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The Glicksberg Theorem on Weakly Compact Sets for Nuclear Groups

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ABSTRACT: By the weak topology on an Abelian topological group we mean the topology induced by the family of all continuous characters. A well-known theorem of I. Glicksberg says that weakly compact subsets of locally compact Abelian (LCA) groups are compact. D. Remus and F.J. Trigos-Arrieta [1993. Proceedings Amer. Math. Soc. **117**] observed that Glicksberg's theorem remains valid for closed subgroups of any product of LCA groups. Here we show that, in fact, it remains valid for all nuclear groups, a class of Abelian topological groups introduced by the first author in the monograph, "Additive subgroups of topological vector spaces" [1991. Lecture Notes in Math. **1466**].

There are several theorems in commutative harmonic analysis which remain valid for certain Abelian topological groups which are not locally compact. For instance, the Bochner theorem on positive-definite functions is true for nuclear locally convex spaces (see [6, Chapter 4, Section 2.3]), while the Pontryagin duality theorem is true for closed subgroups of countable products of locally compact Abelian (LCA) groups (see, e.g., [2] for further references). To treat results of this type from a unified point of view, the first author introduced in [1] the so-called nuclear groups, a class of Abelian topological groups which contains LCA groups and nuclear locally convex spaces, and is closed with respect to the operations of taking subgroups, separated quotients and arbitrary products (a different definition, of a nuclear Lie group, had been given in [6, Chapter 4, Section 5.4]).

Nuclear groups satisfy, among other properties, the Bochner theorem [1, Theorem 12.1] and, under some additional assumptions, also the Pontryagin duality theorem [1, Corollary 17.3]. From the point of view of convergent series and se-

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quences, properties of nuclear groups are similar to those of nuclear spaces (see [1, Section 10] and [3]). For instance, every weakly convergent sequence of points of a nuclear group is convergent in the original topology [3, Theorem 1].

Let G be an Abelian topological group. By a *character* of G we mean a homomorphism of G into the group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. By the *weak topology* on G we mean the topology induced by the family $G^\wedge$ of all continuous characters of G. Following [7], we say that *G respects compactness* if every weakly compact subset of G is compact in the original topology. It was proved in [7] that closed subgroups of any product of LCA groups respect compactness. The aim of this paper is to prove the following generalization of that result:

THEOREM: Nuclear groups respect compactness.

The proof given below is a modification of the proof of the above-mentioned Theorem 1 of [3]. We apply notation and terminology introduced in [3]. The family of neighborhoods of zero in an Abelian topological group G is denoted by $\mathcal{N}_0(G)$. Given a real number x, we denote by $\langle x\rangle$ the number $y \in (-½, ½]$ such that $x - y \in \mathbb{Z}$. For the definitions of a nuclear group and a nuclear vector group we refer the reader to [3] or to [1, (7.1) and (9.2)]. All vector spaces are assumed to be real.

LEMMA 1: Let $(x_s)_{s=1}^{\infty}$ be a sequence of nonzero real numbers with $|x_{s+1}/x_s| \geq 3$ for every s. Then there exists a real number t such that $|\langle tx_s\rangle| \geq \frac{1}{8}$ for every s.

Proof: For each $s = 1, 2, \ldots$, let $A_s = \{t \in \mathbb{R}: |\langle tx_s\rangle| \geq \frac{1}{8}\}$. We have to show that $\bigcap_{s=1}^{\infty} A_s \neq \emptyset$. All components of A_s are closed intervals of length $\frac{3}{4}x_s^{-1}$, hence all components of $\mathbb{R}\backslash A_s$ are open intervals of length $\frac{1}{4}x_s^{-1}$.

Now, choose any component I_1 of A_1. Since $|x_2/x_1| \geq 3$, it follows easily that I_1 must contain some component I_2 of A_2. Similarly, I_2 must contain some component I_3 of A_3, and so on. This allows us to construct inductively a decreasing sequence of closed intervals $(I_s)_{s=1}^{\infty}$ such that I_s is a component of A_s for every s; then $\bigcap_{s=1}^{\infty} A_s \supset \bigcap_{s=1}^{\infty} I_s \neq \emptyset$. ❑

Let $T: E \to F$ be a bounded linear operator acting between Banach spaces. By $d_k(T)$, $k = 1, 2, \ldots$, we denote the Kolmogorov numbers of T (see [9, p. 308]). The distance of a point $u \in F$ to a subset A of F is denoted by $d(u, A)$. By span$A$ we denote the linear subspace of F spanned over A. If K is an additive subgroup of E, then we denote by K_E^* the family of all continuous linear functionals f on E such that $f(K) \subset \mathbb{Z}$.

LEMMA 2: Let E, F be Hilbert spaces and $T: E \to F$ a bounded linear operator such that $\sum_{k=1}^{\infty} kd_k(T) \leq 1$. Let K be an additive subgroup of E. Given arbitrary $a \in E$ and $r > 0$ such that $d(Ta, T(K)) \geq r$, one can find an $f \in K_E^*$ with $|\langle f(a)\rangle| \geq \frac{1}{4}$ and $\|f\| \leq 4r^{-1}$.

This follows directly from Proposition (8.4) of [1]. The condition $\sum_{k=1}^{\infty} kd_k(T) \leq 1$ may be replaced by $\sum_{k=1}^{\infty} d_k(T) \leq c$, where c is some numerical constant; it is enough to apply Theorem 3.1(i) of [4] instead of Proposition (3.11) of [1] in the proof of (8.4) in [1].

LEMMA 3: Let $T: E \to F$ and $S: F \to G$ be bounded linear operators acting between Hilbert spaces. Suppose that $\sum_{k=1}^{\infty} k\,d_k(T) \le 1$ and $d_k(S) \to 0$ as $k \to \infty$. Let K be an additive subgroup of E and $(a_n)_{n=1}^{\infty}$ a sequence in E such that

$$d(ST(a_m - a_n), ST(K)) \ge 1, \quad m \ne n. \tag{1}$$

Then one can choose a subsequence $(a_{n_s})_{s=1}^{\infty}$ of (a_n) satisfying the following condition: to each $u \in E$ there corresponds some $f \in K_E^*$ such that $|\langle f(u - a_{n_s})\rangle| \ge \frac{1}{8}$ for almost all s.

Proof: Suppose that

$$C := \sup_n d(Ta_n, T(K)) < \infty.$$

We can find a sequence $(v_n)_{n=1}^{\infty}$ in K such that $\|Ta_n - Tv_n\| < C + 1$ for every n. Then the set $\{ST(a_n - v_n)\}_{n=1}^{\infty}$ is totally bounded, because the condition $d_k(S) \to 0$ implies that S is a compact operator (see [9, p. 308]). Hence,

$$\liminf_{m,n \to \infty} \|ST(a_m - v_m) - ST(a_n - v_n)\| = 0,$$

which is impossible in view of (1). Thus $C = \infty$, and therefore, we may simply assume that $d(Ta_n, T(K)) \to \infty$ as $n \to \infty$.

The rest of the proof is similar to the proof of Lemma 3(b) in [3]. Let M be the linear subspace of E spanned over K, let N be the orthogonal completion of $\overline{M}$ in E, and let ϕ and ψ be the orthogonal projections of E onto $\overline{M}$ and N, respectively.

Suppose first that $\limsup \|\psi(a_n)\| = \infty$. Then there is a continuous linear functional g on N such that $\limsup |g\psi(a_n)| = \infty$ (weakly bounded subsets of locally convex spaces are bounded). We can choose a subsequence $(a_{n_s})_{s=1}^{\infty}$ of (a_n) such that $|g\psi(a_{n_{s+1}})/g\psi(a_{n_s})| \ge 4$ for every s. Now, take an arbitrary $u \in E$. Then

$$|g\psi(a_{n_{s+1}} - u)|/|g\psi(a_{n_s} - u)| \ge 3$$

for almost all s, say, for $s \ge s_0$. By Lemma 1, we can find some $t \in \mathbb{R}$ such that $|\langle tg\psi(a_{n_s} - u)\rangle| \ge \frac{1}{8}$ for $s \ge s_0$, and we may take $f = tg\psi$.

Next, suppose that $\limsup \|\psi(a_n)\| < \infty$. Choose a sequence $(b_n)_{n=1}^{\infty}$ in M with $b_n - \phi(a_n) \to 0$. For every n, we have

$$d(Ta_n, T(K)) \le \|Ta_n - T\phi(a_n)\| + \|T\phi(a_n) - Tb_n\| + d(Tb_n, T(K)),$$

$$\|Ta_n - T\phi(a_n)\| \le \|T\| \cdot \|a_n - \phi(a_n)\| = \|T\| \cdot \|\psi(a_n)\|,$$

$$\|T\phi(a_n) - Tb_n\| \le \|T\| \cdot \|\phi(a_n) - b_n\|.$$

As $d(Ta_n, T(K)) \to \infty$, it follows that $d(Tb_n, T(K)) \to \infty$.

Choose an index n_1 such that $d(Tb_{n_1}, T(K)) > 2$. By Lemma 2, there is some $g_1 \in K_{\overline{M}}^*$ with $|\langle g_1(b_{n_1})\rangle| \ge \frac{1}{4}$ and $\|g_1\| \le 4 \cdot 2^{-1}$. As $b_{n_1} \in \operatorname{span} K$, we can find a finitely generated subgroup K_1 of K with $b_{n_1} \in M_1 := \operatorname{span} K_1$. Then we can find an index n_2 such that $d(Tb_{n_2}, T(K + M_1)) > 2^2$ and, by Lemma 2, some $g_2 \in (K + M_1)_{\overline{M}}^*$ with $|\langle g_2(b_{n_2})\rangle| \ge \frac{1}{4}$ and $\|g_2\| \le 4 \cdot 2^{-2}$. By repeating this procedure, we construct by induction a sequence $M_1 \subset M_2 \subset \dots$ of finite-dimensional subspaces of M, a subsequence $(b_{n_s})_{s=1}^{\infty}$ of (b_n) and a sequence $g_s \in K_{\overline{M}}^*$ such that $b_{n_s} \in M_s$, $|\langle g_s(b_{n_s})\rangle| \ge \frac{1}{4}$, $g_{s+1}(M_s) = \{0\}$ and $\|g_s\| \le 4 \cdot 2^{-s}$ for every s.

Now, take an arbitrary $u \in E$. We can find a positive integer p such that $\|\phi(u)\| \le 2^{p-7}$ and $\|b_{n_s} - \phi(a_{n_s})\| \le 2^{p-7}$ whenever $s \ge p$. If $x, y \in \mathbb{R}$ and $|\langle y\rangle| \ge \frac{1}{4}$, then there is a coefficient $t = 0, \pm 1$ with $|\langle x + ty\rangle| \geq \frac{1}{4}$. Therefore, we can construct inductively a sequence $t_p, t_{p+1}, \ldots = 0, \pm 1$ such that

$$|\langle t_p g_p(b_{n_s}) + \ldots + t_s g_s(b_{n_s})\rangle| \geq \tfrac{1}{4}$$

for $s = p, p+1, \ldots$. Consider the functional $f_p = \sum_{r=p}^{\infty} t_r g_r$ on $\overline{M}$. It is clear that $f_p(K) \subset \mathbb{Z}$. We have $\|f_p\| \leq \sum_{r=p}^{\infty} \|g_r\| \leq 2^{-p+3}$. If $s \geq p$, then

$$|\langle f_p(b_{n_s})\rangle| = |\langle \textstyle\sum_{r=p}^{\infty} t_r g_r(b_{n_s})\rangle| = |\langle \textstyle\sum_{r=p}^{s} t_r g_r(b_{n_s})\rangle| \geq \tfrac{1}{4},$$

which implies that

$$|\langle f_p\phi(u - a_{n_s})\rangle| = |\langle f_p(b_{n_s}) - f_p(b_{n_s}) + f_p\phi(a_{n_s}) - f_p\phi(u)\rangle|$$

$$\geq |\langle f_p(b_{n_s})\rangle| - \|f_p\| \cdot \|b_{n_s} - \phi(a_{n_s})\| - \|f_p\| \cdot \|\phi(u)\| \geq \tfrac{1}{4} - \tfrac{1}{16} - \tfrac{1}{16} = \tfrac{1}{8}.$$

So, we may take $f = f_p\phi$. ❑

Let p be a seminorm on a vector space E. We write $B_p = \{u \in E : p(u) \leq 1\}$. The quotient space $E/p^{-1}(0)$ endowed with its canonical norm is denoted by E_p, and the canonical projection of E onto E_p by ψ_p. We shall identify E_p with the corresponding subspace of the completion $\tilde{E}_p$. We say that p is a *pre-Hilbert* seminorm if $\tilde{E}_p$ is a Hilbert space. If $q \leq p$ is another seminorm on E, the canonical operator from E_p to E_q is denoted by T_{pq}. By $\tilde{T}_{pq}$: $\tilde{E}_p \to \tilde{E}_q$ we denote the canonical extension of T_{pq}.

Proof of the Theorem: Let G be a nuclear group. Due to Theorem (9.6) of [1], there exist a nuclear vector group F, a subgroup P of F and a closed subgroup K of P such that G is topologically isomorphic to P/K. Naturally, we may identify P/K with a subgroup of F/K. As the property of respecting compactness is evidently inherited by arbitrary subgroups, we may simply assume that $G = F/K$. Let β: $F \to G$ be the canonical projection.

Let X be a weakly compact subset of G. First we shall prove that X is totally bounded. Suppose the contrary. Then we can find some $V \in \mathcal{N}_0(G)$ and some sequence $(g_n)_{n=1}^{\infty}$ in X such that $g_m - g_n \notin V$ whenever $m \neq n$. To obtain a contradiction, we shall construct a subsequence of (g_n) without weak cluster points in G.

Choose $U \in \mathcal{N}_0(F)$ such that $\beta(U) \subset V$. By (9.3) and (2.14) of [1], we can find a linear subspace E of F and pre-Hilbert seminorms $p \geq q \geq r$ on E such that $B_r \subset U$, $B_p \in \mathcal{N}_0(F)$, $\sum_{k=1}^{\infty} k\, d_k(\tilde{T}_{pq}) \leq 1$ and $d_k(\tilde{T}_{qr}) \to 0$ as $k \to \infty$. We have the canonical commutative diagram

$$\begin{array}{ccccc}
E & \xrightarrow{\text{id}} & E & \xrightarrow{\text{id}} & E \\
\downarrow{\scriptstyle \psi_p} & & \downarrow{\scriptstyle \psi_q} & & \downarrow{\scriptstyle \psi_r} \\
E_p & \xrightarrow{T_{pq}} & E_q & \xrightarrow{T_{qr}} & E_r \\
\downarrow{\scriptstyle \text{id}} & & \downarrow{\scriptstyle \text{id}} & & \downarrow{\scriptstyle \text{id}} \\
\tilde{E}_p & \xrightarrow{\tilde{T}_{pq}} & \tilde{E}_q & \xrightarrow{\tilde{T}_{qr}} & \tilde{E}_r
\end{array}$$

Set $H = E \cap K$ and consider the canonical commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{\text{id}} & F \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\ E/H & \xrightarrow{\mu} & F/K \end{array}$$

Since $B_p \in \mathcal{N}_0(F)$, the subspace E spanned over B_p is an open subgroup of F, and $A := \beta(E)$ is an open subgroup of $G = F/K$. Observe that μ is a topological embedding. The canonical projection $\gamma: G \to G/A$ is continuous if both G and G/A are endowed with their weak topologies, hence $\gamma(X)$ is a weakly compact subset of G/A. As G/A is discrete, Glicksberg's theorem implies that $\gamma(X)$ is compact, hence finite. Therefore, we can choose a subsequence $(g_n')_{n=1}^{\infty}$ of (g_n) such that $\gamma(g_n')$ is constant. Consequently, we can find a sequence $(u_n)_{n=1}^{\infty}$ in E such that $g_n' = \beta(u_n) + g_1'$ for all n.

According to our definitions, we have

$$d(\tilde{T}_{qr}\tilde{T}_{pq}(\psi_p(u_m) - \psi_p(u_n)),\ \tilde{T}_{qr}\tilde{T}_{pq}(\psi_p(K))) \geq 1$$

whenever $m \neq n$. Then it easily follows from Lemma 3 that we can choose a subsequence $(u_{n_s})_{s=1}^{\infty}$ of (u_n) such that the sequence $(\alpha(u_{n_s}))_{s=1}^{\infty}$ does not have any weak cluster points in E/H. In other words, the sequence $(\beta(u_{n_s}))_{s=1}^{\infty}$ does not have weak cluster points in $A = \beta(E) = \mu(E/H)$. Being an open subgroup, A is a weakly closed subset of G. Thus, $(\beta(u_{n_s}) + g_1')_{s=1}^{\infty}$ is a subsequence of (g_n) without weak cluster points in G.

Let us identify G with a subgroup of the completion $\tilde{G}$. Let $\overline{X}$ be the closure of X in $\tilde{G}$. As X is weakly compact, it is weakly closed in $\tilde{G}$, which means that $\overline{X} = X$. Then X is compact, being a closed and totally bounded subset of the complete group $\tilde{G}$. ❑

REMARK 1: A nuclear vector group is not necessarily a topological vector space (cf. [1, p.86]). If F above were indeed a topological vector space, then $E = F$ and the proof would be simpler.

REMARK 2: Following [7], let us denote by $\mathfrak{K}$ and $\mathfrak{P}$ the classes of Abelian topological groups which respect compactness and satisfy Pontryagin duality, respectively. Let A be an open subgroup of an Abelian topological group G. It was observed in [7, Proposition 2.7], that if G belongs to $\mathfrak{K}$ (respectively, to $\mathfrak{P}$), then so does A. The converse is also true: an easy argument shows that $A \in \mathfrak{K} \Rightarrow G \in \mathfrak{K}$, while $A \in \mathfrak{P} \Rightarrow G \in \mathfrak{P}$ was proved in [5, (2.3)].

REMARK 3: It was asked in [7] if every group in $\mathfrak{P} \cap \mathfrak{K}$ can be embedded into a product of LCA groups. The answer is negative; this result had been announced in [8]. Here we give another argument. Corollary 1.5 of [7] says that all Montel spaces belong to $\mathfrak{P} \cap \mathfrak{K}$. On the other hand, it easily follows from the structure theorem for LCA groups that if a topological vector space E can be embedded into a product of LCA groups, then it can be embedded into a product of real lines. So, if E is infinite-dimensional, then every neighborhood of 0 in E contains an infinite dimensional linear subspace. Therefore, for instance, the classical Montel spaces $\mathcal{D}$, $\mathcal{E}$, $\mathcal{H}$, $\mathcal{S}$ (see, e.g, [10, Section 8, Chapter III]) cannot be embedded into products of LCA groups.

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