a topological group [8], however the converse does not hold, since for instance, every Banach space is reflexive as a group.

Komura has given an example of a reflexive locally convex space X, which is not complete [7]. Therefore, X is reflexive as a group, i.e. $\alpha_X: X \longrightarrow X^{\wedge\wedge}$ is a topological isomorphism. As mentioned in Remark 1, $X^\wedge$ and $\Gamma_c X$ have the same compact subsets and $\Gamma_c \Gamma_c X$ carries the compact open topology. Thus, if the sets $X^{\wedge\wedge}$ and $\Gamma_c \Gamma_c X$ were equal, they should be equal as topological groups. However, since X is not complete, it is not BB-reflexive as a vector space, nor as a group, i.e. X is not bicontinuously isomorphic to $\Gamma_c \Gamma_c X$.

This proves that the sets $X^{\wedge\wedge}$ and $\Gamma_c \Gamma_c X$ are distinct, and furthermore, in this example $X^{\wedge\wedge}$ is not a closed subgroup of $\Gamma_c \Gamma_c X$.

A *BB*-reflexive group which is not Pontryagin reflexive. If a group G is BB-reflexive, but not Pontryagin reflexive, the canonical embedding $\alpha_G: G \longrightarrow G^{\wedge\wedge}$ cannot be continuous, otherwise, by Theorem 1, $\alpha_G(G)$ can be identified with $\kappa_G(G)$. In [1] (17.6), an example of a group N for which α_N is not continuous is provided. We reproduce here the example and we will see that it is suitable for our purposes. We also give an alternative proof of the non continuity of α_N.

Let $D = \mathscr{D}(\mathbb{R})$ be the space of the test functions on the real line, and let D' be its dual space, i.e. the space of all distributions on $\mathbb{R}$. It is well-known that D is a locally convex topological vector space, which is nuclear and non-metrizable. Smolyanov proved that D contains a closed linear subspace L such that D/L is topologically isomorphic to a nonclosed dense subspace M of the countable product of real lines, $\mathbb{R}^\omega$. In [1] it is proved that $M^\wedge$ coincides with $(\mathbb{R}^\omega)^\wedge$, and since $\mathbb{R}^\omega$ is Pontryagin reflexive, $M^{\wedge\wedge}$ is isomorphic to $\mathbb{R}^\omega$. Thus M is not a Pontryagin reflexive group.

Let $N = L^0$ be the annihilator of L in D', i.e. $N = \{f \in D', f(L) = \{0\}\}$. We have:

$$M \cong D/L \cong D''/L^{00} \cong (L^0)' = N'.$$

The topology of N' coincides with the compact open topology, therefore N' is topologically isomorphic to $N^\wedge$ [8]. Since M is not Pontryagin reflexive we can conclude that neither $N^\wedge$, nor N are Pontryagin reflexive.

On the other hand, N is a closed subspace of the space D', thus, it is a complete, locally convex vector space. As mentioned before, N is a BB-reflexive vector space and equivalently a BB-reflexive group.

We will now state a direct elementary proposition, from which it follows that α_N is not continuous. Two new definitions are required.

A subgroup H of a topological group G is said to be:

dually closed if for every g in $G - H$ there is a character τ in $G^\wedge$ with $\tau(g) \neq 0$ and $\tau(H) = \{0\}$.

dually embedded if every character in $H^\wedge$ can be extended to a character in $G^\wedge$.

Proposition 2. *Let H be a dually closed and dually embedded subgroup of a reflexive topological group. Then $\alpha_H: H \to H^{\wedge\wedge}$ is an open algebraic isomorphism.*

Proof. The following diagramm

$$\begin{array}{ccc} H & \xrightarrow{\;i\;} & G \\ {\scriptstyle\alpha_H}\big\downarrow & & \big\downarrow{\scriptstyle\alpha_G} \\ H^{\wedge\wedge} & \xrightarrow{\;i^{\wedge\wedge}\;} & G^{\wedge\wedge} \end{array}$$

where all the mappings have the obvious meaning, is commutative. Therefore α_H is injective.

Since H is dually embedded, the restriction mapping $i^\wedge: G^\wedge \longrightarrow H^\wedge$ is surjective, and consequently $i^{\wedge\wedge}: H^{\wedge\wedge} \longrightarrow G^{\wedge\wedge}$ is injective. Take now ϕ in $H^{\wedge\wedge}$; by the reflexivity of G we can find g in G such that $\alpha_G(g) = i^{\wedge\wedge}(\phi)$. Observe that g must be in H, otherwise there would exist a character τ in $G^\wedge$ such that $\tau(g) \neq 0$ and $\tau(H) = \{0\}$, which contradicts $\alpha_G(g) = i^{\wedge\wedge}(\phi)$. Thus, $i^{\wedge\wedge}\alpha_H(g) = i^{\wedge\wedge}(\phi)$ implies $\alpha_H(g) = \phi$, and this proves that α_H is surjective.

In order to prove that α_H is open, take an open set O in H, and let U be open in G such that $U \cap H = O$. Then $\alpha_H(O) = i^{\wedge\wedge -1}(\alpha_G(U) \cap \alpha_G(H)) = i^{\wedge\wedge -1}(\alpha_G(U))$ is open in H.

Remark 2. If E is a locally convex topological vector space, since E' and $E^\wedge$ are algebraically isomorphic [8], the annihilator of any subspace L of E, $L^0 = \{f \in E'; f(L) = 0\}$, is a dually closed and dually embedded subgroup of $E^\wedge$. In the previous example, α_N is not a topological isomorphism; thus, by Proposition 2 it cannot be continuous.

Remark 3. Locally compact groups are both Pontryagin reflexive and BB-reflexive. The test function space is an example of a non locally compact group with the same property.

***BB*-reflexivity of sums and products of denumerably many locally compact abelian groups.** First we give some general considerations on product spaces. We recall that the box topology on a product of topological spaces is the one generated by products of open sets. Obviously it is finer than the Tychonoff product topology. On the other hand, the product of convergence spaces is defined as the product of the underlying sets endowed with the initial convergence structure relative to the canonical projections [2]. If the factor spaces are topological, then the "product convergence" is precisely the convergence associated with the product topology.

Let $\{G_n; n \in \mathbb{N}\}$ be a sequence of locally compact abelian groups. Consider the direct sum ΣG_n endowed with the topology τ induced by the box topology of ΠG_n. It is well known that $(\Sigma G_n, \tau)$ is a Pontryagin reflexive group, whose dual may be identified with $\Pi G_n^\wedge$ endowed with the Tychonoff topology τ_p [6]. We establish now the relationship between this group and the BB-dual of ΣG_n, whose underlying set is also $\Pi G_n^\wedge$. We have:

Proposition 3. *The continuous convergence structure Λ on the set $\Pi G_n^\wedge$ is finer than the convergence of the product topology τ_p and coarser than that of the box topology τ_b.*

Proof. The fact that Λ is finer than the convergence of τ_p follows from the previous considerations, since τ_p may be identified with the compact open topology in $\Pi G_n^\wedge$.

In order to prove that Λ is coarser than the convergence of the box topology in $\Pi G_n^\wedge$, we show that the evaluation mapping $w: \Pi G_n^\wedge \times \Sigma G_n \to T$ is continuous if $\Pi G_n^\wedge \times \Sigma G_n$ is endowed with $\tau_b \times \tau$.

Let N be a zero neighborhood in T, and let K_n be a compact neighborhood of zero in $G_n^\wedge$, for every n in $\mathbb{N}$. The set ΠK_n is a zero neighborhood in τ_b, whereas $(\Pi K_n, N) := \{\tau \in (\Pi G_n^\wedge)^\wedge, \tau(\Pi K_n) \subset N\}$ is a zero neighborhood in the compact open topology of

$(\Pi G_n^\wedge)^\wedge$, being τ_p the topology considered for $\Pi G_n^\wedge$. Due to continuity of $\alpha: \Sigma G_n \to (\Pi G_n^\wedge)^\wedge$ we can find a neighborhood of zero in ΣG_n, say U, such that $\alpha(U) \subset (\Pi K_n, N)$. Thus $w(\Pi K_n \times U) \subset N$, which proves that w is continuous and therefore $\Lambda \leqq \tau_b$.

Corollary 2. *The continuous convergence structure in the product of topological dual groups does not coincide in general with the product of the corresponding continuous convergence structure of the factors.*

Proof. Observe that by Proposition 1, $\Gamma_c(\Sigma G_n)$ is locally compact in the sense of convergence. As a set it coincides with $\Pi G_n^\wedge$. Therefore, if the groups G_n are taken to be locally compact but non compact, Λ is different from τ_p, whose convergence structure is precisely the product one.

The above example also shows that the inequality $\tau_p < \Lambda$ is strict. On the other hand, $\Lambda < \tau_b$ is also strict; in fact, if for every $n \in \mathbb{N}$, $G_n^\wedge$ is a compact non trivial group, then Λ coincides with the convergence structure of the compact open topology, which is the product topology τ_p, obviously different from that of the box topology.

Remark 4. From the arguments of Corollary 2 it is clear that the *BB*-dual of a sum is not isomorphic to the product of the duals as it was for Pontryagin duality.

Proposition 4. *The direct sum ΣG_n of a sequence of locally compact abelian groups is BB-reflexive.*

Proof. By Theorem 1, $(\Sigma G_n)^{\wedge\wedge}$ is a topological subgroup of $\Gamma_c\Gamma_c(\Sigma G_n)$. We will prove that in fact they are equal, and the assertion follows from the reflexivity of the sum in the ordinary Pontryagin sense.

Take κ in $\Gamma_c\Gamma_c(\Sigma G_n)$. This means that $\kappa: (\Pi G_n^\wedge, \Lambda) \to T$ is continuous. Denote by $H_m := \prod_{n=1}^{m} G_n^\wedge$. The continuous convergence structure Λ_m on H_m can be identified with the compact open topology, since it is locally compact. For every $m \in \mathbb{N}$, $\kappa_m := \kappa|_{H_m}$ is a continuous character. Taking into account that $\prod_{n=1}^{\infty} G_n^\wedge$ endowed with the product topology is the projective limit of the sequence $\{H_m; m \in \mathbb{N}\}$, we obtain that κ, being the limit of the mappings κ_m, is τ_p-continuous. Therefore $\Gamma_c\Gamma_c(\Sigma G_n) = (\Sigma G_n)^{\wedge\wedge}$.

Proposition 5. *The product ΠG_n of a sequence of locally compact abelian groups is BB-reflexive.*

Proof. An argument similar to the previous one gives the proof.

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Anschriften der Autoren:

M. J. Chasco
E.T.S.I.T.
Universidad de Vigo
36200 Vigo
Spain

E. Martín-Peinador
Facultad de Ciencias Matemáticas
Universidad Complutense
28040 Madrid
Spain