A. Now, $(\{l\} \cup A_i)$ is a sequence in $\overline{\mathcal{F}}[L]^{(\beta)}$ converging to $\{l\} \cup A$, hence $\{l\} \cup A \in \overline{\mathcal{F}}[L]^{(\beta+1)}$ as required. The limit ordinal case is clear. Since $\mathcal{F}[L]^{(\omega^\alpha)} \neq \emptyset$, it follows that $\mathcal{F}[L]^{(\beta)}$ is infinite for each $\beta < \omega^\alpha$, so that $\{l\} \in \overline{\mathcal{F}}[L]^{(\beta)}$ for every $l \in L$, and each $\beta < \omega^\alpha$. Thus $\{l\} \in \overline{\mathcal{F}}[L]^{(\omega^\alpha)}$ for every $l \in L$ and hence $\emptyset \in \overline{\mathcal{F}}[L]^{(\omega^\alpha+1)}$, so that $s(\overline{\mathcal{F}}[L]) > \omega^\alpha + 1$. Finally, we apply the previous case to $\overline{\mathcal{F}}[L]$ to obtain $M = (m_i) \in [L]$ with $\mathcal{S}_\alpha(M) \subseteq \overline{\mathcal{F}}[M]$. Then setting $M' = (m_i)_{i>2}$ we have $\mathcal{S}_\alpha(M') \subseteq \mathcal{F}[M']$ as required. ■

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On Mackey topology for groups

by

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Abstract. The present paper is a contribution to fill in a gap existing between the theory of topological vector spaces and that of topological abelian groups. Topological vector spaces have been extensively studied as part of Functional Analysis. It is natural to expect that some important and elegant theorems about topological vector spaces may have analogous versions for abelian topological groups. The main obstruction to get such versions is probably the lack of the notion of convexity in the framework of groups. However, the introduction of quasi-convex sets and locally quasi-convex groups by Vilenkin [26] and the work of Banaszczyk [1] have paved the way to obtain theorems of this nature. We study here the group topologies compatible with a given duality. We have obtained, among others, the following result: for a complete metrizable topological abelian group, there always exists a finest locally quasi-convex topology with the same set of continuous characters as the original topology. We also give a description of this topology as an $\mathfrak{S}$-topology and we prove that, for the additive group of a complete metrizable topological vector space, it coincides with the ordinary Mackey topology.

Introduction. A vector topology τ in a real topological vector space E is called a *compatible topology* for E if the set of all τ-continuous linear functionals is the same as the set E^* of all continuous linear functionals in the original topology of E. The Mackey–Arens theorem implies that if E is a topological vector space, then there exists a finest locally convex compatible topology for E, called in the literature the *Mackey topology*, and frequently denoted by $\tau(E, E^*)$. Similar assertions are also proved for topological vector spaces over non-Archimedian fields [14]. In the present paper we study the question for topological abelian groups.

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As above, let us say that a group topology τ on an abelian topological group X is a *compatible topology* for X if the set of all τ-continuous characters coincides with the set $X^\wedge$ of all continuous characters in the initial topology of X. Although it is a very natural question to study the group topologies compatible with a given one, only a particular case has been considered in the literature so far: Varopoulos proves in [25] that the least upper bound of all compatible locally precompact topologies is a compatible topology. Our aim is to obtain results of this sort for a class of group topologies as wide as possible.

First of all we show that the least upper bound of all compatible group topologies for a topological abelian group may not be a compatible topology (see Proposition 2.2). Therefore, it is necessary to consider a smaller class of compatible topologies. In this spirit we have chosen the class of all locally quasi-convex compatible group topologies. The notion of locally quasi-convex topological abelian group was introduced by Vilenkin at the begining of the fifties [26], and was given further impetus by Banaszczyk, in his monograph [1]. Our choice is motivated by the fact that the underlying group of a topological vector space is locally quasi-convex if and only if the space is locally convex ([1], (2.4)).

The class of all compatible locally quasi-convex group topologies for a given topological abelian group X is nonempty: it always contains the Bohr topology. The least upper bound of this class in the lattice of all topologies in X, $\tau_g(X, X^\wedge)$, is also a locally quasi-convex topology on X. Our main result, Theorem 4.2, asserts that if X is a complete metrizable topological abelian group, then $\tau_g(X, X^\wedge)$ is a compatible topology for X, which obviously is the finest locally quasi-convex compatible topology. We also obtain a concrete description of $\tau_g(X, X^\wedge)$ for the considered case.

As proved in Proposition 5.4, if X is the additive topological group of a complete metrizable real vector space, then $\tau_g(X, X^\wedge)$ coincides with the ordinary Mackey topology $\tau(X, X^*)$. Thus, in the framework of metrizable complete groups, our topology is a generalization of Mackey topology. We do not know if this assertion holds true without any requirements. As a matter of fact, the following question remains open: Is the topology $\tau_g(X, X^\wedge)$ a compatible topology for an arbitrary topological abelian group?

As a by-product we have obtained a sort of "equicontinuity principle for groups" (Theorem 1.5), which might be of interest in its own right. It states the following: If X is a complete metrizable topological abelian group, then any subset of $X^\wedge$ which is compact in the pointwise convergence topology, $\sigma(X^\wedge, X)$, is equicontinuous.

1. Preliminaries. Let X, Y be abelian groups. The set of all group homomorphisms from X into Y, $\mathrm{Hom}(X, Y)$, endowed with pointwise operation is a group. If X, Y are topological groups, the set of all continuous homomorphisms $\mathrm{CHom}(X, Y)$ is clearly a subgroup of $\mathrm{Hom}(X, Y)$. The symbols $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$ will have the usual meaning. We identify $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ with the multiplicative group of complex numbers with modulus one, endowed with the metric induced by that of $\mathbb{C}$.

For a group X, any group homomorphism $\phi : X \to \mathbb{T}$ is called a *character*. Clearly, $\mathrm{Hom}(X, \mathbb{T})$ is a multiplicative abelian group, which it is natural to call the *algebraic dual* of X. When X is a topological group, the set of all continuous characters $X^\wedge := \mathrm{CHom}(X, \mathbb{T})$ is called the *dual group* of X. If $X^\wedge$ separates the points of X, we will say that X is a *DS-group* (abbreviation of *dually separated group*).

We recall that the *Bohr topology* of X, denoted by $\sigma(X, X^\wedge)$, is the weakest topology in X with respect to which all the elements of $X^\wedge$ are continuous. Clearly, $\sigma(X, X^\wedge)$ is Hausdorff if and only if X is a DS-group. Note also that the topological group $(X, \sigma(X, X^\wedge))$ is always precompact.

In the dual group $X^\wedge$, we denote by $\sigma(X^\wedge, X)$ the topology of pointwise convergence and by $\mathrm{comp}(X^\wedge, X)$ that of uniform convergence on the compact subsets of X. For short, $X^\wedge_\sigma := (X^\wedge, \sigma(X^\wedge, X))$ and $X^\wedge_c := (X^\wedge, \mathrm{comp}(X^\wedge, X))$. Notice that, if $x \in X$, the mapping $\widehat{x} : X^\wedge \to T$ defined by $\phi \mapsto \phi(x)$ is a character which is continuous on $X^\wedge_\sigma$ and *a fortiori* on $X^\wedge_c$. Obviously, the set $\{\widehat{x} : x \in X\}$ separates points in $X^\wedge$, therefore $X^\wedge_c$ and $X^\wedge_\sigma$ are DS-groups. The mapping $\alpha_X : X \to (X^\wedge_c)^\wedge$ defined by $x \mapsto \widehat{x}$ is a canonical group homomorphism. With the Pontryagin duality theorem in mind, we claim that the topology $\mathrm{comp}(X^\wedge, X)$ is the most natural for $X^\wedge$. Therefore, $(X^\wedge_c)^\wedge_c$ is called the *bidual group* of X. Recall that a topological abelian group X is said to be *Pontryagin reflexive* if α_X is a topological isomorphism between the groups X and $(X^\wedge_c)^\wedge_c$.

Let X be a topological abelian group, and let $A \subset X$, $B \subset X^\wedge$ be nonempty subsets. Define

$$A^\triangleright = \{\phi \in X^\wedge : \mathrm{Re}(\phi(x)) \geq 0,\ \forall x \in A\}$$

and

$$B^\triangleleft = \{x \in X : \mathrm{Re}(\phi(x)) \geq 0,\ \forall \phi \in B\},$$

where Re denotes the real part of a complex number. Clearly, $A^\triangleright$ (respectively $B^\triangleleft$) is a closed subset of $X^\wedge_\sigma$ (respectively of $X_\sigma = (X, \sigma(X, X^\wedge))$).

We state for further use the following:

PROPOSITION 1.1. *Let X be a topological abelian group and let $U \subset X$ be a neighborhood of the neutral element. Then*:

(a) $\{f \in \mathrm{Hom}(X, \mathbb{T}) : \mathrm{Re}(f(x)) \geq 0,\ \forall x \in U\} \subset X^\wedge$.
(b) *$U^\triangleright$ is an equicontinuous subset of $X^\wedge$.*
(c) *$U^\triangleright$ is a compact subset of $X^\wedge_\sigma$.*
(d) *$U^\triangleright$ is a compact subset of $X^\wedge_c$.*

Proof. The proof can be found in [1] or in [18]. ■

COROLLARY 1.2. *Let X be a topological abelian group and let τ be a group topology in X such that $(X,\tau)^\wedge \subset X^\wedge$. Then, for any τ-neighborhood U of the neutral element of X, the set $U^\triangleright$ is compact in $X^\wedge_\sigma$.*

Proof. By Proposition 1.1(a), $U^\triangleright \subset (X,\tau)^\wedge$, and by 1.1(c), $U^\triangleright$ is compact in $(X,\tau)^\wedge_\sigma$. Since the natural injection $(X,\tau)^\wedge_\sigma \to X^\wedge_\sigma$ is continuous, $U^\triangleright$ is compact in $X^\wedge_\sigma$. ■

COROLLARY 1.3. *Let X be a topological abelian group and let $B \subset X^\wedge$ be a nonempty subset. Then the following assertions are equivalent:*

(a) *B is equicontinuous.*

(b) *There is a neighborhood U of the neutral element of X such that $B \subset U^\triangleright$.*

(c) *$B^\triangleleft$ is a neighborhood of the neutral element of X.*

Proof. The implications (a)⇒(b)⇒(c) are evident. (c)⇒(a) follows from 1.1(b). ■

Next we state an "equicontinuity principle for groups" (Theorem 1.5), and we prove that Glicksberg's theorem is a consequence of it. Recall that a topological space is said to be a *Baire space* if whenever the union of a sequence of its closed subsets has an interior point, then some of them must have an interior point. We call a topological group *hereditarily Baire* if any of its closed separable subgroups is a Baire space with respect to the induced topology. Observe that any complete metrizable and any Hausdorff locally compact topological group are in a natural way hereditarily Baire. For the sake of completeness, we first include a result of [8], giving a slightly different proof of it.

PROPOSITION 1.4. *Let X, Y be topological groups, and let $u_n : X \to Y$ be a continuous homomorphism for each $n \in \mathbb{N}$. Suppose that the set of all $x \in X$ such that $(u_n(x))$ is a Cauchy sequence in Y is a nonmeager subset A of X. Then $\{u_n : n \in \mathbb{N}\}$ is equicontinuous.*

Proof. Take a neighborhood V of the neutral element e_Y of Y, and denote by W another closed symmetric neighborhood of e_Y such that $WWW \subset V$. Since $(u_n(x))$ is a Cauchy sequence for all $x \in A$, we have

$$A \subset \bigcup_{p=1}^{\infty} \bigcap_{n,m \geq p}^{\infty} \{x \in X : u_n(x).(u_m(x))^{-1} \in W\} =: \bigcup_{p=1}^{\infty} F_p.$$

A being nonmeager and F_p closed for every $p \in \mathbb{N}$, one of them, say F_q, has nonempty interior U. Due to continuity of u_i there is a neighborhood U_1 of the neutral element of X such that $U_1 \subset UU^{-1}$ and $u_i(U_1) \subset W$, $i = 1, \ldots, q$. We now show that $u_n(U_1) \subset V$ for all $n > q$. Fix $n > q$ and $x \in U_1$.

We have $x = x_1.x_2^{-1}$, where $x_1, x_2 \in U$. According to our choice of q and U_1 we have $u_n(x_1).(u_q(x_1))^{-1} \in W$, $u_q(x_2).(u_n(x_2))^{-1} \in W$, $u_q(x) \in W$. Therefore, $u_n(x) = u_n(x_1).(u_q(x_1))^{-1}.u_q(x).u_q(x_2).(u_n(x_2))^{-1} \in WWW \subset V$. Thus, $u_n(W) \subset V$ for all $n \in \mathbb{N}$, and we conclude that $\{u_n : n \in \mathbb{N}\}$ is equicontinuous. ■

THEOREM 1.5. *Let X be a metrizable hereditarily Baire group, let Y be a metrizable topological group, and let $B \subset \mathrm{CHom}(X,Y)$ be any subset which is compact in the topology of pointwise convergence. Then B is equicontinuous.*

Proof. We first claim that X can be taken separable without loss of generality. In fact, the equicontinuity of B will be proved if for any sequence in X convergent to the neutral element, say $x_n \to e_X$, we deduce that $u(x_n) \to e_Y$ uniformly with respect to $u \in B$. Therefore we can restrict ourselves to the closed subgroup of X generated by the sequence (x_n), which is also Baire, by our assumption on X.

In order to prove that B is equicontinuous, it is enough to check that any sequence of elements of B contains an equicontinuous subsequence, and then take into account the metrizability of X.

Consider thus a sequence $\{u_n : n \in \mathbb{N}\}$ in B. Let τ be the topology in $\mathrm{CHom}(X,Y)$ of pointwise convergence on X, and let τ_0 be that of pointwise convergence on a countable dense subset of X. Clearly, $\tau_0 \subset \tau$, therefore B is compact in τ_0, and in fact τ and τ_0 coincide on B. Since τ_0 is metrizable, so is the topology induced by τ on B, and consequently the sequence $\{u_n : n \in \mathbb{N}\}$ has a subsequence $\{u_{k_n} : n \in \mathbb{N}\}$ which converges pointwise to an element $u \in B$. The equicontinuity of $\{u_{k_n} : n \in \mathbb{N}\}$ now follows from Proposition 1.4. ■

REMARK 1. In [19] it was already proved that any pointwise convergent sequence of continuous homomorphisms from a Baire topological group is equicontinuous. The statement of Theorem 1.5 remains valid if the group X is either Baire separable or Čech-complete ([24], Lemma 4.1(2), and [17], Theorem 2.3, respectively). The analogous facts, and even stronger, are well known in the context of topological vector spaces (Banach–Steinhaus theorem). Namely, if X and Y are topological vector spaces, X is a Baire space and $B \subset \mathrm{CHom}(X,Y)$ is bounded in the topology of pointwise convergence, then B is equicontinuous.

COROLLARY 1.6. *Let X be a metrizable hereditarily Baire, or a separable Baire, or a Čech-complete topological abelian group, and let $B \subset X^\wedge_\sigma$ be a compact subset. Then B is equicontinuous, and consequently it is compact in $X^\wedge_c$.*

Proof. This is a direct consequence of Theorem 1.5 and Remark 1. ■

Corollary 1.7. *Let X be a Pontryagin reflexive topological abelian group such that $X_c^\wedge$ is hereditarily Baire and metrizable or separable Baire or Čech-complete. Then any $\sigma(X, X^\wedge)$-compact subset B of X is compact in the original topology of X.*

Proof. Consider B as a compact subset of $(X_c^\wedge)_\sigma^\wedge$. By Corollary 1.6, B is compact in $(X_c^\wedge)_c^\wedge$. Now by Pontryagin reflexivity of X, the topology of $(X_c^\wedge)_c^\wedge$ is precisely the original topology of X. ■

Corollary 1.8 (Glicksberg's theorem). *Let X be a locally compact abelian group. Then any $\sigma(X, X^\wedge)$-compact subset of X is compact in the original topology of X.*

Proof. This is a direct consequence of Corollary 1.7, since $X_c^\wedge$ is locally compact, therefore Čech-complete. ■

Remark 2. Another proof of Glicksberg's theorem is given in [7]. Notice that Corollary 1.7 does not hold for all Pontryagin reflexive groups (see [20]). The assertion of Corollary 1.8 is also valid for nuclear groups (see [2]).

We say that a topological abelian group X is *g-barrelled* if any compact subset of $X_\sigma^\wedge$ is equicontinuous. Thus, Corollary 1.6 asserts that any hereditarily Baire metrizable or separable Baire or Čech-complete topological abelian group is g-barrelled.

Later on we shall clarify the relationship between the notions of g-barrelled group and barrelled in the ordinary sense for topological vector spaces, thus obtaining many other examples of g-barrelled groups.

The following assertion gives a permanence property of the class of g-barrelled groups.

Proposition 1.9. *Let $(X_i)_{i\in I}$ be a family of g-barrelled groups, and for each $i \in I$, let u_i be a group homomorphism from X_i into an abelian group G. Then G, endowed with the finest group topology that makes continuous all the homomorphims u_i for $i \in I$, is a g-barrelled group.*

Proof. Let $K \subset G_\sigma^\wedge$ be a compact subset. We will see that $K^\triangleleft$ is a neighborhood of the neutral element in G, and then apply Corollary 1.3. For any fixed $i \in I$, $u_i : X_i \to G$ is continuous and so is its adjoint, $u_i' : G_\sigma^\wedge \to (X_i)_\sigma^\wedge$; consequently, $u_i'(K)$ is compact in $(X_i)_\sigma^\wedge$. Since X_i is a g-barrelled group, $u_i'(K)$ is equicontinuous. This implies that $(u_i'(K))^\triangleleft$ is a neighborhood of the neutral element in X_i. But $u_i^{-1}(K^\triangleleft) = (u_i'(K))^\triangleleft$, and by the definition of the topology of G, the set $K^\triangleleft$ is a neighborhood of the neutral element in G, hence K is equicontinuous, and the claim is proved. ■

We now establish some notations and results for topological real vector spaces, as we did above for abelian groups.

Let E and F be vector spaces. Denote by $\mathrm{Lin}(E,F)$ the vector space of all linear operators from E into F, and let us call E^{a} the *algebraic dual* of E, i.e. $E^{\mathrm{a}} = \mathrm{Lin}(E,\mathbb{R})$. If E and F are topological vector spaces, then $\mathrm{CLin}(E,F)$ denotes the vector space of all continuous linear operators, and $E^* = \mathrm{CLin}(E,\mathbb{R})$ is called the *dual* of E. If E^* separates the points of E we say that E is a *DS-space*. The weak topologies $\sigma(E,E^*)$ and $\sigma(E^*,E)$ of E and of E^*, respectively, are defined as the corresponding pointwise convergence topologies. In E^* we can also consider $\mathrm{comp}(E^*,E)$, the topology of uniform convergence on the compact subsets of E. The spaces $(E^*,\sigma(E^*,E))$ and $(E^*,\mathrm{comp}(E^*,E))$ will be denoted by E_σ^* and E_c^* respectively. It is clear that E_σ^*, and hence E_c^*, is a DS-space.

A vector space E is also an additive group, and therefore it is possible to consider the group $\mathrm{Hom}(E,\mathbb{T})$. The mapping $p : E^{\mathrm{a}} \to \mathrm{Hom}(E,\mathbb{T})$, defined by the equality

$$p(l) = \exp(2\pi i l) \quad \text{for all } l \in E^{\mathrm{a}},$$

is an injective group homomorphism between the additive group of E^{a} and the multiplicative group $\mathrm{Hom}(E,\mathbb{T})$. It is easy to see that a character $\phi \in \mathrm{Hom}(E,\mathbb{T})$ belongs to the range of p if and only if the restriction of ϕ to one-dimensional vector subspaces of E is continuous. If E is a topological vector space then $p(E^*) = E^\wedge$, where $E^\wedge = \mathrm{CHom}(E,\mathbb{T})$ is the dual group of E considered as an additive topological abelian group. A proof of this simple but important assertion can be seen in [10], (23.32.a). The restriction of p to E^* is also a topological group isomorphism between the topological groups E_c^* and $E_c^\wedge$ (see [1], (2.3)).

Proposition 1.10. *Let E be a topological vector space. Then:*

(a) *The space $(E,\sigma(E,E^*))$ and the group $(E,\sigma(E,E^\wedge))$ have the same family of compact subsets.*

(b) *A subset $B \subset E^*$ is $\sigma(E^*,E)$-compact if $p(B)$ is $\sigma(E^\wedge,E)$-compact.*

Proof. (a) This is Lemma 1.2 of [20].

(b) Apply (a) to E_σ^*. ■

Let E be a topological vector space. If $A \subset E$ and $B \subset E^*$ are nonempty sets, we define

$$A^\circ = \{x^* \in E^* : |x^*(x)| \le 1,\ \forall x \in A\}$$

and

$$^\circ B = \{x \in E : |x^*(x)| \le 1,\ \forall x^* \in B\}.$$

Clearly, A° and $^\circ B$ are closed convex symmetric subsets of E_σ^* and of $(E,\sigma(E,E^*))$ respectively. According to the bipolar theorem, the converse assertions also hold, i.e. if a subset $A_1 \subset E$ is convex symmetric and

$\sigma(E,E^*)$-closed, then $A_1 = {}^\circ B$ for some subset $B \subset E^*$, and the analogue holds for subsets $B_1 \subset E^*$.

PROPOSITION 1.11. *Let E be a topological vector space, and let $p : E^* \to E^\wedge$ be the above defined group isomorphism. For any nonempty $A \subset E$ and $B \subset E^*$, the following assertions hold:*

(a) *$p((4A)^\circ) \subset A^\triangleright$ and if A is balanced, i.e. $tA \subset A$ for all $t \in \mathbb{R}$ with $|t| \le 1$, then $p((4A)^\circ) = A^\triangleright$.*

(b) *${}^\circ(4B) \subset (p(B))^\triangleleft$ and if B is balanced, then ${}^\circ(4B) = (p(B))^\triangleleft$.*

(c) *$(A^\triangleright)^\triangleleft \subset {}^\circ(A^\circ)$. If A is balanced, then $(A^\triangleright)^\triangleleft = {}^\circ(A^\circ)$.*

(d) *$((p(B))^\triangleleft)^\triangleright \subset p(({}^\circ B)^\circ)$. If B is balanced, then $((p(B))^\triangleleft)^\triangleright = p(({}^\circ B)^\circ)$.*

Proof. (a) and (b) need only routine verification; (c) and (d) follow from (a) and (b) together with the bipolar theorem. ■

PROPOSITION 1.12. *For a topological vector space E, the following assertions are equivalent:*

(a) *The additive group E is g-barrelled.*

(b) *The compact subsets of E^*_σ are equicontinuous.*

Proof. (a)⇒(b). Let $B \subset E^*_\sigma$ be compact. Then $p(B)$ is compact in $E^\wedge_\sigma$. According to (a), $p(B)$ is equicontinuous. This implies easily that B is equicontinuous.

(b)⇒(a). Let $C \subset E^\wedge_\sigma$ be compact. By Proposition 1.10(b), $p^{-1}(C)$ is compact in E^*_σ, and now by (b) it is equicontinuous. Thus, C is also equicontinuous. ■

We say that a topological vector space E is *g-barrelled* if every compact subset of E^*_σ is equicontinuous. Proposition 1.12 shows that a topological vector space is g-barrelled if and only if its underlying topological group is a g-barrelled topological group. Recall that a topological vector space is *barrelled* if the bounded subsets of E^*_σ are equicontinuous. Evidently, any barrelled topological vector space is g-barrelled as a topological vector space, but these two notions are different (see Remark 16). Thus, we have many examples of g-barrelled topological vector spaces. In particular, they are g-barrelled topological abelian groups.

A subgroup X_0 of a topological abelian group is said to be *dually embedded* if any continuous character $\phi_0 : X_0 \to \mathbb{T}$ is the restriction to X_0 of a continuous character $\phi : X \to \mathbb{T}$. If X is a locally compact abelian group, then all its subgroups are dually embedded. This is an important consequence of the Pontryagin reflexivity theorem.

The parallel notion for topological vector spaces is the *Hahn–Banach extension property* (*HBEP*). Namely, a vector subspace E_0 of a topological vector space E is said to have HBEP if any continuous linear functional $l_0 : E_0 \to \mathbb{R}$ is a restriction of a continuous linear functional $l : E \to \mathbb{R}$. It is well known that any dense vector subspace has HBEP, and that any vector subspace of a locally convex space also has HBEP (the Hahn–Banach theorem).

For further use, we now study the dual of a group, endowed with a topology which is the supremum of a certain family of group topologies. This was already done in [25] (Proposition 3, p. 481), but we get the result with a weaker assumption.

PROPOSITION 1.13. *Let X be an abelian group (respectively a vector space), τ_i, $i \in I$, a family of group (vector) topologies in X, and let $\tau = \sup\{\tau_i : i \in I\}$. Suppose that for any finite subset $J \subset I$ the diagonal*

$$D_J(X) = \Big\{(x_i) \in \prod_{i\in J}(X,\tau_i) : x_i = x \in X,\ \forall i \in J\Big\}$$

is dually embedded in $\prod_{i\in J}(X,\tau_i)$. Then any τ-continuous character $\phi : X \to \mathbb{T}$ is of the form

$$\phi(x) = \prod_{i\in I_0} \phi_i(x), \qquad x \in X,$$

where $I_0 \subset I$ is a finite subset and $\phi_i : X \to \mathbb{T}$ is a τ_i-continuous character for $i \in I_0$. (Respectively, any τ-continuous linear functional $l : X \to \mathbb{R}$ can be expressed as $l(x) = \sum_{i\in I_0} l_i(x)$, $\forall x \in X$, with the analogous conditions.)

Proof. Fix a τ-continuous character $\phi : X \to \mathbb{T}$. It is easy to check that there is a finite $I_0 \subset I$ such that ϕ is continuous in $\tau_{I_0} := \sup\{\tau_i : i \in I_0\}$ (take into account that τ is a group topology and ϕ a homomorphism). Let $u : X \to \prod_{i\in I_0} X_i$ be the mapping $x \mapsto (x_i)_{i\in I_0}$ where $x_i = x$. The topology τ_{I_0} is the preimage under u of $\prod\{\tau_i : i \in I_0\}$, so u is continuous from (X,τ_{I_0}) to $\prod_{i\in I_0}(X,\tau_i)$. Thus, the mapping

$$\psi_0 : D_{I_0}(X) \to \mathbb{T},$$

defined by $\psi_0(u(x)) = \phi(x)$, is a continuous character. By our assumption $D_{I_0}(X)$ is dually embedded in $\prod_{i\in I_0}(X,\tau_i)$, therefore ψ_0 has an extension $\psi : \prod_{i\in I_0}(X,\tau_i) \to \mathbb{T}$. It is well known that any such extension has the form

$$\psi((x_i)_{i\in I_0}) = \prod_{i\in I_0} \psi_i(x_i),$$

where $\psi_i : (X,\tau_i) \to \mathbb{T}$, $i \in I_0$, are continuous characters. Now it is evident that $\phi = \prod_{i\in I_0} \phi_i$, where $\phi_i = \psi_i u$, $i \in I_0$. ■

REMARK 3. In other words, Proposition 1.13 says that, under mild conditions, a group X, endowed with a topology which is the supremum of a family of group topologies in X, say τ_i, $i \in I$, has as dual the group generated by the union $\bigcup_{i\in I}(X,\tau_i)^\wedge$ inside $\mathrm{Hom}(X,\mathbb{T})$. Similarly, for a vector space X, $(X, \sup_{i\in I}\tau_i)^*$ coincides with the vector subspace of X^{a} generated by the union $\bigcup_{i\in I}(X,\tau_i)^*$.

For a directed family of topologies the picture is simpler.

PROPOSITION 1.14. *Suppose that in the notations of Proposition* 1.13, τ_i, $i\in I$, *is a directed family with respect to set-theoretic inclusion. Then*

$$(X,\tau)^\wedge = \bigcup_{i\in I}(X,\tau_i)^\wedge$$

in the case of groups, and

$$(X,\tau)^* = \bigcup_{i\in I}(X,\tau_i)^*$$

in the case of vector spaces.

Proof. This is straightforward from Proposition 1.13. (For the case of groups it was already stated in [25], p. 483, Corollary.) ■

2. Compatible topologies. We begin with some definitions and results in the framework of topological vector spaces, which will be the model for our work with topological abelian groups.

Let E be a topological vector space. A new vector topology τ is said to be *compatible* for E if $(E,\tau)^* = E^*$. There is at least one locally convex compatible topology for E, the weak topology $\sigma(E,E^*)$; the least upper bound of all locally convex topologies compatible for E will be denoted by $\tau(E,E^*)$. Proposition 1.13 together with the Hahn–Banach theorem implies that $\tau(E,E^*)$ is a compatible topology for E. Consequently, for any topological vector space E, $\tau(E,E^*)$ is the finest locally convex compatible topology for E, and it is called the *Mackey topology* of E. The Mackey–Arens theorem asserts that $\tau(E,E^*)$ is precisely the topology of uniform convergence on all the $\sigma(E^*,E)$-compact convex subsets of E^*. Recall that a locally convex space E is called a *Mackey space* if its original topology coincides with $\tau(E,E^*)$. All barrelled and all metrizable locally convex spaces are Mackey spaces ([3], Sect. IV.2, Prop. 4).

For a locally convex space E, $\tau(E,E^*)$ may be strictly finer than the original topology of E. For instance, if E is an infinite-dimensional Banach space, then the Mackey topology of the locally convex space $(E,\sigma(E,E^*))$ is the original topology of E, which is strictly finer than $\sigma(E,E^*)$. Notice also that if E is a nonlocally convex metrizable topological vector space, then $\tau(E,E^*)$ is coarser than the original topology of E.

In any topological vector space E there is, thus, a finest locally convex compatible topology for E. The assumption of local convexity cannot be removed: as proved in [12], if a topological vector space contains a dense infinite-codimensional vector subspace, then E does not admit a finest compatible vector topology. In [13] a similar result was obtained for locally r-convex ($0<r<1$), locally bounded and locally pseudoconvex topologies. The following assertion can be considered as a refinement of the mentioned result for a less general class of spaces.

PROPOSITION 2.1. *Let E be a metrizable infinite-dimensional topological vector space, and let $\widetilde{\tau}$ be the least upper bound of the family of all compatible vector topologies for E. Then*:

(a) *For any noncontinuous linear functional $f: E\to\mathbb{R}$ there are two vector topologies β_1 and β_2 compatible for E, finer than the original, such that f is* $\sup(\beta_1,\beta_2)$-*continuous.*

(b) $(E,\widetilde{\tau})^* = E^{\mathrm{a}}$.

(c) *The topology $\widetilde{\tau}$ is not compatible for E.*

Proof. Evidently (a) implies (b), and (b) also implies (c) since, in the considered case, $E^*\neq E^{\mathrm{a}}$. Thus, we only have to prove (a). We shall follow the scheme of [13].

Take $f\in E^{\mathrm{a}}\setminus E^*$; then $\ker f = \{e\in E : f(e)=0\}$ is dense in E. Since $\ker f$ is itself a metrizable, infinite-dimensional topological vector space, it contains a dense vector subspace G such that $\dim(\ker f/G)=\aleph_0$ (see [16]). And now, the proof of Proposition 1 of [13] works. However, we write out a self-contained proof, without using Lemma 2.9 of [21].

Let $X := E/G$ and let $q: E\to X$ be the quotient map. Observe that G is also dense in E and $\dim X = \aleph_0$. Define $h\in X^{\mathrm{a}}$ by $hq(e)=f(e)$, and a linear mapping $S: X\to X$ by $S(x) = x - h(x)y$, where y is a fixed element of X such that $h(y)=2$. Straightforward computations show that that $SS(x)=x$, $\forall x\in X$, therefore S is bijective. We can equip X with a Hausdorff, dual-less vector topology α_1 (i.e. $(X,\alpha_1)^*=\{0\}$). Such a topology can be constructed from a well-known dual-less topological vector space, say, for instance, $L_{1/2}[0,1]$. Since the algebraic dimension of X is $\aleph_0$, it follows that X is algebraically isomorphic to a dense vector subspace of $L_{1/2}[0,1]$, and one can take as α_1 the preimage topology of that of $L_{1/2}[0,1]$.

Let α_2 be the image topology of α_1 under S. Evidently, $S:(X,\alpha_1)\to(X,\alpha_2)$ is continuous. This implies that $(X,\alpha_2)^*=\{0\}$.

The topology $\alpha := \sup(\alpha_1,\alpha_2)$ is no longer dual-less. In fact, the sets of the form $S(A_1)\cap A_2$, where A_1, A_2 are neighborhoods of zero in α_1, constitute a fundamental system of neighborhoods of zero in α, and the equality $S^{-1}(S(A_1)\cap A_2) = A_1\cap S(A_2)$ proves that $S:(X,\alpha)\to(X,\alpha)$ is continuous. This, in turn, implies that the mapping $x\mapsto Sx - x = -h(x)y$ is also (α,α)-continuous and, since α is Hausdorff, $h:(X,\alpha)\to\mathbb{R}$ is also continuous.

Denote by τ_0 the original topology of E, and let $\beta_i = \sup(\tau_0, q^{-1}(\alpha_i))$. By means of Proposition 1.13, we will prove that the topologies β_1 and β_2 are compatible for E.

First, we verify that the diagonal $D = \{(e,e) : e \in E\}$ is dense in $(E,\tau_0)\times(E,q^{-1}(\alpha_i))$. Take (e',e'') in $E\times E$. Since $G\times G$ is dense in $(E,\tau_0)\times(E,\tau_0)$, we can pick a net (e'_δ, e''_δ) in $G\times G$ convergent to $(e',-e'')$. Evidently, $(e'_\delta+e''_\delta+e'', e'_\delta+e''_\delta+e'')\in D$ is such that $e'_\delta+e''_\delta+e''\to e'$ in τ_0, and it remains to show that $e'_\delta+e''_\delta+e''\to e''$ in $q^{-1}(\alpha_i)$, i.e. $q(e'_\delta+e''_\delta+e'')\to q(e'')$ in α_i. But e'_δ, e''_δ are in G, hence $q(e'_\delta+e''_\delta+e'') = q(e'')$ for all δ. Applying now Proposition 1.13, we have $(E,\beta_i)^* = E^*$.

Since $\sup(\beta_1,\beta_2)$ is finer than $\sup(q^{-1}(\alpha_1), q^{-1}(\alpha_2))$, we conclude that f is also $\sup(\beta_1,\beta_2)$-continuous. ■

REMARK 4. Let E be a topological vector space and let $\mathcal{T}_{\mathcal{V}}(E)$ be the family of all vector topologies compatible for E. It follows easily from Zorn's lemma and Proposition 1.14 that the partially ordered family $(\mathcal{T}_{\mathcal{V}}(E),\subset)$ has maximal elements. Proposition 2.1(a) implies that if E is an infinite-dimensional metrizable topological vector space, then its original topology is not a maximal element of $\mathcal{T}_{\mathcal{V}}(E)$.

By analogy to the case of vector spaces, we say that a group topology τ on a topological abelian group X is *compatible* for X if $(X,\tau)^\wedge = X^\wedge$. Clearly, the original topology of X is compatible and it is natural to ask whether the least upper bound of the family of all compatible topologies for X is again a compatible topology. The following reinterpretation of Proposition 2.1 gives a negative answer in general.

PROPOSITION 2.2. *Let E be a metrizable infinite-dimensional topological vector space, and let $\widetilde{\tau}_{\mathrm{g}}$ be the least upper bound of the family of all compatible group topologies for the underlying topological abelian group of E. Then*

$$E^\wedge \subsetneqq p(E^{\mathrm{a}}) \subset (E,\widetilde{\tau}_{\mathrm{g}})^\wedge.$$

In particular, $\widetilde{\tau}_{\mathrm{g}}$ is not a compatible topology for the additive topological abelian group of E.

Proof. Since there exists a noncontinuous linear functional, the inclusion

$$E^\wedge = p(E^*) \subset p(E^{\mathrm{a}})$$

is strict. Take any $f\in E^{\mathrm{a}}$. Then $p(f)$ is a $\widetilde{\tau}$-continuous character (see Proposition 2.1), and the inclusion $p(E^{\mathrm{a}})\subset (E,\widetilde{\tau}_{\mathrm{g}})^\wedge$ will follow from the fact that $\widetilde{\tau}_{\mathrm{g}}$ is finer than $\widetilde{\tau}$. ■

REMARK 5. Let X be a topological abelian group and let $\mathcal{T}_G(X)$ be the family of all group topologies compatible for X. It follows easily from Zorn's

lemma and Proposition 1.14 that the partially ordered family $(\mathcal{T}_G(X),\subset)$ has maximal elements.

3. Locally quasi-convex compatible topologies.

The notion of locally quasi-convex group was introduced by Vilenkin in [26]. We refer the reader to [1] for a closer examination of this class of groups. We only state here the facts required for our aims.

A nonempty subset A of a topological abelian group X is said to be *quasi-convex* if for any $x\in X\setminus A$ there is a character $\phi\in X^\wedge$ such that $\phi\in A^\triangleright$ and $\mathrm{Re}(\phi(x))<0$. Observe that the notion of quasi-convex set depends upon the pair $(X,X^\wedge)$ and therefore the family of quasi-convex sets will not change if we equip X with another group topology compatible for X. Obviously, any quasi-convex $A\subset X$ contains the neutral element, and the intersection of any family of quasi-convex sets is again quasi-convex.

The *quasi-convex hull* $Q(A)$ of a subset $A\subset X$ is defined as the intersection of all quasi-convex subsets containing A. This definition makes sense because the whole group X is quasi-convex. It is straightforward to see that $Q(A) = (A^\triangleright)^\triangleleft$. The fact that A is quasi-convex if and only if $A = Q(A)$ immediately implies that any quasi-convex $A\subset X$ is closed in the Bohr topology $\sigma(X,X^\wedge)$. A subgroup $G\subset X$ is quasi-convex if and only if it is closed in $\sigma(X,X^\wedge)$ (see [1], (2.5)).

Let E be a real topological vector space. From the bipolar theorem it easily follows that an absolutely convex $\sigma(E,E^*)$-closed subset of E is quasi-convex in E considered as a group. Conversely, Proposition 1.11(c) implies that if A is a balanced subset of E and A is quasi-convex in the additive group of E, then A is absolutely convex and $\sigma(E,E^*)$-closed. In general, a quasi-convex subset of E may not be convex in the ordinary sense. In fact, a direct verification shows that the set $\{-1,0,1\}$ is quasi-convex in the additive group of $\mathbb{R}$.

If A is a nonempty subset of a topological abelian group X, then $A^\triangleright$ is quasi-convex in $X^\wedge_\sigma$. The following assertion shows that if A is a neighborhood of zero in X, then $A^\triangleright$ has better properties.

LEMMA 3.1. *Let X be a topological abelian group and let V be a neighborhood of the neutral element of X. Then $V^\triangleright$ is quasi-convex in $(X_{\mathrm{d}})^\wedge_\sigma$, where X_{d} denotes the group X endowed with the discrete topology.*

Proof. Take any $\phi\in (X_{\mathrm{d}})^\wedge\setminus V^\triangleright$. It is clear that $\alpha(X)\subset ((X_{\mathrm{d}})^\wedge_\sigma)^\wedge$. Thus, it is sufficient to find an element $a\in X$ such that $a\in (V^\triangleright)^\triangleleft$ and $\mathrm{Re}(\phi(a))<0$. Suppose that such an element does not exist. Then for every $x\in (V^\triangleright)^\triangleleft$ we have $\mathrm{Re}(\phi(x))\ge 0$. In particular, since $V\subset (V^\triangleright)^\triangleleft$, we obtain $\mathrm{Re}(\phi(x))\ge 0$ for all $x\in V$. Therefore, by Proposition 1.1, $\phi\in V^\triangleright$, a contradiction. ■

Corollary 3.2. *Let X be a topological abelian group, and let $K \subset X^\wedge$ be an equicontinuous subset which is quasi-convex in $X^\wedge_\sigma$. Then K is quasi-convex in $(X_{\rm d})^\wedge_\sigma$.*

Proof. By Corollary 1.3, $W := K^\triangleleft$ is a neighborhood of the neutral element in X. Since K is quasi-convex in $X^\wedge_\sigma$, we have $K = W^\triangleright$. By the previous lemma, K is quasi-convex in $(X_{\rm d})^\wedge_\sigma$. ∎

Remark 6. In general, a continuous homomorphic image of a quasi-convex set may not be quasi-convex. In particular, a quasi-convex subset B of $X^\wedge_\sigma$ may not be quasi-convex in $(X_{\rm d})^\wedge_\sigma$.

A topological abelian group (X,τ) is called *locally quasi-convex* if there exists a fundamental system of quasi-convex neighborhoods for the neutral element of X; in this case also the topology τ is said to be locally quasi-convex. Evidently, any Hausdorff locally quasi-convex topological group is a DS-group. A topological vector space E considered as an additive topological group is locally quasi-convex if and only if it is locally convex as a topological vector space (see [1], (2.4)).

Subgroups of locally quasi-convex groups are locally quasi-convex. However, a Hausdorff quotient of a locally quasi-convex group may not be locally quasi-convex ([1], Th. (5.1)(c)).

Proposition 3.3. *Let X be an abelian group, and let τ_j, $j \in J$, be a family of locally quasi-convex group topologies in X. Then $\tau = \sup_{j\in J} \tau_j$ is a locally quasi-convex topology.*

Proof. For every $j \in J$, let V_j be a quasi-convex τ_j-neighborhood of e_X. The sets of the form

$$V = \bigcap_{j\in F} V_j,$$

where $F \subset J$ is finite, are a fundamental system of τ-neighborhoods of e_X. Now it is straightforward to show that V is quasi-convex in τ. ∎

Proposition 3.3 implies that the product of an arbitrary family of locally quasi-convex groups is a locally quasi-convex group. Now we describe another method to produce locally quasi-convex group topologies.

Let Y be a multiplicative group, $B \subset Y$ a nonempty subset and $n \in \mathbb{N}$ a natural number. Here and below $B^{(n)}$ will denote the set $\{y^n : y \in B\}$. Obviously $B^{(n)} \subset B.B \overset{n}{\ldots} B$. A nonempty family $\mathfrak{S}$ of subsets of Y is called *well-directed* if the following conditions hold:

(a) For $B_1, B_2 \in \mathfrak{S}$, there exists $B_3 \in \mathfrak{S}$ such that $B_1 \cup B_2 \subset B_3$.
(b) For $B \in \mathfrak{S}$ and $n \in \mathbb{N}$, there exists $A \in \mathfrak{S}$ such that $B^{(n)} \subset A$.

If $\mathfrak{S}$ is the family of all nonempty finite subsets, or of all compact subsets, of Y, then $\mathfrak{S}$ is well-directed. We do not know if the family of all compact quasi-convex subsets of a topological abelian group is well-directed.

For a given family $\mathfrak{S}$ of subsets of Y, denote by $\overline{\mathfrak{S}}$ the family of all sets of the form $B_1^{(k_1)} \cup \ldots \cup B_n^{(k_n)}$, where $n, k_1, \ldots, k_n$ are natural numbers, and $B_1, \ldots, B_n$ are elements of $\mathfrak{S}$. Clearly, $\overline{\mathfrak{S}}$ is well-directed and contains $\mathfrak{S}$.

In what follows it will be convenient to fix a group duality (X,Y), which consists of an abelian group X and a subgroup Y of $\mathrm{Hom}(X,\mathbb{T})$. If Y separates the points of X, we say that the duality is *separating*. The topologies $\sigma(X,Y)$ and $\sigma(Y,X)$ are defined as in Section 1 for the case $Y = X^\wedge$. For $A \subset X$ and $B \subset Y$, the subsets $A^\triangleright \subset Y$ and $B^\triangleleft \subset X$ are defined analogously.

A topology τ in X is *compatible* with a group duality (X,Y) if $(X,\tau)^\wedge = Y$. Similarly, a topology τ' in Y is compatible with the duality if $(Y,\tau')^\wedge$ coincides with the natural image of X in $\mathrm{Hom}(Y,\mathbb{T})$.

Let (X,Y) be a group duality and $\mathfrak{S}$ a family of nonempty subsets of X. Since $\mathbb{T}$ is a metric space, we can consider in $Y \subset \mathbb{T}^X$ the topology $\tau_{\mathfrak{S}}(Y,X)$ of uniform convergence on the sets $A \in \mathfrak{S}$. It will be called the $\mathfrak{S}$*-topology*, and it is a group topology. If $\mathfrak{S}$ covers X, then $\tau_{\mathfrak{S}}(Y,X)$ is Hausdorff.

In the same fashion, if $\mathfrak{S}'$ is a family of nonempty subsets of Y, and $\alpha : X \to \mathrm{Hom}(Y,\mathbb{T})$ the natural homomorphism, then the preimage topology $\alpha^{-1}(\tau_{\mathfrak{S}'}(\alpha(X),Y))$ will be denoted by $\tau_{\mathfrak{S}'}(X,Y)$ and called the $\mathfrak{S}'$*-topology* of X. Clearly, the $\mathfrak{S}'$-topology in X is a group topology and it is Hausdorff if Y separates the points of X and $\mathfrak{S}'$ covers Y. It can be easily checked that $\tau_{\mathfrak{S}}(Y,X) = \tau_{\overline{\mathfrak{S}}}(Y,X)$ and $\tau_{\mathfrak{S}'}(X,Y) = \tau_{\overline{\mathfrak{S}'}}(X,Y)$.

Proposition 3.4. *Let X be a group, Y a subgroup of* $\mathrm{Hom}(X,\mathbb{T})$, *and $\mathfrak{S}$ and $\mathfrak{S}'$ families of nonempty subsets of X and Y respectively. Then:*

(a) *The collection*

$$\mathcal{B} = \{B^\triangleleft : B \in \overline{\mathfrak{S}'}\}$$

is a fundamental system of neighborhoods of the neutral element e_X in the topology $\tau_{\mathfrak{S}'}(X,Y)$. In particular, $\tau_{\mathfrak{S}'}(X,Y)$ is a locally quasi-convex topology.

(b) *The collection*

$$\mathcal{A} = \{A^\triangleright : A \in \overline{\mathfrak{S}}\}$$

is a fundamental system of neighbourhoods of the neutral element e_Y in the topology $\tau_{\mathfrak{S}}(Y,X)$. In particular, $\tau_{\mathfrak{S}}(Y,X)$ is locally quasi-convex.

Proof. We only indicate the proof of (a). The proof of (b) is similar. It is easy to see that the family

$$V_{B,\varepsilon} := \{x \in X : \sup_{\phi\in B} |1-\phi(x)| < \varepsilon\}, \qquad 0 < \varepsilon \le 2,\ B \in \overline{\mathfrak{S}'},$$

is a fundamental system of neighborhoods of e_X for the topology $\tau_{\mathfrak{S}'}(X,Y)$. Fix $B \in \overline{\mathfrak{S}'}$. If $x \in V_{B,\sqrt{2}}$, then $\mathrm{Re}(\phi(x)) > 0$ for all $\phi \in B$, i.e. $x \in B^{\triangleleft}$. Thus, every member of the family $\mathcal{B}$ is a $\tau_{\mathfrak{S}'}(X,Y)$-neighborhood of e_X. In order to see that $\mathcal{B}$ is a fundamental system of neighborhoods, take a fixed $\varepsilon \in (0,2]$ and $B \in \overline{\mathfrak{S}'}$. Choose $n \in \mathbb{N}$ such that $2/n < \varepsilon$. Taking into account the relationship

$$\{t \in \mathbb{T} : \mathrm{Re}(t) \geq 0, \ldots, \mathrm{Re}(t^n) \geq 0\} \subset \{t \in \mathbb{T} : |1-t| < 2/n\},$$

since $\overline{\mathfrak{S}'}$ is well-directed, there exists $D \in \overline{\mathfrak{S}'}$ such that $B \cup B^{(1)} \cup \ldots \cup B^{(n)} \subset D$. If $x \in D^{\triangleleft}$, then $\mathrm{Re}(\phi(x)) \geq 0, \ldots, \mathrm{Re}(\phi^n(x)) \geq 0$ for all $\phi \in B$. Therefore $\sup_{\phi \in B} |1-\phi(x)| \leq 2/n < \varepsilon$ and $x \in V_{B,\varepsilon}$. Thus, $D^{\triangleleft} \subset V_{B,\varepsilon}$ and the first part of (a) is proved.

Now we show that every member of $\mathcal{B}$ is quasi-convex in $\tau_{\mathfrak{S}'}(X,Y)$. Fix again $B \in \overline{\mathfrak{S}'}$. By Proposition 1.1(a), $(B^{\triangleleft})^{\triangleright} \subset (X, \tau_{\mathfrak{S}'})^{\wedge}$. Let $z \in X \setminus B^{\triangleleft}$. There is $\phi \in B$ such that $\mathrm{Re}(\phi(z)) < 0$. Since $B \subset (B^{\triangleleft})^{\triangleright}$, the quasi-convexity of $B^{\triangleleft}$ is proved. ■

COROLLARY 3.5. *Let the family $\mathfrak{S}'$ be such that $(X, \tau_{\mathfrak{S}'})^{\wedge} \subset Y$. Then, for any $B \in \overline{\mathfrak{S}'}$, the set $(B^{\triangleleft})^{\triangleright}$ is compact in $(Y, \sigma(Y,X))$. In particular, any such B is relatively $\sigma(Y,X)$-compact.*

Proof. By Proposition 3.4 the set $B^{\triangleleft}$ is a neighborhood of e_X in the topology $\tau_{\mathfrak{S}'}(X,Y)$. According to Proposition 1.1(a) and (c), $(B^{\triangleleft})^{\triangleright}$ is compact in $(X, \tau_{\mathfrak{S}'})^{\wedge}_{c}$. Since the natural embedding $(X, \tau_{\mathfrak{S}'})^{\wedge}_{\sigma} \to (Y, \sigma(Y,X))$ is continuous, we deduce that $(B^{\triangleleft})^{\triangleright}$ is compact in $\sigma(Y,X)$. ■

COROLLARY 3.6. *Assume that the group duality (X,Y) is separating and that the topology $\tau_{\mathfrak{S}}(Y,X)$ is compatible with (X,Y). If $A \in \overline{\mathfrak{S}}$, then $(A^{\triangleright})^{\triangleleft}$ is compact in $(X, \sigma(X,Y))$. In particular, A itself is $\sigma(X,Y)$-relatively compact.*

Proof. Apply Corollary 3.5 to the duality $(Y, \alpha(X))$, and use the fact that the natural mapping

$$\alpha : (X, \sigma(X,Y)) \to (\alpha(X), \sigma(\alpha(X), Y))$$

is a topological isomorphism. ■

Proposition 3.4 implies that $\sigma(X,Y)$ is a locally quasi-convex topology in X, since it is the $\mathfrak{S}'$-topology for the family of finite subsets of Y. It is clear that for any family $\mathfrak{S}'$ which covers Y, $\sigma(X,Y) \subset \tau_{\mathfrak{S}'}(X,Y)$. The following assertion states another important property of this topology.

THEOREM 3.7. *Let (X,Y) be a group duality. Then the topology $\sigma(X,Y)$ is compatible with (X,Y), i.e., $(X, \sigma(X,Y))^{\wedge} = Y$.*

Proof. Evidently, $Y \subset (X, \sigma(X,Y))^{\wedge}$. We prove the reverse inclusion. Fix any $\phi \in (X, \sigma(X,Y))^{\wedge}$. By Proposition 3.4 there exist $n \in \mathbb{N}$ and $\phi_1, \ldots, \phi_n \in Y$ such that

$$x \in (\{\phi_1, \ldots, \phi_n\})^{\triangleleft} \Rightarrow \mathrm{Re}(\phi(x)) \geq 0.$$

Consider the homomorphism $x \mapsto u(x) := (\phi_1(x), \ldots, \phi_n(x))$ and put $G = u(X)$. Clearly, $u : X \to \mathbb{T}^n$ is continuous and G is a subgroup of $\mathbb{T}^n$. Define a mapping $\psi_0 : G \to \mathbb{T}$ by

$$\psi_0(u(x)) = \phi(x), \qquad x \in X,$$

It is straightforward to check that ψ_0 is a well-defined continuous character. Since $\mathbb{T}^n$ is a compact abelian group, its subgroup G is dually embedded in $\mathbb{T}^n$. Hence, ψ_0 has an extension $\psi \in (\mathbb{T}^n)^{\wedge}$. Since $(\mathbb{T}^n)^{\wedge} = \mathbb{Z}^n$, there are integers $m_1, \ldots, m_n$ such that

$$\phi(x) = \psi_0(u(x)) = \psi(u(x)) = \prod_{k=1}^{n} (\phi_k(x))^{m_k}.$$

Consequently, $\phi = \prod_{k=1}^{n} \phi_k^{m_k} \in Y$. ■

REMARK 7. The assertion of Theorem 3.7 is well known for the case of a separating duality (see [25], p. 481, Corollary; see also [7], p. 37, Theorem 2.3.4, where an interesting proof without using the Pontryagin duality is given). We have included the proof to underline that it also holds without requiring that Y separates the points of X.

COROLLARY 3.8. *Let (X,Y) be a group duality. Then the topology $\sigma(Y,X)$ is compatible with (X,Y), i.e.*

$$(Y, \sigma(Y,X))^{\wedge} = \alpha(X).$$

In other words, for any $\psi \in (Y, \sigma(Y,X))^{\wedge}$, there is an element $x \in X$ such that

$$\psi(\phi) = \phi(x), \qquad \forall \phi \in Y.$$

If Y separates the points of X, the element x with that property is unique.

Proof. Apply Theorem 3.7 to the duality $(Y, \alpha(X))$. ■

Consider now the topology $\mathrm{comp}(X^{\wedge}, X)$, i.e. the $\mathfrak{S}$-topology on the compact subsets of X. Again by Proposition 3.4 we find that $X^{\wedge}_{c} = (X^{\wedge}, \mathrm{comp}(X^{\wedge}, X))$ is a locally quasi-convex group. In particular, the group $(X^{\wedge}_{c})^{\wedge}_{c}$ is also locally quasi-convex, therefore any Pontryagin reflexive group is locally quasi-convex. All Hausdorff locally compact abelian groups are locally quasi-convex, and so are Hausdorff locally precompact topological abelian groups. The latter can be embedded in the corresponding locally compact Hausdorff abelian group. It is easy to see that a non-Hausdorff topological abelian group X whose associated Hausdorff abelian group is locally quasi-convex must also be locally quasi-convex. From this remark it follows that any locally precompact (Hausdorff or not) topological abelian group is locally quasi-convex.

So far we have seen that any $\mathfrak{S}$-topology is a locally quasi-convex topology. Conversely, let us see now that any locally quasi-convex topology is an $\mathfrak{S}$-topology for a suitable family $\mathfrak{S}$. Denote by $\tau_e(X, X^\wedge)$ the $\mathfrak{S}_e$-topology, where $\mathfrak{S}_e$ is the family of all the equicontinuous subsets of $X^\wedge$.

PROPOSITION 3.9. *Let X be a topological abelian group, and let τ_X be its original topology. Then*:

(a) $\sigma(X, X^\wedge) \subset \tau_e(X, X^\wedge) \subset \tau_X$.

(b) *X is locally quasi-convex if and only if $\tau_X = \tau_e(X, X^\wedge)$.*

(c) *If X is locally quasi-convex, then $\tau_X = \tau_{\mathfrak{S}}(X, X^\wedge)$, where $\mathfrak{S}$ is the family of those equicontinuous subsets of $X^\wedge$ which are compact and quasi-convex in $X^\wedge_\sigma = (X^\wedge, \sigma(X^\wedge, X))$.*

Proof. (a) The first inclusion is evident, and the second follows from Proposition 3.4(a), together with the fact that the family $\mathfrak{S}_e$ is well-directed, and Corollary 1.3.

(b) Assume that X is locally quasi-convex, and that V is a quasi-convex neighborhood of the neutral element e_X. By Proposition 1.1(b), $V^\triangleright \in \mathfrak{S}_e$. Thus, $(V^\triangleright)^\triangleleft$ is a neighborhood of e_X in $\tau_e(X, X^\wedge)$. Since V is quasi-convex, we have $V = (V^\triangleright)^\triangleleft$, and therefore $\tau_X \subset \tau_e(X, X^\wedge)$. Now by (a) we have the desired equality.

(c) By (b) we have $\tau_{\mathfrak{S}}(X, X^\wedge) \subset \tau_X$. On the other hand, let V be a quasi-convex neighborhood of e_X in τ_X. We have $V = (V^\triangleright)^\triangleleft$. By Proposition 1.1, $V^\triangleright \subset X^\wedge$ is equicontinuous $\sigma(X^\wedge, X)$-compact and, being a polar set, it is quasi-convex. This implies that $\tau_X \subset \tau_{\mathfrak{S}}(X, X^\wedge)$. ∎

REMARK 8. For a topological abelian group X the topology $\tau_e(X, X^\wedge)$ is the finest locally quasi-convex topology coarser than the original topology of X. The family of quasi-convex hulls of all neighborhoods of the neutral element of X is a fundamental system of neighborhoods (see [4]).

Consider again a group duality (X, Y). As we have seen, the locally quasi-convex topology $\sigma(X, Y)$ is compatible for (X, Y). In fact, it is even precompact. A complete description of all locally precompact compatible topologies is contained in [25], in the following terms. Let $\mathcal{L}$ be a locally compact topology in Y such that $\sigma(Y, X) \subset \mathcal{L}$; then the topology $\mathcal{L}_c$ of uniform convergence on all $\mathcal{L}$-compact subsets of Y is a locally precompact topology in X compatible with the duality. Furthermore, all locally precompact topologies in X compatible with the duality can be obtained in this way (see [25], p. 480, Proposition 1). Denote now by $\tau_{\text{var}}(X, Y)$ the least upper bound of the family of all such topologies in X. The main results in this direction are the following: $\tau_{\text{var}}(X, Y)$ is an $\mathfrak{S}$-topology in X, where

$K \in \mathfrak{S}$ if and only if K is a compact subset of Y for some locally compact topology $\mathcal{L}(K)$ finer than $\sigma(Y, X)$. Also, if $\mathcal{P}$ is a projective limit of locally precompact topologies in X, then $\mathcal{P}$ is compatible with the duality if and only if $\sigma(X, Y) \subset \mathcal{P} \subset \tau_{\text{var}}(X, Y)$ (see [25], p. 483, Proposition 5). In what follows we shall study similar problems for the wider class of locally quasi-convex topologies.

We denote by $\mathfrak{Q}(X, Y)$ (or just $\mathfrak{Q}$) the family of all locally quasi-convex topologies in X compatible with a group duality (X, Y).

PROPOSITION 3.10. *In the partially ordered set $(\mathfrak{Q}(X, Y), \subset)$, there always exist maximal elements.*

Proof. This follows from Zorn's lemma and Propositions 1.14 and 3.3. ∎

In a natural way, the Mackey–Arens theorem suggests the following questions:

QUESTION 1. Is there a finest locally quasi-convex topology in X compatible with a group duality (X, Y)?

QUESTION 2. If there is such a topology, may it be described as an $\mathfrak{S}$-topology for an appropriate family $\mathfrak{S}$ of subsets of Y?

We first study both questions separately, and later on we shall provide answers for some group dualities.

Denote by $\tau_g(X, Y)$ the least upper bound of the family of all locally quasi-convex topologies compatible with a group duality (X, Y). In other words, $\tau_g(X, Y)$ is the l.u.b. of $\mathfrak{Q}(X, Y)$ in the family of all topologies in X. Proposition 3.3 implies that $\tau_g(X, Y)$ is a locally quasi-convex topology in X. Therefore Question 1 is reduced to finding out when the topology $\tau_g(X, Y)$ is compatible with the duality (X, Y), which in turn can be reformulated as we indicate now.

PROPOSITION 3.11. *Let (X, Y) be a group duality. The following assertions are equivalent*:

(a) *The topology $\tau_g(X, Y)$ is compatible with (X, Y).*

(b) *For any two locally quasi-convex topologies in X, τ_1 and τ_2, which are compatible with the duality (X, Y), the topology $\sup(\tau_1, \tau_2)$ is also compatible with (X, Y).*

Proof. (a)⇒(b) is evident. On the other hand, condition (b) implies that the family $\mathfrak{Q}(X, Y)$ is directed with respect to set-theoretic inclusion. Applying Proposition 1.14 we obtain (b)⇒(a). ∎

PROPOSITION 3.12. *Let (X, Y) be a group duality such that the topology $\tau_g(X, Y)$ is compatible with (X, Y). Then $\tau_g(X, Y)$ is an $\mathfrak{S}$-topology, where $\mathfrak{S}$ is a well-directed family of compact quasi-convex subsets of $(Y, \sigma(Y, X))$.*

Proof. We have $(X,\tau_{\rm g}(X,Y))^\wedge = Y$ and it remains to apply Proposition 3.9(c). ■

Remark 9. The authors do not know if the family $\mathfrak{S}$ in the above proposition can be taken to be the family of all compact quasi-convex subsets of $(Y,\sigma(Y,X))$.

The topology $\tau_{\rm g}(X,Y)$ is in a sense a "descriptive candidate" to be the Mackey topology for the group duality (X,Y). Motivated again by the Mackey–Arens theorem, we now introduce a "constructive candidate".

Denote by $\mathfrak{S}_{\rm qc}(Y,X)$ or just $\mathfrak{S}_{\rm qc}$ the family of all compact quasi-convex subsets of $(Y,\sigma(Y,X))$, and by $\tau_{\rm qc}(X,Y)$ the $\mathfrak{S}_{\rm qc}$-topology on X. According to Proposition 3.4(a), $\tau_{\rm qc}(X,Y)$ is a locally quasi-convex topology.

Proposition 3.13. *Let (X,Y) be a group duality. Then*

$$\tau_{\rm g}(X,Y)\subset\tau_{\rm qc}(X,Y).$$

Proof. By Proposition 3.9(c), any locally quasi-convex topology τ on X is an $\mathfrak{S}$-topology, with $\mathfrak{S}\subset\mathfrak{S}_{\rm qc}$. ■

Proposition 3.14. *Let (X,Y) be a group duality. The following statements are equivalent:*

(a) *The topology $\tau_{\rm qc}(X,Y)$ is compatible for (X,Y) and in that case $\tau_{\rm g}(X,Y)=\tau_{\rm qc}(X,Y)$.*

(b) *The family $\mathfrak{S}_{\rm qc}$ is well-directed and any of its members is quasi-convex in $(X_{\rm d})^\wedge_\sigma$.*

Proof. (a)⇒(b). By Proposition 3.9 the family $\mathfrak{S}_{\rm qc}$ coincides with the family of all $\tau_{\rm qc}(X,Y)$-equicontinuous compact quasi-convex subsets of $(Y,\sigma(Y,X))$. This implies that $\mathfrak{S}_{\rm qc}$ is well-directed and according to Corollary 3.2 the second part of (b) also holds.

(b)⇒(a). Take any $\phi\in(X,\tau_{\rm qc}(X,Y))^\wedge$. We show that $\phi\in Y$. Since $\mathfrak{S}_{\rm qc}$ is well-directed, by the continuity of ϕ and Proposition 3.4(a), there exists $K\in\mathfrak{S}_{\rm qc}$ such that $\mathrm{Re}(\phi(x))\ge 0$ for all $x\in K^\triangleleft$. Consequently, ϕ belongs to the quasi-convex hull of K in $(X_{\rm d})^\wedge_\sigma$. According to (b) the set K itself is quasi-convex in $(X_{\rm d})^\wedge_\sigma$, thus we obtain $\phi\in K\subset Y$. Therefore $(X,\tau_{\rm qc}(X,Y))^\wedge\subset Y$. The reverse inclusion follows from the fact that $\sigma(X,Y)\subset\tau_{\rm qc}(X,Y)$. ■

4. The main theorems. For a given group duality (X,Y), we do not know in general if the topologies $\tau_{\rm g}(X,Y)$ and $\tau_{\rm qc}(X,Y)$ are compatible. We now study a related question. As we observed in Section 2, any barrelled locally convex space is a Mackey space. In Section 1 we have seen that a notion of barrelledness can be introduced for groups. Now a natural question is the following: Do barrelled groups behave the same as barrelled spaces with respect to the Mackey topology? Later on we shall see that the answer is positive.

We say that a topological abelian group (X,τ) is *pre-Mackey* if any compact and quasi-convex subset of $X^\wedge_\sigma$ is equicontinuous in τ; in this case the topology τ is also said to be a *pre-Mackey topology*. Recall that (X,τ) is g-barrelled if any compact subset of $X^\wedge_\sigma$ is equicontinuous. This implies that a g-barrelled group is a pre-Mackey group. We can now formulate the main result:

Theorem 4.1. *Let (X,Y) be a group duality. Suppose that τ_0 is a group topology (not necessarily locally quasi-convex) in X, compatible with (X,Y) and such that one of the following conditions is satisfied:*

(1) *(X,τ_0) is a pre-Mackey group.*
(2) *(X,τ_0) is a g-barrelled group.*
(3) *(X,τ_0) is a hereditarily Baire metrizable or a Čech-complete group.*
(4) *(X,τ_0) is a complete metrizable group.*

Then:

(a) *$\tau_{\rm qc}(X,Y)\subset\tau_0$.*
(b) *The topology $\tau_{\rm qc}(X,Y)$ is compatible with (X,Y).*
(c) *$\tau_{\rm g}(X,Y)=\tau_{\rm qc}(X,Y)$.*
(d) *A locally quasi-convex topology τ in X is compatible with (X,Y) if and only if $\sigma(X,Y)\subset\tau\subset\tau_{\rm qc}(X,Y)$.*

Proof. By Corollary 1.6 we see that (4)⇒(3)⇒(2)⇒(1). Evidently, (a)⇒(b). By Proposition 3.13 and the definition of $\tau_{\rm g}(X,Y)$, (b) implies (c). Statement (d) follows from (c) and the definition of $\tau_{\rm g}(X,Y)$. So, we only need to prove that (1) implies (a). This follows from Proposition 3.9(a). In fact, we have $(X,\tau_0)^\wedge=Y$, and consequently, any member of $\mathfrak{S}_{\rm qc}(Y,X)$ is a compact quasi-convex subset of $(X,\tau_0)^\wedge_\sigma$. Since (X,τ_0) is a pre-Mackey group, any member of $\mathfrak{S}_{\rm qc}(Y,X)$ is τ_0-equicontinuous. Hence, $\tau_{\rm qc}(X,Y)\subset\tau_{\rm e}(X_0,(X_0)^\wedge)$, where $X_0:=(X,\tau_0)$. By Proposition 3.9(a) we obtain $\tau_{\rm qc}(X,Y)\subset\tau_{\rm e}(X_0,(X_0)^\wedge)\subset\tau_0$, i.e. (a) holds. ■

We now reformulate the theorem in terms of the natural duality of a topological abelian group.

Theorem 4.2. *Let X be a topological abelian group. Suppose that one of the following conditions is satisfied:*

(1) *X is a pre-Mackey group.*
(2) *X is a g-barrelled group.*
(3) *X is a hereditarily Baire metrizable or a Čech-complete group.*
(4) *X is a complete metrizable group.*

Then:

(a) *The topology* $\tau_{qc}(X, X^\wedge)$ *is coarser than the original topology of* X.
(b) *The topology* $\tau_{qc}(X, X^\wedge)$ *is a compatible topology for* X.
(c) $\tau_g(X, X^\wedge) = \tau_{qc}(X, X^\wedge)$.
(d) $\tau_{qc}(X, X^\wedge)$ *is the finest locally quasi-convex topology compatible for* X.
(e) *If* X *is locally quasi-convex, then* $\tau_{qc}(X, X^\wedge)$ *is the original topology of* X.

Proof. The assertions (a), (b), (c), (d) are the same as those of Theorem 4.1. It is straightforward that (a) and (d) imply (e). ■

REMARK 10. Condition (1) is not necessary for $\tau_g(X, X^\wedge)$ to be a compatible topology. In fact, if $X = (\mathbb{R}, \sigma(\mathbb{R}, \mathbb{R}^\wedge))$, then $\tau_g(\mathbb{R}, \mathbb{R}^\wedge)$ is the usual topology of $\mathbb{R}$.

COROLLARY 4.3. *Let* X *be an abelian group, and let* τ_1 *and* τ_2 *be locally quasi-convex pre-Mackey topologies in* X *such that*

$$(X, \tau_1)^\wedge = (X, \tau_2)^\wedge.$$

Then $\tau_1 = \tau_2$. *Equivalently, there is at most one locally quasi-convex pre-Mackey topology compatible with a group duality.*

Proof. Put $X^\wedge := (X, \tau_1)^\wedge$. By Theorem 4.2 we have $\tau_1 = \tau_{qc}(X, X^\wedge) = \tau_2$. ■

The fact that a locally compact topology on an abelian group X is determined by the set of all continuous characters on X (see [9]) can be obtained as a consequence of the above corollary.

COROLLARY 4.4 ([9], [25]). *Let* τ_1 *and* τ_2 *be two locally compact group topologies on an abelian group* X *such that* $(X, \tau_1)^\wedge = (X, \tau_2)^\wedge$. *Then* $\tau_1 = \tau_2$.

Proof. As remarked before, locally compact groups are locally quasi-convex. On the other hand, according to Corollary 1.6, a locally compact abelian group is g-barrelled and consequently it is also a pre-Mackey group. Thus, the assertion follows from Corollary 4.3. ■

REMARK 11. For locally compact topologies more is known. Namely, if $\tau_1 \subset \tau_2$ and $\tau_1 \neq \tau_2$ are locally compact group topologies on an abelian group X, then there exist at least $2^{\aleph_1}$ τ_2-continuous characters that are τ_1-discontinuous (see [7], p. 118).

COROLLARY 4.5. *Let* X *be a locally quasi-convex pre-Mackey group. Suppose that there is a locally compact group topology compatible for* X. *Then* X *is locally compact.*

Proof. Any locally compact group topology is locally quasi-convex and pre-Mackey. Therefore the conclusion follows from Corollary 4.3. ■

REMARK 12. Corollary 4.5 implies that if X is the underlying additive group of an infinite-dimensional Banach space, there is no locally compact compatible group topology. A direct proof of this assertion is not evident.

REMARK 13. As shown in [5], the conclusion of Corollary 4.3 does not hold under the assumption that (X, τ_1) and (X, τ_2) are both Pontryagin reflexive groups. Thus, a Pontryagin reflexive group may not be a pre-Mackey group.

Let X be a topological abelian group. Denote by $\mathfrak{S}_{fc}$ the family of all $\sigma(X^\wedge, X)$-compact subsets of $X^\wedge$ and by $\tau_{fc}(X, X^\wedge)$ the $\mathfrak{S}_{fc}$-topology in X. Evidently, $\tau_{qc}(X, X^\wedge) \subset \tau_{fc}(X, X^\wedge)$. These two topologies coincide if X is g-barrelled. We can further formulate the following:

THEOREM 4.6. *Let* X *be a topological abelian group. Suppose that one of the following conditions is satisfied:*

(1) X *is a g-barrelled group.*
(2) X *is a metrizable hereditarily Baire or a Čech-complete group.*
(3) X *is a complete metrizable group.*

Then:

(a) *The topology* $\tau_{fc}(X, X^\wedge)$ *is coarser than the original topology of* X.
(b) *The topology* $\tau_{fc}(X, X^\wedge)$ *is a compatible topology for* X.
(c) $\tau_g(X, X^\wedge) = \tau_{qc}(X, X^\wedge) = \tau_{fc}(X, X^\wedge)$.
(d) $\tau_{fc}(X, X^\wedge)$ *is the finest locally quasi-convex topology compatible for* X.
(e) *If* X *is locally quasi-convex, then* $\tau_{fc}(X, X^\wedge)$ *is the original topology of* X.

Proof. This is a particular case of Theorem 4.2, since now $\tau_{qc}(X, X^\wedge) = \tau_{fc}(X, X^\wedge)$. ■

5. Connections with the usual Mackey topology of a topological vector space. The topologies $\tau_g(X, X^\wedge)$ and $\tau_{qc}(X, X^\wedge)$, introduced and studied in the previous section, are natural candidates to be the "Mackey topology" for an arbitrary topological abelian group X. We have defined them using only tools of the theory of topological groups. Now we shall prove that, rather unexpectedly, for the underlying group of a wide class of topological vector spaces, these two topologies coincide with the ordinary Mackey topology. We do not know if this happens in general.

Recall that for a topological vector space E, the *Mackey topology* $\tau(E, E^*)$ is the topology of uniform convergence on the absolutely convex compact subsets of E^*_σ. Denote by $\tau_{\mathrm{fc}}(E,E^*)$, as in the case of groups, the topology of uniform convergence on compact subsets of E^*_σ.

PROPOSITION 5.1. *Let E be a topological vector space. Then*

$$\tau(E,E^*)\subset\tau_{\mathrm{g}}(E,E^\wedge)\subset\tau_{\mathrm{qc}}(E,E^\wedge)\subset\tau_{\mathrm{fc}}(E,E^\wedge)=\tau_{\mathrm{fc}}(E,E^*).$$

Proof. By the Mackey–Arens theorem $(E,\tau(E,E^*))^*=E^*$, and consequently $(E,\tau(E,E^*))^\wedge=E^\wedge$. Since $\tau(E,E^*)$ is a locally convex topology, it is a locally quasi-convex group topology in E, so $\tau(E,E^*)\subset\tau_{\mathrm{g}}(E,E^\wedge)$. The inclusion $\tau_{\mathrm{g}}(E,E^\wedge)\subset\tau_{\mathrm{qc}}(E,E^\wedge)$ is valid for an arbitrary topological abelian group E (see Proposition 3.13). On the other hand, the inclusion $\tau_{\mathrm{qc}}(E,E^\wedge)\subset\tau_{\mathrm{fc}}(E,E^\wedge)$ is evident. Finally, $\tau_{\mathrm{fc}}(E,E^\wedge)\subset\tau_{\mathrm{fc}}(E,E^*)$ follows from the fact that any compact subset of $E^\wedge_\sigma$ has the form $p(B)$, where B is a compact subset of E^*_σ (see Proposition 1.10). In order to show $\tau_{\mathrm{fc}}(E,E^*)\subset\tau_{\mathrm{fc}}(E,E^\wedge)$, it is enough to prove that, for any fixed balanced compact set $B\subset E^*_\sigma$, ${}^\circ B$ is a neighborhood of zero in $\tau_{\mathrm{qc}}(E,E^\wedge)$. Clearly, $\frac14 B$ is also a balanced subset of E^*. By Proposition 1.11 we have ${}^\circ B=\left(p\left(\frac14B\right)\right)^{\triangleleft}$. Taking into account that $p\left(\frac14B\right)$ is a compact subset of $E^\wedge_\sigma$, we deduce that ${}^\circ B$ is a neighborhood of zero in $\tau_{\mathrm{fc}}(E,E^\wedge)$. ∎

PROPOSITION 5.2. *Let E be a topological vector space such that E^*_σ has the convex compactness property, i.e., the absolutely convex hull of any compact subset of E^*_σ is relatively compact in E^*_σ. Then*

$$\tau(E,E^*)=\tau_{\mathrm{g}}(E,E^\wedge)=\tau_{\mathrm{qc}}(E,E^\wedge).$$

Consequently, the Mackey topology $\tau(E,E^*)$ is the finest among all locally quasi-convex topologies compatible for E defined on the additive group of E.

Proof. The convex compactness property of E^*_σ obviously implies that $\tau(E,E^*)=\tau_{\mathrm{fc}}(E,E^*)$. An application of Proposition 5.1 gives the stated equalities. The last conclusion follows from the definition of $\tau_{\mathrm{g}}(E,E^\wedge)$ and from the Mackey–Arens theorem. ∎

REMARK 14. We shall see below that the convex compactness property holds for a wide class of spaces. Notice, however, that Proposition 5.2 provides examples of topological abelian groups E which are not pre-Mackey, but still $\tau_{\mathrm{g}}(E,E^\wedge)=\tau_{\mathrm{qc}}(E,E^\wedge)$ is the finest compatible locally quasi-convex topology. Consider, more precisely, an infinite-dimensional Banach space equipped with its weak topology.

Denote by $\beta(E^*,E)$ the strong topology in the dual E^* of a topological vector space E; that is, $\beta(E^*,E)$ is the topology of uniform convergence on all $\sigma(E,E^*)$-bounded subsets of E. As usual, $E^*_\beta:=(E^*,\beta(E^*,E))$. Recall that E is said to be *infrabarrelled* if the bounded subsets of E^*_β are equicontinuous. Evidently, any barrelled space is infrabarrelled. Also, if E is a metrizable topological vector space, then E is infrabarrelled. The following statement may be of independent interest:

PROPOSITION 5.3. *Let E be a topological vector space. Consider the assertions:*

(a) *E is barrelled.*

(b) *E is g-barrelled.*

(c) *The space E^*_σ has the convex compactness property.*

(d) *The absolutely convex hull of any null sequence of E^*_σ is relatively compact in E^*_σ.*

(e) *Any bounded subset of E^*_σ is bounded in E^*_β.*

Then (a)⇒(b)⇒(c)⇒(d)⇒(e). *If E is infrabarrelled, in particular, if E is metrizable, then* (e)⇒(a).

Proof. The implications (a)⇒(b) and (c)⇒(d) are evident. It is also clear that, if E is infrabarrelled, then (e) implies (a).

(b)⇒(c). Let $B\subset E^*_\sigma$ be a compact subset. Since E is g-barrelled, B is equicontinuous. Consequently, ${}^\circ B$ is a neighborhood of zero in E. According to the Alaoglu–Bourbaki theorem, $({}^\circ B)^\circ$ is compact in E^*_σ. Now, the absolutely convex hull of B is contained in $({}^\circ B)^\circ$.

(d)⇒(e). Suppose, by contradiction, that a bounded subset $B\subset E^*_\sigma$ is not bounded in E^*_β. Then we can find a $\sigma(E,E^*)$-bounded set $A\subset E$ and a sequence $(x^*_n)\subset B$ such that

$$\sup_{x\in A}|x^*_n(x)|>n^2,\qquad\forall n\in\mathbb{N}.$$

Since B is bounded in E^*_σ, we have $\lim\frac1n x^*_n(x)=0$ for all $x\in E$, i.e. $\left(\frac1n x^*_n\right)$ is a null sequence in E^*_σ. Let K be its closed absolutely convex hull. According to (d), K is compact in E^*_σ. Thus, K is an absolutely convex complete subset of E^*_σ, which by the Banach–Mackey theorem is bounded in E^*_β. In particular, the sequence $\left(\frac1n x^*_n\right)$ is bounded in E^*_β. But by the above condition, we have

$$\sup_{x\in A}\sup_{n\in\mathbb{N}}\left|\frac1n x^*_n(x)\right|=\infty.\ ∎$$

REMARK 15. For a general locally convex space E, assertion (c) may not imply (b). For instance, let E be an infinite-dimensional Banach space. Consider $E_\sigma=(E,\sigma(E,E^*))$. Evidently, $E^*_\sigma=(E_\sigma)^*_\sigma$. So, according to the implication (a)⇒(c), the space E_σ satisfies (c), but it is not g-barrelled.

REMARK 16. As J. Mendoza has pointed out, in general (b) does not imply (a) in Proposition 5.3. For instance, if E is a nonreflexive Banach space, then $(E^*,\tau(E^*,E))$ is a g-barrelled space which is not barrelled.

REMARK 17. The notion of g-barrelled locally convex space seems not to have been considered earlier in the literature. A related notion has been introduced in [27]: a locally convex space E is called *sequentially barrelled* if any null sequence in E^*_σ is equicontinuous. It is shown in [27] (Proposition 4.1) that Proposition 5.3(e) is valid for sequentially barrelled spaces. The original topology of a sequentially barrelled space may not coincide with the Mackey topology $\tau(E,E^*)$ (see [11], p. 250, where sequentially barrelled spaces are named c_0-barrelled). Thus, a sequentially barrelled space may not be g-barrelled.

Propositions 5.2 and 5.3 imply:

PROPOSITION 5.4. *Let E be a topological vector space. The Mackey topology $\tau(E,E^*)$ is the finest among all locally quasi-convex compatible group topologies for the additive group of E if E satisfies one of the following conditions*:

(a) *E is barrelled.*
(b) *E is g-barrelled.*
(c) *E is a Baire space.*
(d) *E is complete and metrizable.*

Finally, we show that the topology considered in [25] very seldom has the property stated in Proposition 5.4.

PROPOSITION 5.5. *Let E be a topological vector space, and let $\tau_{\rm var}(E)$ be the least upper bound of the family of all locally precompact compatible group topologies for the additive topological group of E. Then $\tau_{\rm var}(E)=\sigma(E,E^*)$.*

Proof. We first show that $\tau_{\rm var}(E)$ is finer than $\sigma(E,E^*)$. Fix $f\in E^*$ and denote by τ_f the preimage of the topology of $\mathbb{R}$ under f. It is sufficient to prove that τ_f is coarser than $\tau_{\rm var}(E)$. Obviously, τ_f is a precompact topology and

$$(E,\tau_f)^\wedge=\{p(tf):t\in\mathbb{R}\}.$$

Put $\nu_f=\sup(\tau_f,\sigma(E,E^\wedge))$. It is easy to check that ν_f is a compatible group topology for E. Hence ν_f and *a fortiori* τ_f is coarser than $\tau_{\rm var}(E)$.

In order to show that $\sigma(E,E^*)$ is finer than $\tau_{\rm var}(E)$, fix a locally precompact compatible group topology τ in E; we prove that $\tau\subset\sigma(E,E^*)$. We can assume that there is a locally compact abelian group Y and a group homomorphism $u:E\to Y$ such that τ is the preimage of the topology of Y under u. Further, we can assume that Y is a closed subgroup of a product $\mathbb{R}^n\times K\times D$, where n is a natural number, and K and D are respectively a compact and a discrete topological abelian group ([1], (1.9)). There are thus τ-continuous homomorphisms $u_1:E\to\mathbb{R}^n$, $u_2:E\to K$ and $u_3:E\to D$ such that $u(x)=(u_1(x),u_2(x),u_3(x))$, for all $x\in E$. We now prove that

u_i are continuous with respect to $\sigma(E,E^*)$, and therefore τ is coarser than $\sigma(E,E^*)$.

First, it is easy to see that we can find continuous linear forms $f_i:E\to\mathbb{R}$, $i=1,\ldots,n$, such that $u_1(x)=(f_1(x),\ldots,f_n(x))$, $x\in E$. Thus u_1 is $\sigma(E,E^*)$-continuous.

In the second place, τ-continuity of u_2 implies that it is continuous with respect to $\sigma(E,E^\wedge)$ and $\sigma(K,K^\wedge)$. But the latter is the original topology of K, and $\sigma(E,E^\wedge)$ is coarser than $\sigma(E,E^*)$. Therefore, u_2 is $\sigma(E,E^*)$-continuous.

Finally, $u_3:E\to D$ is τ-continuous, and therefore $G:=\ker u_3$ is a τ-open subgroup of E. Since τ is locally precompact, it is locally quasi-convex. By Proposition 5.1, $\tau\subset\tau_{\rm lc}(E,E^*)$. Consequently, G is an open subgroup of E. Since E is connected, $G=E$, and u_3 is the null homomorphism, which is obviously $\sigma(E,E^*)$-continuous. ■

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On a vector-valued local ergodic theorem in L_∞

by

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Abstract. Let $T = \{T(u) : u \in \mathbb{R}_d^+\}$ be a strongly continuous d-dimensional semigroup of linear contractions on $L_1((\Omega,\Sigma,\mu);X)$, where (Ω,Σ,μ) is a σ-finite measure space and X is a reflexive Banach space. Since $L_1((\Omega,\Sigma,\mu);X)^* = L_\infty((\Omega,\Sigma,\mu);X^*)$, the adjoint semigroup $T^* = \{T^*(u) : u \in \mathbb{R}_d^+\}$ becomes a weak*-continuous semigroup of linear contractions acting on $L_\infty((\Omega,\Sigma,\mu);X^*)$. In this paper the local ergodic theorem is studied for the adjoint semigroup T^*. Assuming that each $T(u)$, $u \in \mathbb{R}_d^+$, has a contraction majorant $P(u)$ defined on $L_1((\Omega,\Sigma,\mu);\mathbb{R})$, that is, $P(u)$ is a positive linear contraction on $L_1((\Omega,\Sigma,\mu);\mathbb{R})$ such that $\|T(u)f(\omega)\| \le P(u)\|f(\cdot)\|(\omega)$ almost everywhere on Ω for every $f \in L_1((\Omega,\Sigma,\mu);X)$, we prove that the local ergodic theorem holds for T^*.

1. Introduction. Define $\mathbb{P}_d = \{u = (u_1,\ldots,u_d) : u_i > 0,\ 1 \le i \le d\}$ and $\mathbb{R}_d^+ = \{u = (u_1,\ldots,u_d) : u_i \ge 0,\ 1 \le i \le d\}$, and denote by $\mathcal{I}_d$ the class of all bounded intervals in $\mathbb{P}_d$ and by λ_d the d-dimensional Lebesgue measure. Let X be a reflexive Banach space and (Ω,Σ,μ) be a σ-finite measure space. We consider a strongly continuous d-dimensional semigroup $T = \{T(u) : u \in \mathbb{P}_d\}$ of linear contractions on $L_1(\Omega;X) = L_1((\Omega,\Sigma,\mu);X)$, where $L_1((\Omega,\Sigma,\mu);X)$ is the usual Banach space of all X-valued strongly measurable functions on Ω for which the norm is given by

$$\|f\|_1 = \int_\Omega \|f(\omega)\|\,d\mu < \infty.$$

Since X is reflexive by hypothesis, it follows (cf. Chapter IV of [4]) that $L_1(\Omega;X)^* = L_\infty(\Omega;X^*)$, where $L_\infty(\Omega;X^*)$ is the Banach space of all X^*-valued strongly measurable functions on Ω for which the norm is given by

$$\|f\|_\infty = \operatorname{ess\,sup}\{\|f(\omega)\| : \omega \in \Omega\} < \infty.$$

Thus the adjoint semigroup $T^* = \{T^*(u) : u \in \mathbb{P}_d\}$ becomes a weak*-continuous d-dimensional semigroup of linear contractions acting on $L_\infty(\Omega;X^*)$.

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