Topology and its Applications 159 (2012) 2290–2309

Contents lists available at SciVerse ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

A survey on reflexivity of abelian topological groups ☆

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ARTICLE INFO

MSC:
22A05
22D45
43A40
46A20
54H11

Keywords:
Pontryagin duality theorem
Dual group
Reflexive group
Strongly reflexive group
Metrizable group
Čech-complete group
ω-Bounded group
P-group

ABSTRACT

The Pontryagin duality theorem for locally compact abelian groups (briefly, LCA groups) has been the starting point for many different routes of research in Mathematics. From its appearance there was a big interest to extend it in a context broader than LCA groups. Kaplan in the 40's proposed—and it still remains open—the problem of characterization of all abelian topological groups for which the canonical mapping into its bidual is a topological isomorphism, assuming that the dual and the bidual carry the compact-open topology. Such groups are called reflexive.

In this survey we deal with results on reflexivity of certain classes of groups, with special emphasis on the smaller class which better reflects the properties of LCA groups, namely that of strongly reflexive groups. A topological abelian group is said to be strongly reflexive if all its closed subgroups and its Hausdorff quotients as well as the closed subgroups and the Hausdorff quotients of its dual group are reflexive.

0. Introduction

One instance of the spectacular interplay between topology and algebra is Pontryagin–van Kampen duality theorem for locally compact abelian groups. Undoubtedly, it is one of the masterpieces in Mathematics. This explains why the abelian topological groups satisfying the Pontryagin–van Kampen duality, the so called *reflexive groups*, have received considerable attention starting from the late 40's of the past century.

Locally compact abelian groups (LCA groups) were initially studied by Pontryagin as the natural class of groups embracing Lie groups. In his remarkable book "Topological groups" (the first English edition from 1939) he already touches the main topics involved in what is commonly understood by "duality theory for abelian groups". Roughly speaking, the duality by him established consists on assigning to an LCA group another LCA group called the dual group. The good knowledge of categorical language nowadays available permits us to describe Pontryagin's approach as follows. Take first the circle group of the complex plane $\mathbb{T}$, with its natural topology, as dualizing object. Then assign to a group G in the class LCA the group $G^\wedge := CHom(G, \mathbb{T})$ of continuous homomorphisms, and endow it with the compact open topology. This is precisely the dual group of G. After observing that the dual of a compact group is discrete and conversely, he proved that the dual of a group G in LCA is again in LCA.

The celebrated Theorem of Pontryagin and van Kampen establishes that the natural evaluation mapping from an LCA group into its bidual is a topological isomorphism (see [93, Theorem 32] or Theorem 1.1). The contribution of van Kampen

☆ Partially supported by MICINN of Spain MTM2009-14409-C02-01 and MTM2009-14409-C02-02.
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0166-8641/$ – see front matter
http://dx.doi.org/10.1016/j.topol.2012.04.012

was to remove the "separability" constraint in Pontryagin's first claim. A topological abelian group G is called *reflexive* if the canonical mapping α_G from G into its bidual $G^{\wedge\wedge}$ is a topological isomorphism. Since the "dual groups" $CHom(G,\mathbb{T})$ are abelian and Hausdorff, reflexivity only makes sense within the class of abelian Hausdorff groups.

The first examples of reflexive groups out of the class of LCA groups were found by Kaplan in a very deep paper (see [79], 1948) where he established the duality between arbitrary products of abelian topological groups and direct sums of their duals. To this end, he first defined the so called asterisk topology for direct sums of topological groups, which is a group topology made "ad casum", in order to get the mentioned duality. With this instrument at hand he proved that arbitrary products of reflexive groups (in particular of LCA groups) are reflexive, which stimulated further research in order to find new classes of reflexive groups. As pointed out in [79]: *an as yet unsolved problem is to characterize the class of topological abelian groups for which the Pontryagin duality holds, that is those groups which are the character groups of their character groups*. Several authors have claimed that they had solved this problem: however their proofs either have gaps, or the statements are too complicated to deserve the name of "intrinsic characterization of reflexive groups" [82,83,105,106,70].

To date many reflexive groups have been found within different classes of topological groups. For instance, in the class of locally convex vector spaces, in the class of free topological groups, in the class of metrizable groups and very recently in the class of precompact groups. (See e.g. [1,2,5,12,17,25,27,31,64,66,67,70,75,76,79,80,90,92,100].)

The simple observation that closed subgroups and Hausdorff quotients of LCA groups are again LCA, and therefore reflexive, leads to a more strict point of view for extending the Pontryagin–van Kampen duality theory: just to consider classes of reflexive groups in which the closed subgroups and the Hausdorff quotients are again reflexive. In a remarkable paper by Brown, Higgins and Morris (see [18]) "strong duality" is considered for the first time. A precise definition, after eliminating several non-independent requirements in [18], can be stated as follows. A topological abelian group G is *strongly reflexive* if the closed subgroups and the Hausdorff quotients of G and of its dual group $G^\wedge$ are reflexive (see [12]).

Varopoulos in [104] already studied the duality properties of subgroups and quotients of a class of reflexive non-locally compact groups. Noble [91] proved that closed subgroups of countable products of LCA groups are reflexive and Leptin [85] showed that this cannot be extended to arbitrary products.

Another sort of reflexivity has been originated by recourse to convergence groups. For a topological abelian group G, define the convergence dual as $CHom(G,\mathbb{T})$ endowed with the continuous convergence structure (instead of the compact-open topology). In general this is no longer a topological group: however, if G is locally compact the convergence dual is exactly the same as the ordinary dual. The duality thus originated by an excursion to convergence groups, may be considered as an extension of Pontryagin duality (see [16,19,20,22,26,30]).

Some reflexivity theories for non-abelian groups have been also developed (see [35,65,71,72]) but we will not treat on them here.

In this survey we bring together the main results beyond reflexivity known to hold for distinct classes of abelian topological groups. Some of them are very recent and unexpected, for instance those referred to precompact groups. We do not pretend to be exhaustive: a difficult task in a growing field. We have tried to give the flavor of the topic and a good number of references.

A survey on duality theory of abelian topological groups with sections on reflexive groups and strongly reflexive groups can be found in [103].

1. Preliminaries

All groups considered are abelian, therefore we usually omit this word in the sequel. The symbol $\mathbb{T}$ denotes the multiplicative group of complex numbers with modulus 1, with its natural topology. The set $\mathbb{T}_+ := \{x \in \mathbb{T}\colon \operatorname{Re} x \geqslant 0\}$ is a particular neighborhood of $1 \in \mathbb{T}$ which plays a pivot role in duality. For a topological group G, $G^\wedge$ denotes the group of all continuous homomorphisms from G into $\mathbb{T}$, also called continuous characters. If $G^\wedge$ is endowed with the compact-open topology, it is a Hausdorff topological group which is defined to be the *dual group* of G. We shall use the symbol τ_{co} to denote the compact-open topology on $G^\wedge$ when a distinction is necessary. Frequently, $G^\wedge$ already denotes the dual with the corresponding compact-open topology. If G has sufficiently many continuous characters (that is, $G^\wedge$ separates the points of G) then G is said to be *maximally almost periodic* or *MAP*.

The bidual group $G^{\wedge\wedge}$ is $(G^\wedge)^\wedge$ and the canonical evaluation mapping $\alpha_G : G \to G^{\wedge\wedge}$ is defined by $\alpha_G(g)(\kappa) := \kappa(g)$, for all $g \in G$ and $\kappa \in G^\wedge$.

Theorem 1.1 *(Pontryagin–van Kampen, 1935). If G denotes a locally compact abelian group, the canonical mapping $\alpha_G : G \to G^{\wedge\wedge}$ is a topological isomorphism.*

Non-reflexive abelian groups occur frequently. A natural easy example is the group of rational numbers $\mathbb{Q}$ endowed with the Euclidean topology (see Lemma 5.7 and Theorem 5.5).

A subgroup H of a topological group G is said to be:

- *dually closed* if, for every element x of $G \setminus H$, there is a continuous character φ in $G^\wedge$ such that $\varphi(H) = 1$ and $\varphi(x) \neq 1$.
- *dually embedded* if every continuous character defined on H can be extended to a continuous character on G.
- *h-embedded* if every character defined on H can be extended to a continuous character on G.

Dually closed and dually embedded subgroups already appear in Kaplan's writing, but Noble was the first to call them in this way [91]. On the other hand Tkachenko introduced the h-embedded subgroups in [102].

It is easy to prove that a closed subgroup H of a topological group G is dually closed if and only if the quotient group G/H has sufficiently many continuous characters to separate points.

The *annihilator of* a subgroup $H \subset G$ is defined as the subgroup $H^{\perp} := \{\varphi \in G^{\wedge}\colon\ \varphi(H) = \{1\}\}$. If L is a subgroup of $G^{\wedge}$, the *inverse annihilator* is defined by $L^{\perp} := \{g \in G\colon\ \varphi(g) = 1,\ \forall \varphi \in L\}$. Although the inverse annihilator is frequently denoted by ${}^{\perp}L$, we shall simply warn the reader if we are taking a direct annihilator of the subgroup L in $G^{\wedge\wedge}$.

Annihilators are the specializations for subgroups of the more general notion of polars of subsets. Namely, for $A \subset G$ and $B \subset G^{\wedge}$, the polar of A is $A^{\rhd} := \{\varphi \in G^{\wedge}\colon\ \varphi(A) \subset \mathbb{T}_+\}$ and the inverse polar of B is $B^{\lhd} := \{g \in G\colon\ \varphi(g) \in \mathbb{T}_+,\ \forall \varphi \in B\}$. For a topological abelian group G, it is not difficult to prove that a set $M \subset G^{\wedge}$ is *equicontinuous* if there exists a neighborhood U of the neutral element in G such that $M \subset U^{\rhd}$.

Let $f : G \to E$ be a continuous homomorphism of topological groups. The *dual mapping* $f^{\wedge} : E^{\wedge} \to G^{\wedge}$ defined by $(f^{\wedge}(\chi))(g) := (\chi \circ f)(g)$ is a continuous homomorphism. If f is onto, then $f^{\wedge}$ is injective. For a closed subgroup H of a topological group G, denote by $p : G \to G/H$ the canonical projection and by $i : H \to G$ the inclusion. The dual mappings $p^{\wedge}$ and $i^{\wedge}$ give rise to the natural continuous homomorphisms $\varphi : (G/H)^{\wedge} \to H^{\perp}$ and $\psi : G^{\wedge}/H^{\perp} \to H^{\wedge}$. Observe that if H is dually embedded, ψ is onto. In general φ and ψ are not **topological** isomorphisms: they are under certain conditions that we will study later, and in such case they produce a natural connection between closed subgroups and Hausdorff quotients of the corresponding dual group.

2. Locally quasi-convex groups

Reflexive groups lie in a wider class of groups, the so called *locally quasi-convex groups*. Vilenkin had the seminal idea to define a sort of convexity for abelian topological groups. Inspired by the Hahn–Banach theorem for locally convex spaces, the following definitions are given in [107]:

Definition 2.1. A subset A of a topological group G is called *quasi-convex* if for every $g \in G \setminus A$, there is some $\chi \in A^{\rhd}$ such that $\operatorname{Re} \chi(g) < 0$.

It is easy to prove that for any subset A of a topological group G, $A^{\rhd\lhd}$ is a quasi-convex set. It will be called the *quasi-convex hull* of A since it is the smallest quasi-convex set that contains A. Obviously, A is quasi-convex iff $A^{\rhd\lhd} = A$.

If A is a subgroup of G, A is quasi-convex if and only if A is dually closed.

Remark 2.2. The definition of a quasi-convex subset A of G relies on the topology of G, since the characters in $A^{\rhd}$ are required to be continuous.

Definition 2.3. A Hausdorff topological group G is *locally quasi-convex* if it has a basis of zero neighborhoods whose elements are quasi-convex subsets.

The Hausdorff assumption in the definition of locally quasi-convex groups makes possible the claim that they are MAP groups. Examples of locally quasi-convex groups are provided by any dual group, say $G^{\wedge}$. In fact, it is easy to prove that the sets $K^{\rhd}$ where $K \subset G$ is compact, constitute a quasi-convex zero-neighborhood basis for the compact-open topology in $G^{\wedge}$. Thus, any reflexive group is locally quasi-convex, as it is the dual of its character group.

It is straightforward to prove that any subgroup of a locally quasi-convex group is locally quasi-convex; a Hausdorff quotient of a locally quasi-convex group may not be locally quasi-convex [5, 12.8]. However, in the class of locally compact groups, and in the more general class of nuclear groups, every Hausdorff quotient is locally quasi-convex, see [12, 7.5]. Also quotients of locally quasi-convex groups by compact subgroups are locally quasi-convex [9].

An easy example of a non-reflexive locally quasi-convex topological group is the group of rational numbers $\mathbb{Q}$ with the topology induced by $\mathbb{R}$. As a subgroup of the locally quasi-convex group $\mathbb{R}$, it also has this property and it is non-reflexive as mentioned above.

Since topological vector spaces are special instances of abelian topological groups, it is natural to compare quasi-convexity and convexity for subsets of a topological vector space as well as the corresponding local properties. There is a subtle difference between the convex subsets and the quasi-convex subsets of a topological vector space E. To begin with, convexity is a merely algebraic property, while quasi-convexity involves the topology of E. It is amazing that there are finite or countably infinite quasi-convex subsets [5,45–47], [84, Chapter 7]. Nevertheless, if $A \subset E$ is a balanced subset and $co(A)$ denotes its convex hull, then $co(A) = A^{\rhd\lhd}$ [19, 6.3.1]. On the other hand, a Hausdorff topological vector space E is locally convex if and only if E is locally quasi-convex as an additive group [12, 2.4]. Thus, local quasi-convexity is an extension for groups of the notion of local convexity in vector spaces.

Reflexive locally convex spaces constitute a well-established topic in Functional Analysis. Smith in 1952 was the first to relate reflexivity in the sense of Pontryagin for a locally convex space with the by now traditional concept of reflexivity in the Functional Analysis sense. As a result she obtained in [100] a new class of Pontryagin reflexive non-LCA groups, from a point of view totally different of that of Kaplan. She was inspired by a paper of Arens of 1947, where the reflexivity of topological vector spaces is treated for the first time [3]. We briefly describe her approach in the next paragraph.

For a topological vector space E, denote by E^* the vector space of all continuous linear forms on E. Arens introduced the term "reflexive topology" to denote a topology t on E^* such that the continuous linear forms on (E^*, t) were precisely the "evaluations" at the elements of E [3]. The current notion of reflexive space is much stronger at present. By the dual of E it is commonly understood E^* endowed with the topology of uniform convergence on the family $\mathcal{B}$ of all the bounded subsets of E. If E_b^* denotes the dual so topologized, then E is said to be *reflexive* if the canonical mapping from E into $(E_b^*)_b^*$ is a topological isomorphism. After proving that $E^\wedge$ and E^* are algebraically isomorphic as groups, Smith points out that, $(E_b^*)_b^* \approx E$ implies $(E_c^\wedge)_c^\wedge \approx E$, although there is no obvious reason why this is true (here c denotes the compact open topology and $\approx$ topological isomorphism as spaces in the first case and as groups in the second one). Therefore, reflexive locally convex spaces are reflexive as topological groups.

In [100] it is also proved that all Banach spaces are reflexive as topological groups, thus reflexivity in Pontryagin sense is a property strictly weaker than reflexivity in the sense of Functional Analysis. The result is valid for complete metrizable locally convex spaces as well. With the tools of locally quasi-convex groups—which were not available to M. Smith—the proof is much easier, and the use of a norm can be avoided. In fact, in [12, (15.7)] it is proved that even a complete metrizable locally convex **vector group** is reflexive. Nevertheless, the structure of vector space is essential here, and the result does not have a counterpart for topological groups. As proved in [5, (11.15)], the group $G := L^p_{\mathbb{Z}}([0,1])$ of the almost everywhere integer-valued functions, with $p > 1$ and topology induced by the classical of $L^p([0,1])$, is a locally quasi-convex complete metrizable group which is not reflexive. Later on we will see that for nuclear groups the claim holds even in a stronger sense: that is, a nuclear complete metrizable group is strongly reflexive.

Along the past 20 years there has been intensive research in order to give counterparts for abelian locally quasi-convex groups of results known to hold for locally convex spaces. For instance the theorems of Grothendieck about completeness [21], of Mackey–Arens [34], of Dunford–Pettis [88], of Eberlein–Smulyan [24], and some others.

The weak topology $w(E, E^*)$ on a topological vector space E has a parallel for topological groups, as we express below after collecting a few properties of precompact topological groups. (Recall that an abelian topological group G is precompact if for every neighborhood of zero V there exists a finite subset $F \subset G$ such that $G = F + V$.)

Let $Hom(G, \mathbb{T})$ be the group of all characters on an abstract abelian group G, and let H be a subgroup of $Hom(G, \mathbb{T})$. The weak topology induced by H on G is a precompact group topology which is Hausdorff whenever H separates the points of G, and will be denoted by $w(G, H)$. As proved in [38] (see also [34]), $(G, w(G, H))^\wedge = H$ for separating H. On the other hand, as a consequence of the Peter–Weyl theorem, any precompact Hausdorff group G carries the weak topology corresponding to its character group. Thus, precompact Hausdorff groups are simply subgroups of compact Hausdorff groups.

For a topological group G, $w(G, G^\wedge)$ is called the *Bohr topology of* G. It is weaker than the original topology of G, and the notation $G^+ := (G, w(G, G^\wedge))$ is quite extended in the Literature. As said above, G^+ is precompact and $(G^+)^\wedge = G^\wedge$. In particular, since the additive group of real numbers $\mathbb{R}$ is not precompact, $\mathbb{R}$ and $\mathbb{R}^+$ are not topologically isomorphic groups. Taking into account that $w(\mathbb{R}, \mathbb{R}^*)$ as a vector space topology coincides with the usual topology of $\mathbb{R}$, we obtain that $w(\mathbb{R}, \mathbb{R}^\wedge) \neq w(\mathbb{R}, \mathbb{R}^*)$ and the same happens for any topological vector space E.

3. The canonical mapping α_G

The canonical mapping $\alpha_G : G \to G^{\wedge\wedge}$ is the backbone for the reflexivity of a topological group G. It is the mapping associated to the evaluation $e : G^\wedge \times G \to \mathbb{T}$, defined by $e(\phi, x) = \phi(x)$ ($\phi \in G^\wedge$, $x \in G$), in the following sense: $\alpha_G(x)(\phi) = e(\phi, x)$. If $G^\wedge$ carries the compact-open topology and $G^\wedge \times G$ the product topology, it is well known that the continuity of e implies that of α_G, but the converse does not hold. Observe also that for any locally compact abelian group e is continuous. There is an amazing result which allows us to distinguish the class of LCA groups in the framework of reflexive groups. Namely, if G is a reflexive group and e is continuous, then G must be locally compact [87]. However one could go a step further to unveal this property: reflexivity is not needed in its full strength. We will introduce below the quasi-convex compactness property and come back to the question.

We first study when is α_G 1–1, onto, continuous or open without imposing any assumption on G.

Proposition 3.1. *Let G be a topological group. The canonical mapping $\alpha_G : G \to G^{\wedge\wedge}$ is a homomorphism such that*:

(1) *α_G is injective iff G is MAP.*
(2) *α_G is continuous if and only if the compact subsets of $G^\wedge$ are equicontinuous.*
(3) *α_G is k-continuous* (*i.e., the restriction of α_G to any compact subset $K \subset G$ is continuous*); *in particular, α_G is sequentially continuous.*
(4) *If the group G is a k-space, then α_G is continuous.*
(5) *If G is locally quasi-convex, α_G is relatively open and one-to-one.*
(6) *$(G^\wedge, w(G^\wedge, G))^\wedge = \alpha_G(G)$ algebraically.*
(7) *If the compact subsets of G are finite, then α_G is onto.*

Proof. The assertion (1) is straightforward, (2) is [5, Proposition 5.10].

Item (3) derives from well-known topological results (see [81,91]), and (4) is a consequence of (3).

In order to prove (5), observe that for a quasi-convex zero-neighborhood V, $\alpha_G(V) = V^{\triangleright\triangleright} \cap \alpha_G(G)$, where $V^{\triangleright}$ is compact in the compact-open topology of $G^\wedge$.

(6) is a consequence of the fact that $\alpha_G(G)$ separates points of $G^\wedge$.

Under the assumption (7) the compact-open topology on $G^\wedge$ coincides with the pointwise convergence topology $w(G^\wedge, G)$, so (6) applies. □

Since all the groups we are dealing with are abelian, we call a topological group *complete* if it is complete for its unique uniformity as a topological group.

The next fact follows from the conjunction of the notions of k-space and the compact open topology:

Fact A. *If a topological group G is a k-space, then its dual group $G^\wedge$ is complete.*

A topological group G is said to have the quasi-convex compactness property (briefly, **qcp**) if for every compact $K \subset G$ its quasi-convex hull $K^{\triangleright\triangleleft}$ is again compact. In the framework of locally quasi-convex groups, this property is related to completeness and with the mapping α_G as follows:

Proposition 3.2. *Let G be a locally quasi-convex group. The following assertions hold*:

(1) *If G is complete, then G has the qcp. The converse also holds if G is moreover metrizable.*
(2) *If α_G is onto, G has the qcp. The converse does not hold, even for metrizable groups.*
(3) *If $e : G^\wedge \times G \to \mathbb{T}$ is continuous and G has the qcp, then G is locally compact.*
(4) *If α_G is continuous, then $G^\wedge$ has the qcp.*
(5) *If G is reflexive, both G and $G^\wedge$ have the qcp.*

Proof. The proofs are not hard. They can be seen in [19, Chapter 6] (where the qcp property is formally introduced for the first time), the second part of (2) in [70] and in [23]. The converse of (3) holds without the assumption of the qcp. A similar assertion to (3) (without qcp) was known to hold replacing the first factor $G^\wedge$ by the set $C(G, \mathbb{T})$ of continuous functions from G to $\mathbb{T}$. □

We give now some examples which can be quoted later on for distinct properties.

Example 3.3. Let G be the group of rational numbers with the usual Euclidean topology. Since G is metrizable non-complete, α_G is continuous, but G does not have the qcp by (1) (for a direct argument, use the fact that the compact set $K = \{0\} \cup \{2^{-n}\colon n \in \mathbb{N}\}$ in $\mathbb{Q}$ has quasi-convex hull $[-1, 1]$ in $\mathbb{R}$ [44, Example 4.5], hence the quasi-convex hull $[-1, 1] \cap \mathbb{Q}$ of K in $\mathbb{Q}$ is not compact). The evaluation mapping $e : G^\wedge \times G \to \mathbb{T}$ is also continuous, because $G^\wedge$ is precisely $\mathbb{R}$ endowed with the usual topology, and the product $\mathbb{R} \times \mathbb{Q}$ is a k-space by Whitehead's theorem. This example proves that $G^\wedge$ cannot be replaced by G in Proposition 3.2(4).

Example 3.4. Let $G := \mathbb{R}^{(\mathbb{N})} \times \mathbb{R}^{\mathbb{N}}$, where $\mathbb{R}^{\mathbb{N}}$ carries the ordinary Tychonoff topology, and $\mathbb{R}^{(\mathbb{N})}$ is the countable direct sum of real lines with the box topology. Then G is a reflexive self-dual topological group: in particular, it is locally quasi-convex and α_G is continuous. It is not a k-space (e.g. [12]), and consequently non-metrizable. As a product of the two complete groups, $\mathbb{R}^{\mathbb{N}}$ and its dual $\mathbb{R}^{(\mathbb{N})}$, G is complete.

Example 3.5. Let $G := L^2_{\mathbb{Z}}[0, 1]$ be the group of the almost everywhere integer-valued functions, with the topology induced by the classical norm of $L^2([0, 1])$. This example appears in [5], where the dual is calculated obtaining that $G^\wedge = L^2([0, 1])$. Therefore $G \leqslant L^2([0, 1])$ is a closed subgroup which has the same dual as the whole group. As said above, it is locally quasi-convex complete metrizable and non-reflexive. Further it has the qcp by Proposition 3.2(1).

4. Strong reflexivity and related notions

The LCA groups are the best behaved from the point of view of reflexivity as closed subgroups and Hausdorff quotients are still LCA groups, so reflexive. Furthermore, there is a formidable connection between closed subgroups of an LCA group G (resp. of its dual $G^\wedge$) and Hausdorff quotients of $G^\wedge$ (resp. of G). The next proposition describes more precisely these properties. We shall see in the sequel which of them is shared by other classes of groups.

Proposition 4.1. *([12, 17.1]) For a topological abelian group G, the following claims—which may or may not hold for G—can be related as we indicate below*:

1) *Closed subgroups and Hausdorff quotients of G and of $G^\wedge$ are reflexive.*

2) *All the closed subgroups of G and of $G^\wedge$ are dually closed and dually embedded.*
3) *For every pair H and L of closed subgroups of G and of $G^\wedge$ respectively, the natural homomorphisms*

$$\varphi:(G/H)^\wedge \to H^\perp, \qquad \psi: G^\wedge/H^\perp \to H^\wedge; \qquad \varphi':\left(G^\wedge/L\right)^\wedge \to L^\perp, \qquad \psi': G/L^\perp \to L^\wedge$$

are topological isomorphisms.

Then, 1) *implies* 2) *and* 3).

Definition 4.2. An abelian topological group G is called *strongly reflexive* (*s.r.*) if every closed subgroup and every Hausdorff quotient of G and of $G^\wedge$ is reflexive.

Countable products and sums of real lines and circles were the first examples of non-locally compact strongly reflexive groups [18]. Banaszczyk extended this result proving that all countable products and sums of LCA groups are strongly reflexive [11], and observed that these examples were included in a larger class of groups, which he defined and studied in [12], calling them nuclear groups. Although we will deal with the class of nuclear groups in Section 6, we anticipate that it contains the locally convex nuclear vector spaces and the locally compact abelian groups, and it is closed under forming products, subgroups and Hausdorff quotients.

Strong reflexivity was obtained in [12] for complete metrizable nuclear groups, in [5, 20.40] for Čech-complete nuclear groups and in [8] for k_ω nuclear groups. As a matter of fact, we do not know any example of strongly reflexive group outside these classes. Außenhofer constructed in 2007 [7] a non reflexive quotient of the uncountable product $\mathbb{Z}^{\mathbb{R}}$. With this result she answered in the negative the question posed by Banaszczyk in 1990 if uncountable products of real lines are strongly reflexive.

Let us bisect strong reflexivity in the two weaker properties, introduced next:

Definition 4.3. A topological group G will be called:

(i) *s-reflexive* if all closed subgroups of G are reflexive;
(ii) *q-reflexive* if all Hausdorff quotients of G are reflexive.

They are notions stronger than reflexivity as the following example shows:

Example 4.4. The space $L^2[0,1]$ is a reflexive group, but fails to be either s-reflexive or q-reflexive. Indeed, by a theorem of Banaszczyk (see [12, (5.3)] or [13]), every infinite dimensional Banach space has a quotient which does not have non-null characters, witnessing that $L^2[0,1]$ is not a q-reflexive group. On the other hand, $L^2[0,1]$ is neither an s-reflexive group since its closed subgroup $L^2_{\mathbb{Z}}[0,1]$ is not reflexive (Example 3.5).

The following open question arises:

Question 4.5. Let G be a topological group.

(a) If G is s-reflexive and q-reflexive, is it strongly reflexive?
(b) If G is s-reflexive and $G^\wedge$ is s-reflexive, is G strongly reflexive?
(c) If G is q-reflexive and $G^\wedge$ is q-reflexive, is G strongly reflexive?

We shall see in the sequel, that if G is q-reflexive and $G^\wedge$ is s-reflexive, then G need not be strongly reflexive (Proposition 9.1).

Closed subgroups and Hausdorff quotients of strongly reflexive group are strongly reflexive (see [12, (17.1)]), however even finite products of strongly reflexive groups need not be strongly reflexive (the self-dual group $G := \mathbb{R}^{(\mathbb{N})} \times \mathbb{R}^{\mathbb{N}}$ is not strongly reflexive as proved in [12, (17.7)]). In the sequel we study general properties of closed subgroups and Hausdorff quotients and what is missing in some cases, in order that a reflexive group be strongly reflexive.

Proposition 4.6. *Let G be a reflexive group, H a closed subgroup of G and L a closed subgroup of $G^\wedge$. Then the following facts hold*:

(1) *The mapping α_H (resp., α_L) is relatively open and injective.*
(2) *The evaluation mapping $\alpha_{G/H}$ is continuous.*
(3) *H is dually closed iff $\alpha_G(H) = H^{\perp\perp}$.*
(4) *H is dually closed if and only if $\alpha_{G/H}$ is injective.*
(5) *If H is dually closed and dually embedded, α_H is open and bijective.*
(6) *$\alpha_{G/H}$ surjective implies $H^\perp$ is dually embedded.*

(7) *If H is dually closed and $\alpha_{G^\wedge/H^\perp}$ surjective, then H is dually embedded.*
(8) *If L is dually closed, there exists a closed subgroup N of G such that $N^\perp = L$.*

Proof. Item (1) follows from the fact that subgroups of locally quasi-convex groups are locally quasi-convex and Proposition 3.1 (1) and (5). The proof of (2) is straightforward.

(3) Assume that $\alpha_G(H) = H^{\perp\perp}$. To check that H is dually closed, pick an $x \in G \setminus H$. Then $\alpha_G(x) \notin \alpha_G(H)$ by the injectivity of α_G. So $\alpha_G(x) \notin H^{\perp\perp}$. Hence there exists a $\chi \in H^\perp$ with $\chi(x) \neq 1$.

Now assume that H is dually closed and pick $y \in H^{\perp\perp}$. Since α_G is surjective, it has the form $y = \alpha_G(x)$, for some $x \in G$. As $y \in H^{\perp\perp}$, $1 = \chi(x) = \alpha_G(x)(\chi) = y(\chi)$ for every $\chi \in H^\perp$.

(4) follows from the definition of dually closed subgroup.

According to (1), in order to prove (5), we have to only check that α_H is surjective. To this end pick $\gamma \in H^{\wedge\wedge}$ and consider the following commutative diagram

$$\begin{tikzcd} H \arrow[r, "\iota"] \arrow[d, "\alpha_H"] & G \arrow[d, "\alpha_G"] \\ H^{\wedge\wedge} \arrow[r, "\iota^{\wedge\wedge}"] & G^{\wedge\wedge} \end{tikzcd} \tag{1}$$

Since G is reflexive, $\gamma \circ \iota^\wedge = \alpha_G(x)$, for some $x \in G$. If $x \in H$, we are done. Assume that $x \notin H$. Then there exists $\chi \in G^\wedge$ with $\chi(H) = \{1\}$ and $\chi(x) \neq 1$. Hence $\iota^\wedge(\chi) = 1$ but $\gamma \circ \iota^\wedge(\chi) = \alpha_G(x)(\chi) = \chi(x) = 1$, which is a contradiction. Item (5) is also proved in [91].

(6) and (7) are in [32, 1.4] and (8) in [12, 14.2]. $\square$

The open subgroups and the compact subgroups of a topological group G characterize reflexivity (or strong reflexivity) of the original group in the following way:

Proposition 4.7. *Let G be a topological group $H \subset G$ an open subgroup, and let $K \subset G$ be a compact subgroup. Then the following assertions hold*:

(1) *G is reflexive* (*strongly reflexive*) *iff H is reflexive* (*strongly reflexive*).
(2) *If G has sufficiently many continuous characters, G is reflexive* (*strongly reflexive*) *iff G/K is reflexive* (*strongly reflexive*).
(3) *The statements* (1) *and* (2) *also hold if* "*reflexive*" *is replaced by s-reflexive or by q-reflexive.*

Proof. Item (1) is [14, 2.3 and 3.3] and the proof of (2) follows from [14, 2.6, 3.3, 3.4] and [22, 1.4]. The definitions of s-reflexive and q-reflexive are new. The proof of (3) will appear elsewhere. $\square$

Remark 4.8. Hofmann and Morris in [78] define another strengthening of the concept of reflexivity as follows: a reflexive topological group G has *sufficient duality* if every closed subgroup H of G is dually closed and the quotient $G^\wedge/H^\perp$ is reflexive. Clearly, a strongly reflexive group has sufficient duality, and in the next proposition we analyze how much of the converse holds.

Proposition 4.9. *If G is a topological group with sufficient duality, all its closed subgroups are dually closed, dually embedded, and have reflexive dual.*

Proof. Let H be a closed subgroup of G. By (7) of Proposition 4.6 H is dually embedded, and by [20, (14.8)] there is an isomorphism between $G^\wedge/H^\perp$ and $H^\wedge$. Thus $H^\wedge$ is reflexive. $\square$

Proposition 4.10. *If the group G is a k-space with sufficient duality then G is an s-reflexive group.*

Proof. As in the previous proposition, any closed subgroup $H \leqslant G$ is dually closed and dually embedded. The results of Propositions 4.6(5) and 3.1(4) apply to give that α_H is a topological isomorphism. $\square$

5. Metrizable and almost metrizable groups

Metrizable groups are a distinguished class of groups from the point of view of Pontryagin duality theory. As pointed out in Proposition 3.1(4), if G is a metrizable group α_G is continuous. If moreover G is locally quasi-convex, then α_G is injective and relatively open. Thus only surjectivity must be worked out in order to have reflexivity in the class of all metrizable locally quasi-convex groups. The following assertion obtained independently in [27] and [5] is a fundamental result for its many consequences:

Fact B. *If G is a metrizable group, $G^\wedge$ is a k-space. Therefore, $G^{\wedge\wedge}$ is complete.*

Thus, a reflexive metrizable group must be complete. Example 3.5 shows that completeness is not a sufficient condition for the surjectivity of α_G.

Recall that a topological space X is said to be:

(a) *hemicompact* if X has a sequence $\{K_n\}_{n=1}^{\infty}$ of compact subsets such that every compact subset of the space lies inside some K_n.

(b) *almost metrizable* if every $x \in X$ is contained in a compact subset having a countable neighborhood basis in X.

(c) k_ω if it is a hemicompact k-space.

(d) locally k_ω if every point of X has an open neighborhood of zero which is a k_ω space.

A topological group G is almost metrizable if and only if it has a compact subgroup K such that G/K is metrizable [99]. Čech-complete groups are instances of almost metrizable groups. More precisely, a topological group G is Čech-complete if and only if G is almost metrizable and **complete** [5, (2.21)]. Locally k_ω spaces and locally k_ω groups have been recently defined in [69]. As proved there, a topological abelian group is locally k_ω if it has an open subgroup which is a k_ω group [69, Proposition 5.3].

If G is metrizable, then $G^\wedge$ is hemicompact, and this in turn implies that $G^{\wedge\wedge}$ is metrizable. Thus, the square of the duality functor applied to the subcategory of metrizable groups $\mathcal{M}$ leads again to $\mathcal{M}$. This remains true in the broader class of almost metrizable groups (see Proposition 5.1(2)), although the dual of an almost metrizable (e.g., compact, non-metrizable) group need not be hemicompact. In the realm of reflexive groups there is duality between Čech-complete groups and locally k_ω groups [69, Corollary 7.2 and Remark 7.3].

Since many reflexivity properties of metrizable groups are also valid for almost metrizable groups, we study them in this broader class.

Proposition 5.1. *Let G be an almost metrizable topological group. Then*:

(1) *G is a k-space* [5, (1.24)].
(2) *The dual group $G^\wedge$ is a k-space and $G^{\wedge\wedge}$ is complete and almost metrizable* [5, (5.20)].
(3) *The canonical homomorphisms α_G and $\alpha_{G^\wedge}$ are continuous* [*Proposition* 3.1(4)].
(4) *If G is furthermore reflexive, every dually closed and dually embedded subgroup of G is reflexive* [*Propositions* 3.1(4) *and* 4.6(5)].
(5) *If G is reflexive, closed subgroups of G and of $G^\wedge$ are dually embedded and Hausdorff quotients of G and of $G^\wedge$ are locally quasi-convex, then G is strongly reflexive* [32].

The following questions were raised by Außenhofer in [5]:

Question 5.2.

(a) If G is a metrizable locally quasi-convex group, must $G^\wedge$ be reflexive?
(b) Let G be a group such that α_G is an embedding. Does then the quasi-convex hull of $\alpha_G(G)$ coincide with $G^{\wedge\wedge}$?

Theorem 5.3. *([32]) For an almost metrizable topological group, the following assertions are equivalent*:

(i) *G is strongly reflexive.*
(ii) *G and $G^\wedge$ are q-reflexive.*

This answers Question 4.5(c) for almost metrizable groups.

Question 5.4. ([32]) Does the above theorem hold if "q-reflexive" is replaced by "s-reflexive"?

An important feature of metrizable groups is that dense subgroups determine the dual in the following sense:

Theorem 5.5. *([5,27]) Let G be a metrizable topological group and H a dense subgroup of G. Then the dual homomorphism of the inclusion mapping $j: H \hookrightarrow G$ is a topological isomorphism.*

In other words the restriction mapping from $G^\wedge$ to $H^\wedge$ is a topological isomorphism whenever G is metrizable. It seems natural to extend this theorem to the larger class of almost metrizable groups, in particular to *compact* groups. This originated intensive research to obtain other classes of groups with the same property. To this end the following notion was proposed in [36]:

Definition 5.6. Let G be a topological group and let H be a dense subgroup of G. We say that:

(1) *H determines G*, if the continuous isomorphism $G^\wedge \to H^\wedge$, induced by the restriction to G, is a topological isomorphism.
(2) G is a *determined group*, if every dense subgroup of G determines G.

It was proved [36], under the assumption of CH, that a compact abelian group is determined if and only if it is metrizable. A proof in ZFC was given in [74], and a much shorter ZFC-proof was provided in [53]. Other classes of groups with the same property are studied in [29,54].

One can use this notion to obtain easy examples of non-reflexive groups as follows. If a dense proper subgroup H of a reflexive group G determines G, then H is not reflexive. Indeed, if $\iota : H \to G$ is the inclusion, then α_G and $\iota^{\wedge\wedge}$ are isomorphisms in the commutative diagram (1), while ι is not surjective. Hence α_H cannot be surjective either, so H is not reflexive. Putting the above claim in counter-positive form we obtain a useful (necessary) criterion for reflexivity of dense subgroups of reflexive groups:

Lemma 5.7. *If H is a proper dense reflexive subgroup of a reflexive group G, then H does not determine G.*

Applying this lemma to a metrizable (locally) precompact group G, and taking into account Theorem 5.5, we conclude that for such a group the next four properties are equivalent: reflexive, strongly reflexive, (locally) compact, complete. Hence, every (locally) precompact non-complete metrizable group is non-reflexive. Further examples of (non-metrizable, precompact) non-reflexive groups come in the next example:

Example 5.8.

(a) Let $K = \mathbb{Z}_2^\kappa$ for some infinite cardinal κ. Then the direct sum $S = \bigoplus_\kappa \mathbb{Z}_2$ determines K. Indeed, let X be the set of generators of the copies of $\mathbb{Z}_2$ in S. Then the subset $C = \{0\} \cup X$ of S is compact and its polar is 0, as every $\chi \in C^\triangleright$ must vanish on X, hence on S as well.

(b) In the notation of (a), no proper subgroup H of K containing S is reflexive. Indeed, by item (a), H determines K, so Lemma 5.7 applies.

6. The class of nuclear groups

The class of nuclear groups was formally introduced by Banaszczyk in [12]. A source for inspiration was his previous work [10], where he studied the behavior of closed subgroups and quotients by closed subgroups of nuclear vector spaces. Earlier he had studied similar questions for Banach spaces, and he was aware that, from some point of view, nuclear spaces—rather than Banach spaces—are natural generalization of finite dimensional vector spaces. (Let us recall that Banach spaces and nuclear space are “transversal” generalizations of the finite dimensional spaces: a Banach space is nuclear precisely when it is finite dimensional.)

So he set out to find a class of topological groups embracing nuclear spaces and locally compact abelian groups (as natural generalizations of finite dimensional vector spaces). This was the origin of the class of nuclear groups: the definition of the latter in [12] is very technical, as could be expected from its virtue of joining together objects of so different classes. A nice survey on nuclear groups is also provided by L. Außenhofer in [6] (see also [63]). The following are relevant facts concerning the class of nuclear groups:

(Nuc_1) Nuclear groups are locally quasi-convex [12, 8.5].
(Nuc_2) Nuclear groups form a *Hausdorff variety* (i.e., products, subgroups and quotients of nuclear groups are again nuclear) [12, 7.5].
(Nuc_3) Every locally compact abelian group is nuclear [12, 7.10].
(Nuc_4) A nuclear locally convex space is a nuclear group [12, 7.4]. Furthermore, if a topological vector space E is a nuclear group, then it is a locally convex nuclear space [12, 8.9].
(Nuc_5) If G is a nuclear group, every $w(G, G^\wedge)$-compact subset is compact in the original topology of G [15].

The class of nuclear groups properly contains the Hausdorff variety generated by all LCA groups and all nuclear spaces (see [6, Theorem 7.7]). The counterpart of (Nuc_5) for LCA groups is known as *the Glicksberg theorem*. Clearly, (Nuc_5) gives rise to the following:

Fact C. *If G is a nuclear group, then $G^\wedge$ and $(G^+)^\wedge$ coincide not only algebraically but also topologically.*

As a consequence from Fact C, we can obtain a family of precompact groups for which the canonical mapping into the bidual is not continuous.

Example 6.1. If G is a nuclear reflexive nonprecompact group, and $G^+ := (G, w(G, G^\wedge))$, then α_{G^+} is non-continuous. In particular, if G is a non-compact LCA group, then α_{G^+} is not continuous.

Proof. The topologies of G and of G^+ are distinct since G is nonprecompact. By fact C the duals and hence the biduals of G and of G^+ coincide, therefore $\alpha_{G^+} = \alpha_G$ as mappings. Since $\alpha_G : G \to G^{\wedge\wedge}$ is a topological isomorphism, α_{G^+} cannot be a topological isomorphism. By Proposition 3.1(5) α_{G^+} is open, therefore it is **not** continuous. □

Proposition 6.2. *Let G be a nuclear group and $H \subset G$ a closed subgroup. The following duality results hold*:

(1) *The canonical homomorphisms α_G and α_H, and $\alpha_{G/H}$ are injective and relatively open.*
(2) *Closed subgroups of G are dually closed and dually embedded* [12, 8.3 *and* 8.6].
(3) *If G is moreover complete, α_G is an open isomorphism* [5, 21.5].
(4) *If G is a complete k-space, it is reflexive and its closed subgroups are also reflexive. Hence nuclear complete k-spaces are s-reflexive.*
(5) *If G is Čech-complete, its dual group $G^\wedge$ is also nuclear* [5, 20.36] (*see also* [12, 16.1]) *and strongly reflexive* [5, 20.35].
(6) *If G is moreover almost metrizable, the following equivalences hold*:

$$G \text{ is complete} \quad \Leftrightarrow \quad G \text{ is reflexive} \quad \Leftrightarrow \quad G \text{ is strongly reflexive}.$$

Proof. Item (1) derives from (Nuc_1), (Nuc_2), and Proposition 3.1(5). Item (4) can be obtained from (Nuc_1), (3) and Proposition 3.1(4). Item (6) is consequence of (5) and Proposition 5.1(2). □

Remark 6.3. (i) In spite of the good stability properties of the class of nuclear groups, the dual of a nuclear group need not be nuclear. The constraint of (5) in Proposition 6.2 cannot be completely removed.

(ii) Observe that metrizability as well as nuclearity are essential in Proposition 6.2(6). Examples of non-complete reflexive P-groups (therefore nuclear) are provided in [40]. On the other hand $L^2_{\mathbb{Z}}[0,1]$ is complete metrizable and non-reflexive (see Example 3.5).

Let us conclude with some open questions.

Question 6.4. ([6, Question 11.10])

(a) Is every strongly reflexive group nuclear?
(b) Is every strongly reflexive group a k-space?

The self-dual LCA groups have been studied by many authors [60,94,101], this motivates the following general problem:

Problem 6.5. Study the self-dual nuclear groups.

7. Precompact groups

The class of precompact Hausdorff abelian groups can be identified with the class of subgroups of the powers $\mathbb{T}^\kappa$ of $\mathbb{T}$ (i.e., with the Hausdorff variety generated by $\mathbb{T}$). Thus, a precompact Hausdorff group is nuclear (by (Nuc_2) and (Nuc_3)), so locally quasi-convex. The topology of a precompact abelian group G is precisely $w(G, G^\wedge)$.

A "sort of reflexivity" can be considered for the class of precompact abelian Hausdorff groups. In fact, by Proposition 3.1(6), taking the pointwise convergence topology in the character groups instead of the compact-open topology, a precompact group G is topologically isomorphic to $(G^\wedge, w(G^\wedge, G))^\wedge$. (See also [96], or [89] for a more general result for categories of topological modules.)

We turn now to the "standard" reflexivity. It follows from Fact B that a precompact reflexive metrizable group must be compact. Locally compact, non-compact abelian groups endowed with their Bohr topology are examples of precompact non-metrizable non-reflexive groups (see Example 6.1). Observe that the dual groups of the latter are locally compact.

Since the class of precompact groups is included in that of nuclear groups, the statements of Proposition 6.2 apply for them. Further results specific for this subclass are the following:

Proposition 7.1. *Let G be a precompact group. Then the following assertions hold*:

(1) *The equicontinuous subsets in $G^\wedge$ are finite.*
(2) *The mapping α_G is continuous if and only all the compact subsets of $G^\wedge$ are finite.*
(3) *If the compact sets of G and of $G^\wedge$ are finite, then G is reflexive.*

Proof. In order to prove (1), consider an equicontinuous subset A in $G^\wedge$. Then $A^\triangleleft$ is a neighborhood of zero in G and its closure in the completion $\tilde{G}$ of G is also is a neighborhood of zero in $\tilde{G}$, which we call $\overline{A^\triangleleft}$. The set $(\overline{A^\triangleleft})^\triangleright = (A^\triangleleft)^\triangleright$ is compact in $(\tilde{G})^\wedge$ and thus finite.

Item (2) is a corollary of (1) and Proposition 3.1(2).

Item (3) yields from (1), (2) and Proposition 3.1 (5) and (6). □

As mentioned above, a precompact reflexive metrizable group must be compact. This suggests to ask whether "metrizable" can be replaced by "Frèchet–Urysohn" here (or even with "sequential"). This question can be pushed still further, by formulating it for k-spaces:

Question 7.2. Is a reflexive precompact group, that is also a k-space, necessarily compact?

The precompact reflexive groups have been paid a special attention in [33], where the problem of finding non-compact precompact reflexive groups is explicitly formulated. Such groups are produced in [2], where one can find even non-pseudocompact precompact reflexive groups (see [2, Theorem 3.3]).

Let us recall two other questions on reflexivity of precompact groups.

Question 7.3. ([25, Problem 5.2]) Do there exist countable precompact reflexive groups?

Actually, Tkachenko asked this question in the case of the group $\mathbb{Z}$: *Is there a precompact Hausdorff group topology τ on $\mathbb{Z}$ such that $(\mathbb{Z}, \tau)$ is reflexive* [86, Problem 2.1]?

We shall see in Section 10 that $\mathbb{Z}$ admits non-discrete (sequential) reflexive group topologies.

Question 7.4.

(a) Are there strongly reflexive precompact non-compact groups?
(b) Are there precompact non-compact groups that are s-reflexive and q-reflexive?

8. Pseudocompact groups

A Hausdorff topological group G is said to be pseudocompact if it is pseudocompact as a topological space, that is if every continuous real function defined on G is bounded. This property matched with the algebraic structure of the supporting set produces the highly interesting class of pseudocompact groups intensively studied by many authors. The first relevant properties of this class of groups are the following:

(Psc_1) Every pseudocompact group is precompact [39, 1.1].
(Psc_2) A precompact group G is pseudocompact iff it is G_δ-dense in its (compact) completion $\bar{G}$ [39, Theorem 4.1] (which coincides also with its Stone–Čech compactification βG [77, Theorem 28]).

In checking G_δ-density in (Psc_2) one makes use of the fact that every G_δ-set containing 0 in $\bar{G}$, contains a G_δ-subgroup N of $\bar{G}$ (every G_δ-subgroup N of $\bar{G}$ is closed [43] and $\bar{G}/N$ is metrizable). Hence, to check the G_δ-density of G in $\bar{G}$ it suffices to check that $G + N = \bar{G}$ for every G_δ-subgroup N of $\bar{G}$.

Examples of pseudocompact groups are the Σ-products of uncountable families of compact groups.

It is easily seen that a metrizable pseudocompact group is compact (actually, every paracompact pseudocompact space is compact, since a locally finite family of open sets in a pseudocompact space must be finite). Infinite pseudocompact groups have size $\geqslant \mathfrak{c}$ [59].

The following characterization of pseudocompact groups within the class of precompact groups is provided in [73, (3.4)]. Since this result will be the backbone of our exposition in Sections 8, 9, we offer a proof for the benefit of the reader.

Proposition 8.1. *Let G be a precompact group. Then G is pseudocompact if and only if every countable subgroup of $G^\wedge$ is h-embedded in $(G^\wedge, w(G^\wedge, G))$.*

Proof. $\Rightarrow$) Assume that G is pseudocompact, hence G is G_δ-dense in its compact completion $\bar{G}$. Let H be a countable subgroup of $G^\wedge = \bar{G}^\wedge$ and $H^\perp$ the annihilator of H in $\bar{G}$. Since H is countable, $\bar{G}/H^\perp$ is metrizable, hence the closed subgroup $H^\perp$ of $\bar{G}$ is a G_δ-subgroup. By (Psc_2), the pseudocompactness of G gives

$$G + H^\perp = \bar{G}. \qquad (*)$$

Now let $\phi : H \to \mathbb{T}$ be any character. Since the compact group $\bar{G}/H^\perp$ is isomorphic to the dual of the discrete group H, the character ϕ can be considered as the evaluation at some $y + H^\perp \in \bar{G}/H^\perp$, where $y \in \bar{G}$. By $(*)$ there exists $g \in G$ such that $y + H^\perp = g + H^\perp$. This means that $\phi(\psi) = \psi(y) = \psi(g) = \alpha_G(g)(\psi)$ for all $\psi \in H$ (the second equality follows from $y - g \in H^\perp$). Consequently, ϕ can be extended to $\alpha_G(g)$, a continuous character on $(G^\wedge, w(G^\wedge, G))$.

$\Leftarrow$) In order to prove that G is pseudocompact, it is enough to see that G is G_δ-dense in its completion $\bar{G}$. Let N be a G_δ-subgroup of $\bar{G}$. Then $\bar{G}/N$ is metrizable, hence $H := N^\perp = (\bar{G}/N)^\wedge$ is a countable subgroup of $G^\wedge$ and $N = H^\perp$ (now the annihilator is taken in $\bar{G}$). For $p \in \bar{G}$, define a character $\xi : H \to \mathbb{T}$ by $\xi(\psi) = \psi(p)$. By our hypothesis, ξ can be extended to a continuous character η on $(G^\wedge, w(G^\wedge, G))$. Therefore it exists $g \in G$ such that $\eta = \alpha_G(g)$. In particular, $\alpha_G(g)_{|H} = \eta_{|H} = \xi$. Thus, $\psi(g) = \alpha_G(g)(\psi) = \xi(\psi) = \psi(p)$ for every $\psi \in H = N^\perp$. So $p - g \in N = H^\perp$. This proves that G is G_δ-dense in $\bar{G}$. $\square$

We will use frequently the following assertion proved in [2, 2.1]:

Fact D. *If a topological group has the property that its countable subgroups are h-embedded, then its compact subsets must be finite.*

We enumerate now some properties of pseudocompact groups related to reflexivity. The novelty in this class, with respect to that of precompact groups, is the continuity of α_G (see item (2) below).

Proposition 8.2. *Let G be a pseudocompact group. Then the following assertions hold*:

(1) *Every $w(G^\wedge, G)$-compact subset of $G^\wedge$ is finite. Consequently, every compact subset of $(G^\wedge, \tau_{co})$ is also finite.*
(2) *The mapping α_G is continuous, injective and relatively open. Thus, G is reflexive if and only if α_G is surjective.*
(3) *If the compact sets of G are finite, then the group G is reflexive.*
(4) *G is a dual group. In fact, G is topologically isomorphic to $(G^\wedge, w(G^\wedge, G))^\wedge$.*
(5) *If the countable subgroups of G are h-embedded, then $G^\wedge$ is also pseudocompact with countable subgroups h-embedded. Moreover, G is reflexive.*
(6) *G is topologically isomorphic to a Hausdorff quotient of a reflexive pseudocompact group.*

Proof. The first assertion of (1) follows from Proposition 8.1 and Fact D. Another proof is provided in [73, 4.4].

The continuity of α_G in (2) follows from (1) and Proposition 7.1(2). Since G is locally quasi-convex, α_G is injective and relatively open by Proposition 3.1(5). Thus, the reflexivity of G is reduced to check that α_G is surjective.

Under the assumption of (3), α_G is surjective by Proposition 3.1(7). Now the reflexivity of G follows from (2).

In order to prove (4) observe first that $(G^\wedge, w(G^\wedge, G))^\wedge$ may be algebraically identified with G for any topological group G. Since $w(G^\wedge, G)$-compact subset of $G^\wedge$ are finite by (1), it follows that the compact-open topology in $(G^\wedge, w(G^\wedge, G))^\wedge$ coincides with $w(G, G^\wedge)$.

The assumption of (5) implies that the compact subsets of G are finite (Fact D), and the dual of G is $(G^\wedge, w(G^\wedge, G))$. By (4), $(G^\wedge, w(G^\wedge, G))^\wedge = (G, w(G, G^\wedge))$, and the "only if" part of Proposition 8.1 implies that $G^\wedge$ is pseudocompact.

The assertion (6) is [2, 4.4]. $\square$

Example 8.3. Here we provide examples of pseudocompact reflexive groups that are neither q-reflexive nor s-reflexive. In particular, a pseudocompact reflexive group need not be strongly reflexive.

Observe first that any pseudocompact group G, can be identified with a quotient H/L, where H is a pseudocompact group such that all its countable subgroups are h-embedded and L is a closed pseudocompact subgroup of H [55, 5.5]. By (5) of Proposition 8.2, H is reflexive, $H^\wedge$ is pseudocompact and the quotient H/L need not be reflexive. This already proves that there are reflexive pseudocompact groups which are not q-reflexive.

On the other hand the closed subgroups of a reflexive pseudocompact group need not inherit reflexivity. In fact, take a pseudocompact group G with non-reflexive dual and let H and L be as above. Observe that $L^\perp$ is a closed subgroup of $H^\wedge$ and $(L^\perp)^\wedge \cong \frac{H^{\wedge\wedge}}{L^{\perp\perp}}$. Since L is dually closed and H is reflexive, $\frac{H^{\wedge\wedge}}{L^{\perp\perp}} \cong H/L \cong G$ and consequently $H^\wedge$ is a pseudocompact reflexive group with a non-reflexive closed subgroup $L^\perp$.

Let now $P = H \times H^\wedge$, then P is reflexive, but P is neither q-reflexive nor s-reflexive.

According to [41], a topological group G is *hereditarily pseudocompact*, if every closed subgroup of G is pseudocompact. Obviously, every countably compact group is hereditarily pseudocompact.

Example 8.4. The fact that pseudocompact reflexive groups need not be s-reflexive motivates the following statements:

(a) Every hereditarily pseudocompact reflexive group is s-reflexive. Indeed, assume that G is a reflexive hereditarily pseudocompact group and H is a closed subgroup of G. Since G is precompact, H is dually closed and dually embedded, hence α_H is open and bijective by Proposition 4.6(5). By Proposition 8.2(2), α_H is continuous, since H is also pseudocompact. This proves that H is reflexive, so G is s-reflexive.
(b) Every countably compact reflexive group is s-reflexive, according to (a).
(c) No ZFC-examples of countably compact groups without infinite compact subset are known. Under the assumption of MA such topologies were built in [56], this was done also for a larger class of group [51] using forcing.

Question 8.5. Are there strongly reflexive pseudocompact non-compact groups? Are there q-reflexive pseudocompact non-compact groups?

Observe that if the first question above had a positive answer, witnessed by a pseudocompact group G, then not all countable subgroups of G can be h-embedded. Indeed, if all countable subgroups of G were h-embedded, they should be closed by Fact D, with dual groups compact (as they are endowed with the maximal precompact topology). The strong reflexivity of G implies now that its countable subgroups are also reflexive, thus discrete. It remains to note that a precompact group has no infinite discrete subgroups. Let us finally observe, that if all countable subgroups of a group G are h-embedded, then G must be sequentially complete by [1, 2.1].

Remark 8.6. Let us compare Propositions 7.1(3) and 8.2(3). If a topological group G is precompact and the compact subsets of G and of $G^\wedge$ are finite then G is reflexive. Under the stronger assumption that the group is pseudocompact, reflexivity is obtained only requiring that the compact subsets of G are finite.

In Section 9 we will give examples of reflexive pseudocompact groups whose compact subsets are not finite.

On the other hand there are precompact non-pseudocompact groups G such that the compact subsets of $(G^\wedge, w(G^\wedge, G))$ are finite. Any proper non-measurable subgroup of $\mathbb{T}$ is such an example. Indeed, as noticed in [37], the weak topology $w(G^\wedge, G)$ on $\mathbb{Z} = G^\wedge$ has no non-trivial converging sequences, so it has no infinite compact subsets as $\mathbb{Z}$ is countable.

Example 8.7 *(A family of precompact non-pseudocompact groups).* Let G denote a nonprecompact nuclear reflexive group (in particular, G may be a non-compact locally compact group). Then G^+ is a precompact non-pseudocompact group.

Proof. In Example 6.1 it is proved that α_{G^+} is not continuous. By (2) in Proposition 8.2, G^+ is not pseudocompact. □

9. ω-Bounded groups and P-groups

In 2008 Nickolas asked about the existence of non-discrete reflexive P-groups. A positive answer was provided in [67], a very suggestive paper, where it is also proved that the dual of a P-group is ω-bounded, in particular pseudocompact. Thus, the discovery of non-discrete reflexive P-groups in [67] gave as a by-product also examples of non-compact pseudocompact reflexive groups.

In this section we recall the definitions and collect some properties of the two classes of groups mentioned in the title, which are related by duality. As pointed out in [67], loosely speaking the class of P-groups is close to the class of discrete groups, and the same happens with their duals, the class of ω-bounded groups is close to that of compact groups.

A topological group G is said to be *ω-bounded* if every countable subset $M \subset G$ is contained in a compact subset of G. Clearly, "countable subset" may be replaced by "countable subgroup" in the definition of ω-bounded group. If G is ω-bounded and separable, then G is compact. The following fact (with a straightforward proof) will be often used in the sequel:

Fact E. *Every ω-bounded group is pseudocompact, and hence precompact.*

It follows from this fact and Example 8.4, that reflexive ω-bounded groups are s-reflexive.

We recall that a topological space X is a *P-space* if all of its G_δ-sets are open. An abelian topological group which is a P-space is called a *P-group*. For general properties on P-spaces and P-groups the reader can consult [4]. We only mention here what is needed for our aims, and for this reason the P-groups in the sequel are assumed to be Hausdorff.

A P-group has a basis of neighborhoods of the neutral element consisting of open subgroups and hence, it can be embedded in a product of discrete groups. Consequently, from (Nuc_3) and (Nuc_2) it follows that the class of P-groups is included in that of nuclear groups and therefore we can freely apply the results about nuclear or locally quasi-convex groups so far stated to this new subclass of groups.

Any topological group (G, τ) gives rise to a P-group in a natural way. In fact, let $P\tau$ denote the topology generated by the G_δ subsets of τ. It is a group topology, and the pair $(G, P\tau)$ will be called the *P-modification* of (G, τ) (simply, the P-modification of G and PG, if the original topology of G is clear). The P-modification is a tool to obtain P-groups. In fact, the first example of a reflexive P-group, given in [67], is the P-modification of a product of discrete groups. Later on, in [68] the same authors prove that the P-modification of any locally compact group is also reflexive.

We now state the facts about reflexivity known to hold for P-groups.

Proposition 9.1. *Let G be a P-group. Then*:

(1) *The countable subgroups of G are discrete, thus closed and h-embedded.*
(2) *The compact subsets of G are finite, and hence the compact-open topology in $G^\wedge$ coincides with $w(G^\wedge, G)$.*
(3) *The mapping α_G is bijective and open. Therefore, G is reflexive if and only if α_G is continuous.*
(4) *The closure of every countable subset of $G^\wedge$ is equicontinuous and therefore compact in the compact-open topology. Consequently, the dual group $G^\wedge$ is ω-bounded.*
(5) *A countable union of equicontinuous subsets of $G^\wedge$ is equicontinuous.*
(6) *The evaluation mapping $\alpha_{G^\wedge}$ is continuous.*

Furthermore, if G is reflexive, then:

(7) *G is q-reflexive (i.e., all Hausdorff quotients of G are reflexive) and $G^\wedge$ is s-reflexive.*
(8) *G need not be strongly reflexive.*

Proof. (1) Pick a countable subgroup H of G. Since G is a P-group, one can find an open subgroup W of G with $W \cap H = 0$. In particular, H is discrete. Since H embeds into the discrete quotient group G/W, every character on H can be continuously extended to G/W, hence to G as well. So H is h-embedded.

(2) follows from (1) and Fact D.

(3) Bijectivity of α_G follows from (2) and Proposition 3.1(7), openness from Proposition 3.1(5).

In order to prove (4) take a countable subset of $G^\wedge$, say $S := \{\psi_n,\ n \in \mathbb{N}\}$. For every $n \in \mathbb{N}$, $\psi_n^{-1}(\mathbb{T}_+)$ is a neighborhood of zero in G, and $V := \bigcap_{n\in\mathbb{N}} \psi_n^{-1}(\mathbb{T}_+)$ is also a neighborhood of zero in the P-group G. Hence $V^\triangleright$ is an equicontinuous subset of $G^\wedge$ which contains S. By Ascoli theorem $V^\triangleright$ is compact in the compact-open topology, which coincides with $w(G^\wedge, G)$ by (2). Thus $\overline{S}$ is equicontinuous and compact, and this also proves the last assertion of (4), in a different way of that given in [67]. The proof of (5) is similar. Pick for each $n \in \mathbb{N}$ an equicontinuous subset $L_n \subset G^\wedge$. If V_n is a neighborhood of zero in G with $L_n \subset V_n^\triangleright$, then $\bigcup_{n\in\mathbb{N}} L_n \subset \bigcup_{n\in\mathbb{N}} V_n^\triangleright \subset (\bigcap_{n\in\mathbb{N}} V_n)^\triangleright$. Since $\bigcap_{n\in\mathbb{N}} V_n$ is also a neighborhood of zero, the assertion (5) follows.

(6) It follows from (4) that $G^\wedge$ is ω-bounded, in particular pseudocompact. Therefore, (2) of Proposition 8.2 applies.

The first assertion of (7) is proved in [67]. The second one follows from (4) and the fact that reflexive ω-bounded groups are s-reflexive as will be seen in Proposition 9.10. Item (8) can be derived from a classical example of Leptin in [85], recalled also in [67]. The example consists on a non-reflexive group H which is a closed subgroup of the P-modification G of the product $\mathbb{Z}_2^{\mathfrak{c}}$. Hence, G itself is a reflexive non-strongly reflexive P-group. □

The following example of a non-reflexive ω-bounded group is given in [2, Example 2]:

Example 9.2. Let $G = \{x \in \mathbb{Z}_2^{\mathfrak{c}}\colon\ |\operatorname{supp}(x)| \leqslant \omega\}$ be the $\sum$-product of uncountably many copies of $\mathbb{Z}_2$. Then G is ω-bounded and non-reflexive by Example 5.8(b).

We propose now a stronger version of Nickolas' question:

Question 9.3. Are there non-discrete strongly reflexive P-groups?

Proposition 9.4. *Let (G, τ) be an LCA group. If $Pw(G, G^\wedge) \leqslant \tau$, then G is discrete.*

Proof. It suffices to prove that $G^\wedge$ is compact. Since $G^\wedge$ is a LCA group, it suffices to show that $G^\wedge$ is ω-bounded. Let S be a countable subset of $G^\wedge$. Clearly, $S^\triangleleft$ is a neighborhood of zero in $Pw(G, G^\wedge)$. Hence, by our assumption, $S^\triangleleft$ is a neighborhood of zero in τ. Since $G \cong G^{\wedge\wedge}$, there exists a compact subset K of $G^\wedge$, with $K^\triangleleft \subset S^\triangleleft$. Now taking polars,

$$K^{\triangleleft\triangleright} \supset S^{\triangleleft\triangleright} \supset S.$$

As G is locally compact, it has the qcp (see Section 3.2), and $K^{\triangleleft\triangleright}$ is again compact in G and contains S. This proves that $G^\wedge$ is ω-bounded. □

Fact E implies that every ω-bounded group G carries the topology $w(G, G^\wedge)$ and it can be identified with the dual of $(G^\wedge, w(G^\wedge, G))$, as stated in Proposition 8.2(4). We relate the compact-open topology in $G^\wedge$ with the P-modification of $w(G^\wedge, G)$ as follows:

Lemma 9.5. *Let G be an ω-bounded group. Then*:

$$w\big(G^\wedge, G\big) \leqslant Pw\big(G^\wedge, G\big) \leqslant \tau_{co}.$$

Moreover, if every compact subset of G is separable, then $Pw(G^\wedge, G) = \tau_{co}$.

Proof. The first inequality is obvious, and in order to prove the second one take

$$V := \bigcap_{n\in\mathbb{N}} F_n^\triangleright = \left(\bigcup_{n\in\mathbb{N}} F_n\right)^\triangleright,$$

with $F_n \subset G$ finite. Clearly, V is a standard basic neighborhood of zero in $Pw(G^\wedge, G)$. Since $\bigcup_{n\in\mathbb{N}} F_n$ is contained in a compact subset $K \subset G$, we get that $V = (\bigcup F_n)^\triangleright \supset K^\triangleright$ is a zero neighborhood in $(G^\wedge, \tau_{co})$.

For the last assertion, pick a compact subset $K \subset G$. By the assumption, there exists a countable set $S \subset G$ such that $K \subset \overline{S}$. Taking polars, $\overline{S}^\triangleright = S^\triangleright \subset K^\triangleright$. Consequently $K^\triangleright$ is a neighborhood of zero in $Pw(G^\wedge, G)$, which proves that $\tau_{co} \leqslant Pw(G^\wedge, G)$. □

If the original group is not ω-bounded the last inequality above stated may fail, as the next example shows.

Example 9.6. If G is a locally compact non-compact topological group, then $Pw(G^\wedge, G) \not\leq \tau_{co}$ (τ_{co} is the compact-open topology in $G^\wedge$). Indeed, $(G^\wedge, \tau_{co})$ is a non-discrete LCA group, so $Pw(G^\wedge, G) \not\leq \tau_{co}$ by Proposition 9.4.

In [40] it was proved that if G is ω-bounded, $(G^\wedge, w(G^\wedge, G))$ and $(G^\wedge, Pw(G^\wedge, G))$ have the same continuous characters. More is true:

Proposition 9.7. *Let G be an ω-bounded group. Then the underlying groups of $(G^\wedge, w(G^\wedge, G))^\wedge$ and $(G^\wedge, Pw(G^\wedge, G))^\wedge$ may be algebraically and topologically identified with G.*

Proof. The comment preceding this proposition yields that the underlying groups of $(G^\wedge, w(G^\wedge, G))^\wedge$ and $(G^\wedge, Pw(G^\wedge, G))^\wedge$ coincide. Since $(G^\wedge, w(G^\wedge, G))$ is precompact, both of them can be algebraically identified with G as well.

By Proposition 8.2(1), the $w(G^\wedge, G)$-compact subsets of $G^\wedge$ are finite, and the $Pw(G^\wedge, G)$-compact subsets are finite as well (Proposition 9.1). Therefore the corresponding compact-open topologies in the dual G are equal, and coincide with the pointwise convergence topology $w(G, G^\wedge)$. □

According to Proposition 9.7, any ω-bounded group is the dual of some P-group, while Proposition 9.1(4), says that the dual of a P-group is ω-bounded. Let us resume this as follows:

Corollary 9.8. *A topological group G is ω-bounded if and only if it is the dual of a P-group.*

Remark 9.9. In contrast with Corollary 9.8, the dual of an ω-bounded group G is a P-group only under some additional assumption, for instance if the compact subsets of G are separable (see Lemma 9.5).

We outline now some other facts concerning reflexivity of ω-bounded groups.

Proposition 9.10. *Let G be an ω-bounded group. Then*:

(1) *The evaluation mapping α_G is continuous, open and injective. Therefore G is reflexive if and only if α_G is surjective.*
(2) *G is reflexive provided that every compact subset of G is separable.*
(3) *If G is reflexive, then it is also s-reflexive; moreover, if all compact subsets of G are separable, then $G^\wedge$ is a P-group, hence q-reflexive.*
(4) *If every compact subset of G is separable, then G has sufficient duality.*
(5) *A reflexive non-compact ω-bounded group need not be strongly reflexive.*

Proof. The proof of (1) is covered by Fact E and Proposition 8.2(2).

In order to prove (2), we must only check that α_G is surjective, according to (1). To that end, take a continuous character $\varphi : G^\wedge \to \mathbb{T}$. There exists then a compact subset K in G such that $\varphi(K^\triangleright) \subset \mathbb{T}_+$. By our hypothesis there exists a countable $S \subset G$ such that $K \subset \overline{S}$. Then $S^\triangleright$ is a neighborhood of zero in $Pw(G^\wedge, G)$, and $\varphi(S^\triangleright) \subset \varphi(K^\triangleright) \subset \mathbb{T}_+$ implies that φ is $(G^\wedge, Pw(G^\wedge, G))$-continuous. By Proposition 9.7, $\varphi \in (G^\wedge, w(G^\wedge, G))^\wedge = G$ and hence $\varphi = \alpha_G(g)$ for some $g \in G$.

(3) The first assertion follows from Example 8.4(b) and the obvious fact that ω-bounded groups are countably compact. Lemma 9.5 implies that $G^\wedge$ is a P-group and Proposition 9.1(7) yields that $G^\wedge$ is q-reflexive.

(4) Clearly, every closed subgroup H of G is dually closed. On the other hand, G is reflexive by (2), and this implies that $G^\wedge$ is also reflexive. From Lemma 9.5 we obtain that $G^\wedge$ is a P-group, and (7) of Proposition 9.1 yields that all the Hausdorff quotients of $G^\wedge$ are reflexive. Thus, G has sufficient duality.

In order to prove (5) take into account that the dual group of a strongly reflexive group is also strongly reflexive. In (8) of Proposition 9.1 an example of a reflexive non-strongly reflexive P-group G is presented. Then $G^\wedge$ is an ω-bounded reflexive group, which is not strongly reflexive. □

Item (2), combined with Example 9.2, implies that the ω-bounded group from that example has a non-separable compact subset X, although it is not immediately clear how to find that X (one can take the compact set X from Example 5.8(a)).

A positive answer to item (a) in the next question would provide a positive answer to Questions 7.4 and 8.5.

Question 9.11.

(a) Are there non-compact strongly reflexive ω-bounded groups?
(b) Are there non-compact ω-bounded groups that are s-reflexive and q-reflexive?

10. The algebraic structure of (strongly) reflexive groups

It is well known that the algebraic structure of a topological group may determine properties of topological nature. In the present section we gather some results in this line which have to do with reflexivity.

Quite unexpectedly, the following question, posed by Shakhmatov and the second named author (see [51, Question 14.15(i)], [52, Question 25(i)]), gave rise to a large source of reflexive pseudocompact groups:

Question 10.1. Does every pseudocompact group admit a pseudocompact group topology with no infinite compact subsets?

The question was partially answered by Galindo and Macario [66]. They proved that every pseudocompact abelian group G with $|G| \leqslant 2^{2^{\mathfrak{c}}}$ admits a pseudocompact group topology τ for which all the countable subgroups are h-embedded. By Proposition 8.2(5), such pseudocompact group is reflexive, with pseudocompact dual. Under (GCH) one has the following more impressive result:

Theorem 10.2. *([66]) Under the assumption of GCH, every pseudocompact abelian group admits another pseudocompact group topology with no infinite compact subsets (hence reflexive).*

The paper [62] faces the following general question:

Question 10.3. Does every infinite abelian group admit a non-discrete reflexive group topology?

Among the impressive results obtained by Gabriyelyan in this direction, we think the following are upmost interesting:

Example 10.4. ([61]) The group $\mathbb{Z}$ admits a non-discrete reflexive group topology.

Theorem 10.5. *([62]) Every abelian group G of infinite exponent admits a non-discrete reflexive group topology.*

For the sake of further reference, we isolate the following corollary (although, it was used as one of the main ingredients of the proof, see the comments below).

Corollary 10.6. *The Prüfer group $\mathbb{Z}(p^\infty)$ admits a non-discrete reflexive group topology for every prime p.*

The proof of Theorem 10.5 is based on Proposition 4.7(1). If H is a subgroup of G and τ is a group topology on G such that (H, τ) is (strongly) reflexive, then extending τ to a group topology $\widetilde{\tau}$ on G in the standard way (by making (H, τ) an open subgroup of $(G, \widetilde{\tau})$), we obtain a (strongly) reflexive topology $\widetilde{\tau}$ on G. This immediately reduces the proof of Theorem 10.5 to the case of countably infinite groups. The next step is to choose some "typical" countably infinite groups H, that one may find in *any* countably infinite group of infinite exponent. These are: $\mathbb{Z}$, the Prüfer groups $\mathbb{Z}(p^\infty)$ and direct sums of cyclic groups $\bigoplus_{n=1}^\infty \mathbb{Z}_{m_n}$, with $m_n \to \infty$. Therefore, Example 10.4 and Corollary 10.6 are the main ingredients of the proof of Theorem 10.5 given in [62].

Due to the above theorem, the general Question 10.3 can be reduced to the following problem:

Question 10.7. ([62, Problem 2]) Does every infinite abelian group G of finite exponent (in particular $\mathbb{Z}(p)^{(\mathbb{N})}$ for a prime p) admit a non-discrete reflexive group topology?

Note that for every group G of bounded exponent a locally quasi-convex (hence, a reflexive) group topology τ has a basis of zero-neighborhoods formed by open subgroups, hence (G, τ) is nuclear.

One can prove the following partial consistent positive answer of this problem

Theorem 10.8. *Under the assumption of Martin Axiom every infinite abelian group G of prime exponent and size $\mathfrak{c}$ admits a countably compact non-compact s-reflexive group topology.*

As shown in the proof of [56, Theorem 3.9], every abelian group of size $\mathfrak{c}$ and prime exponent admits, under the assumption of Martin Axiom, a countably compact group topology without infinite compact subsets. By Example 8.4(b), such a group G is s-reflexive. This gives the following immediate corollary:

Corollary 10.9. *Under the assumption of Martin Axiom every finite exponent abelian group G of size $\geqslant \mathfrak{c}$ admits a non-locally compact, locally countably compact reflexive group topology.*

Clearly, such a group G contains a group of the form $H = \bigoplus_{\mathfrak{c}} \mathbb{Z}_p$ for a prime p. Taking a countably compact non-compact reflexive group topology τ on H granted by Theorem 10.8 and declaring (H, τ) to be an open topological subgroup of G, one obtains the desired topology on G.

Now using this corollary and Theorem 10.5 we obtain a partial answer to Question 10.3:

Corollary 10.10. *Under the assumption of Martin Axiom every abelian group G of size $\geqslant \mathfrak{c}$ admit a non-discrete, reflexive group topology.*

The version under CH sounds even more impressive: *under the assumption of CH every uncountable abelian group admits a non-discrete, reflexive group topology*.

Theorem 10.5 suggests now the following:

Question 10.11. Does every abelian group G of infinite exponent admit a non-discrete strongly reflexive group topology?

For the particular case of the groups $\mathbb{Z}$ and $\mathbb{Z}(p^\infty)$ we have partial answers:

Example 10.12. (a) The group $\mathbb{Z}$ admits an s-reflexive and q-reflexive non-discrete group topology. Indeed, let τ be a non-discrete reflexive group topology on $\mathbb{Z}$, whose existence is ensured by Example 10.4. Let us prove that closed subgroups and Hausdorff quotients of $(\mathbb{Z}, \tau)$ are reflexive. Since the proper quotients of $\mathbb{Z}$ are finite, they are reflexive and $(\mathbb{Z}, \tau)$ is q-reflexive. It is also s-reflexive. In fact, take a closed non-zero subgroup H. As it has finite index, it is open and by [21], reflexive.

(b) For every prime number p, the Prüfer group $\mathbb{Z}(p^\infty)$ admits an s-reflexive and q-reflexive non-discrete group topology. By Corollary 10.6, there exists a non-discrete reflexive group topology τ on $\mathbb{Z}(p^\infty)$. Pick a proper closed subgroup H of $G = (\mathbb{Z}(p^\infty), \tau)$. Then H is finite, hence reflexive. The quotient G/H is reflexive as well, again by [21]. This proves that $\mathbb{Z}(p^\infty)$ is s-reflexive and q-reflexive.

A more careful look at the above arguments shows that every reflexive group topology on the groups $\mathbb{Z}$ and $\mathbb{Z}(p^\infty)$ is also s-reflexive and q-reflexive. This is a typical instance of the strong impact of the algebraic structure of the group on the behavior of its group topologies.

Acknowledgments

It is a pleasure to thank Lydia Außenhofer, Dmitri Shakhmatov, Gàbor Lukàcs, Saak Gabriyelyan and Vaja Tarieladze for their careful reading of the survey and useful suggestions.

Appendix. Global aspects of Pontryagin–van Kampen duality

So far we were interested mainly in the local aspect of reflexivity—the characterization of reflexive groups in various classes of groups, properties of their dual, etc. According to Kaplan's theorem, the class of reflexive groups is stable under arbitrary products. This suggests to pay more attention also to the global properties of this class.

In the sequel we denote by $\mathcal{L}qc$ the category of locally quasi-convex groups and continuous homomorphisms, and by $\mathcal{N}uc$ (resp., $\mathcal{L}$) the full subcategory of $\mathcal{L}qc$ having as objects all nuclear (resp., locally compact) groups. Finally, let $\mathcal{N}uc^\wedge$ denote the full subcategory of $\mathcal{L}qc$, formed by the duals of all nuclear groups. Then the Pontryagin duality functor $^\wedge : \mathcal{L}qc \to \mathcal{L}qc$, can be studied at two levels:

(a) restricted to $\mathcal{L}$;
(b) restricted to $\mathcal{N}uc \cap \mathcal{N}uc^\wedge$.

The classical evergreen case (a) offers some interesting problems related to uniqueness that we briefly discuss in the sequel.

For topological abelian groups G, H denote by $Chom(G, H)$ the group of all continuous homomorphisms $G \to H$ equipped with the compact-open topology. It was pointed out already by Pontryagin that the group $\mathbb{T}$ is the unique locally compact group L with the property

$$Chom\big(Chom(\mathbb{T}, L), L\big) \cong \mathbb{T} \tag{2}$$

(note that (2) is much weaker than asking $Chom(-, L)$ to define a duality of $\mathcal{L}$). Much later Roeder [97] proved that Pontryagin–van Kampen duality is the unique functorial duality of $\mathcal{L}$, i.e., the unique involutive contravariant endofunctor $\mathcal{L} \to \mathcal{L}$. Several years later Prodanov [95] rediscovered this result in the following much more general setting. Let R be a discrete commutative ring and $\mathcal{L}_R$ be the category of locally compact topological R-modules. A *functorial duality* $^\# : \mathcal{L}_R \to \mathcal{L}_R$ is a contravariant functor such that $^\# \cdot {}^\#$ is naturally equivalent to the identity of $\mathcal{L}_R$ and for each morphism $f : M \to N$ in $\mathcal{L}_R$ and $r \in R$ $(rf)^\# = rf^\#$ (where, as usual, rf is the morphism $M \to N$ defined by $(rf)(x) = rf(x)$). It is easy to see that the restriction of the Pontryagin–van Kampen duality functor on $\mathcal{L}_R$ is a functorial duality, since the Pontryagin–van Kampen dual $M^\wedge$ of an $M \in \mathcal{L}_R$ has a natural structure of an R-module. So *there is always a functorial duality in* $\mathcal{L}_R$. This motivated Prodanov to raise the question *how many* functorial dualities can carry $\mathcal{L}_R$.

Surprisingly this turned out to be quite complicated. For each functorial duality $^\# : \mathcal{L}_R \to \mathcal{L}_R$ the module $T = R^\#$ (the *torus* of the duality $^\#$) is compact and for every $X \in \mathcal{L}_R$ the module $\Delta_T(X) := Chom_R(X, T)$ of all continuous R-module homomorphisms $X \to T$, equipped with the compact-open topology, is algebraically isomorphic to $X^\#$. The duality $^\#$ is called *continuous* if for each X this isomorphism is also topological, otherwise $^\#$ is *discontinuous*. Clearly, continuous dualities are classified by their tori, which in turn can be classified by means of the Picard group $Pic(R)$ of R. In particular, the unique continuous functorial duality on $\mathcal{L}_R$ is the Pontryagin–van Kampen duality if and only if $Pic(R) = 0$ (see [49, Theorem 5.17]). Prodanov [95] (see also [50, §3.4]) proved that every functorial duality on $\mathcal{L} = \mathcal{L}_{\mathbb{Z}}$ is continuous, which in view of $Pic(\mathbb{Z}) = 0$ gives another proof of Roeder's theorem of uniqueness. While the Picard group provides a good tool to measure the failure of uniqueness for continuous dualities, there is still no efficient way to capture it for discontinuous ones. The first example of a discontinuous duality was given in [49, Theorem 11.1]. Discontinuous dualities of $\mathcal{L}_{\mathbb{Q}}$ and its subcategories are discussed in [48]. It was conjectured by Prodanov that in case R is an algebraic number ring uniqueness of dualities is available if and only if R is a principal ideal domain. This conjecture was proved to be true for real algebraic number rings, but Prodanov's conjecture was shown to fail in case R is an order in an imaginary quadratic number field (e.g., the ring $R = \mathbb{Z}[i]$ of Gauss integers, now the R-modules are precisely the abelian groups G provided with an automorphism $\vartheta_G : G \to G$ with $\vartheta^2 = -id_G$ and morphisms the group homomorphisms $f : G \to H$ such that $f \circ \vartheta_G = \vartheta_H \circ f$) [42].

Dimov and Tholen [57,58] offered an abstract setting of Prodanov's approach to dualities.

As far as the case (b) is concerned, the difficulties come from the lack of good understanding of the class $\mathcal{N}uc^\wedge$. Apparently, a reasonable approach can be the restriction to a smaller class $\mathfrak{C}$ contained in $\mathcal{N}uc \cap \mathcal{N}uc^\wedge$ and containing $\mathcal{L}$, that is preserved under the functor $^\wedge : \mathcal{L}qc \to \mathcal{L}qc$. Here one can study the counterpart of the problem of uniqueness: *when uniqueness of* $^\wedge : \mathfrak{C} \to \mathfrak{C}$ *is available*?

Finally, we mention two papers that concern the categorical aspect of Pontryagin duality, although they are not related to uniqueness. In Roeder [98] offers a streamlined categorical proof of the Pontryagin duality theorem.

Various topologies on the direct sum of topological abelian groups have been used in duality theory. It is shown in [28] that the asterisk topologies of topological abelian groups, used by Kaplan and Banaszczyk, are distinct. However, the authors show that in the category $\mathcal{L}qc$ these two topologies coincide with the coproduct topology [28, Proposition 17]. Hence, the coproduct of reflexive groups is reflexive in $\mathcal{L}qc$.

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