Weakly pseudocompact subsets of nuclear groups

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Abstract

Let \(G\) be an Abelian topological group and \(G^+\) the group \(G\) endowed with the weak topology induced by continuous characters. We say that \(G\) respects compactness (pseudocompactness, countable compactness, functional boundedness) if \(G\) and \(G^+\) have the same compact (pseudocompact, countably compact, functionally bounded) sets. The well-known theorem of Glicksberg that LCA groups respect compactness was extended by Trigos-Arrieta to pseudocompactness and functional boundedness. In this paper we generalize these results to arbitrary nuclear groups, a class of Abelian topological groups which contains LCA groups and nuclear locally convex spaces and is closed with respect to subgroups, separated quotients and arbitrary products.

Let \(G\) be an Abelian topological group. By a character of \(G\) we mean a homomorphism of \(G\) into the group \(\mathbb{R}/\mathbb{Z}\). By the weak topology on \(G\) we mean the topology induced by the family \(G^\sim\) of all continuous characters. It is convenient to denote by \(G^+\) the group \(G\) endowed with its weak topology. We say that \(G\) respects compactness if \(G\) and \(G^+\) have the same compact sets. Similarly, we say that \(G\) respects pseudocompactness (countable compactness, functional boundedness) if \(G\) and \(G^+\) have the same pseudocompact (countably compact, functionally bounded) sets (a subset \(X\) of \(G\) is said to be functionally bounded in \(G\) if every real-valued continuous function on \(G\) is bounded on \(X\)).

That LCA groups respect compactness is a well-known theorem of Glicksberg. It was proved in [6, 7] that LCA groups respect functional boundedness and pseudo-
compactness. Next, it was proved in [5] that limits of inverse sequences of LCA groups respect compactness. This result was generalized in [3] to arbitrary nuclear groups, a class of Abelian topological groups introduced in [1]. In the present paper we show that nuclear groups respect pseudocompactness, countable compactness and functional boundedness provided that they are complete.

We have to introduce some notation and terminology. Let $X$, $Y$ be two symmetric convex subsets of a vector space $F$ (all vector spaces are assumed to be real). Suppose that $X \subset Y$. The Kolmogorov diameters of $X$ with respect to $Y$ are defined by

$$d_k(X, Y) = \inf \inf \{ t > 0 : X \subset tY + L \}, \quad k = 1, 2, \ldots,$$

where the infimum is taken over all linear subspaces $L$ of $F$ with $\dim L < k$.

Let $G$ be an Abelian topological group. By $\mathcal{N}_0(G)$ we denote the family of all neighbourhoods of zero in $G$. We say that $G$ is a nuclear group if it is separated and satisfies the following condition: given arbitrary $U \in \mathcal{N}_0(G)$, $c > 0$ and $m = 1, 2, \ldots$, there exist a vector space $F$, two symmetric and convex subsets $X$, $Y$ of $F$ with $d_k(X, Y) \leq ck^{-m}$ for every $k$, a subgroup $K$ of $F$ and a homomorphism $\varphi : K \to G$ such that $\varphi(K \cap X) \in \mathcal{N}_0(G)$ and $\varphi(K \cap Y) \subset U$. Motivations for introducing nuclear groups and the theory of such groups are presented in [1]. Let us recall here several basic facts; the proofs can be found in Section 7 of [1].

(F$_1$) every LCA group is nuclear;
(F$_2$) every nuclear locally convex space, treated as an additive topological group, is a nuclear group;
(F$_3$) every subgroup of a nuclear group is nuclear;
(F$_4$) every separated quotient group of a nuclear group is nuclear;
(F$_5$) the product of an arbitrary family of nuclear groups is nuclear;
(F$_6$) the direct sum of a countable family of nuclear groups is nuclear;
(F$_7$) every group locally isomorphic (in particular, topologically isomorphic) to a nuclear group is nuclear.

Let us also mention that every nuclear group is maximally almost periodic (see Lemma 7 below).

Assertions (F$_3$)-(F$_6$) say that the permanence properties of nuclear groups are similar to those of nuclear locally convex spaces. In particular, inverse limits of nuclear groups are nuclear.

In connection with (F$_2$) it is worth mentioning that if a topological vector space $E$ is a nuclear group, then $E$ is a locally convex space (Proposition (8.9) of [1]). This means, for instance, that if a Banach space $E$ is a nuclear group, then $E$ must be finite dimensional. On the other hand, every locally convex space over an ultrametric field is a nuclear group (Proposition (7.11) of [1]).

Let $F$ be a vector space and $\tau$ a topology on $F$ such that $F_\tau$ is an additive topological group. We say that $F_\tau$ is a locally convex vector group if it is separated and has a base at zero consisting of symmetric convex sets. A locally convex vector group $F$ is called a nuclear vector group if to each symmetric convex $U \in \mathcal{N}_0(F)$ there corresponds
some symmetric convex $V \in \mathcal{M}_0(F)$ with $d_k(V,U) \leq k^{-1}$ for every $k$. Every nuclear vector group is a nuclear group (Proposition (9.4) of [1]).

**Lemma 1.** An Abelian topological group $G$ is nuclear if and only if there exist a nuclear vector group $F$, a subgroup $H$ of $F$ and a closed subgroup $K$ of $H$ such that $G$ is topologically isomorphic to $H/K$.

The “if” part follows from (F$_7$), (F$_3$) and (F$_4$). The “only if” part is the main assertion of Theorem (9.6) of [1].

**Lemma 2.** Let $\varphi : G \to H$ be a continuous homomorphism of Abelian topological groups. Then $\varphi : G^+ \to H^+$ is continuous, too.

This simple fact is a direct consequence of the definitions.

**Remark.** Theorem 1.2 of [6] says that if $G$ and $H$ are LCA groups, then a homomorphism $\varphi : G \to H$ is continuous if and only if $\varphi : G^+ \to H^+$ is continuous. The analysis of the proof given in Remark 1.8 of [6] shows that it is enough to assume here that the dual group $G^\sim$ respects compactness and that $G$ is reflexive (i.e. that the canonical mapping $G \to G^\sim$ is a topological isomorphism). That nuclear groups respect compactness was proved in [3]; see also theorem below. A detailed analysis of reflexivity for nuclear groups is given in Ch. 5 of [1]. In particular, countable products of LCA groups and nuclear Fréchet spaces are reflexive, together with their closed subgroups, Hausdorff quotients and dual groups. In general, however, for a nuclear group $G$ the continuity of $\varphi : G^+ \to H^+$ does not imply the continuity of $\varphi : G \to H$. Here is an example.

Let $H$ be an Abelian topological group with $H^\lor \neq H$ (e.g. let $H = \mathbb{R}$). The group $G = H^+$ is nuclear, being a subgroup of the Bohr compactification of $H$ (see (F$_1$) and (F$_3$)). Let $\varphi : G \to H$ be the identity homomorphism. Then $\varphi : G^+ \to H^+$ is a topological identity, while $\varphi : G \to H$ is not continuous. It seems to be an interesting question if there exists a similar example with $G$ nuclear and complete.

Let $X$ be a subset of a topological space $S$. We say that $X$ is discretely embedded in $S$ if every real-valued function on $X$ can be extended to a continuous real-valued function on $S$. It is clear that every discretely embedded and countable subset of a Hausdorff space must be closed and relatively discrete.

**Lemma 3.** Let $X$ be an infinite subset of a discrete Abelian group $G$. Then $X$ contains a denumerable subset which is discretely embedded in $G^+$.

This is a direct consequence of Theorem 1.1.3 of [8].

Let $A$ be a subset of a normed space $E$. The distance of a point $u \in E$ to $A$ is denoted by $d(u,A)$. The closed unit ball of $E$ is denoted by $B_E$. If $T : E \to F$ is a bounded linear operator acting between normed spaces, then we write $d_k(T) = d_k(T(B_E),B_F)$ for $k = 1, 2, \ldots$.
Lemma 4. Let \( T : E \rightarrow F \) and \( S : F \rightarrow G \) be bounded linear operators acting between pre-Hilbert spaces. Suppose that \( \sum_{k=1}^{\infty} k d_k(T) \leq 1 \) and \( d_k(S) \rightarrow 0 \) as \( k \rightarrow \infty \). Let \( K \) be an additive subgroup of \( E \) and \( (a_n)_{n=1}^{\infty} \) a sequence in \( E \) such that

\[
d(ST(a_n - a_m), ST(K)) \geq 1, \quad m \neq n.
\]

Then one can choose a subsequence \( (a_{n_k})_{k=1}^{\infty} \) of \( (a_n) \) satisfying the following condition: to each \( u \in E \) there corresponds a bounded linear functional \( f \) on \( E \) with \( f(K) \subset \mathbb{Z} \), such that \( \cos 2\pi f(u - a_{n_k}) \leq \sqrt{2}/2 \) for almost all \( i \).

This is Lemma 3 of [3]. The completeness of the spaces \( E, F \) and \( G \), assumed there, is not essential (cf. the proofs of Lemmas 3 and 4 in [2]).

Let \( p \) be a seminorm on a vector space \( E \). We write \( B_p = \{ u \in E : p(u) \leq 1 \} \). The quotient space \( E/p^{-1}(0) \) endowed with its canonical norm is denoted by \( E_p \), and the canonical projection of \( E \) onto \( E_p \) by \( \psi_p \). We say that \( p \) is a pre-Hilbert seminorm if \( E_p \) is a pre-Hilbert space. If \( q \leq p \) is another seminorm on \( E \), then the canonical operator from \( E_p \) to \( E_q \) is denoted by \( T_{pq} \).

Let \( D \) be a subgroup of an Abelian topological group \( E \). It is not hard to see that \( D \) is weakly closed in \( E \) if and only if \( E/D \) is maximally almost periodic, i.e. if and only if \( (G/H)^+ \) is separated. If \( E \) is a topological vector space, this holds if and only if \( H \) is closed in the weak topology induced on \( E \) by continuous linear functionals (see e.g. Proposition (2.5) of [1]). In the proof of Lemma 5 below by the weak topology we shall always mean the topology induced by continuous characters, even when dealing with subgroups and quotient groups of normed spaces.

Lemma 5. Let \( X \) be a subset of a nuclear group \( G \). If \( X \) is not totally bounded, then it contains a denumerable subset which is discretely embedded in \( G^+ \).

Proof. By Lemma 1, we may assume that \( G = H/K \) where \( H \) is a subgroup of some nuclear vector group \( F \), and \( K \) is a closed subgroup of \( H \). Consider the canonical diagram

\[
\begin{array}{c}
H \xrightarrow{id} F \\
\downarrow \quad \downarrow \\
H/K \xrightarrow{i} F/K
\end{array}
\]

Here \( i : H/K \rightarrow F/K \) is a topological embedding, and Lemma 2 implies that \( i : (H/K)^+ \rightarrow (F/K)^+ \) is continuous. Therefore, without loss of generality, we may assume that \( G = F/K \).

If \( X \) is not totally bounded, then we can find some \( U \in \mathcal{M}_0(G) \) and some sequence \( (x_n)_{n=1}^{\infty} \) in \( X \) such that \( x_m - x_n \notin U \) if \( m \neq n \). Choose \( V \in \mathcal{M}_0(F) \) with \( \gamma(V) \subset U \). As in the proof of Theorem 1 in [2], we can find a linear subspace \( E \) of \( F \) and pre-Hilbert seminorms \( p \geq q \geq r \geq s \) on \( E \) such that \( B_s \subset V, \ B_p \in \mathcal{M}_0(F), \sum_{k=1}^{\infty} k d_k(T_{qr}) \leq 1, \)
$d_k(T_{pq}) \to 0$ and $d_k(T_{r^2}) \to 0$ as $k \to \infty$. We have the canonical diagram

$$
\begin{array}{cccc}
E & \xrightarrow{id} & E & \xrightarrow{id} & E \\
\downarrow{\psi_r} & \downarrow{\psi_q} & \downarrow{\psi_r} & \downarrow{\psi_q} & \downarrow{\psi_q} \\
E \rho & \xrightarrow{T_{pq}} & E \rho & \xrightarrow{T_{pq}} & E \rho \\
\end{array}
$$

Since $d_k(T_{pq}) \to 0$ as $k \to \infty$, and $T_{pq}$ maps $E \rho$ onto $E \rho$, it follows that $E \rho$ is separable. Let $H = E \cap K$ and let $D$ be the weak closure of $\psi_q(H)$ in $E \rho$. Consider the canonical diagram

$$
\begin{array}{cccc}
E \rho & \xleftarrow{\psi_q} & E & \xrightarrow{id} & F \\
\downarrow{\gamma} & \downarrow{\beta} & \downarrow{\gamma} & \downarrow{\gamma} \\
E \rho / D & \xrightarrow{\mu} & E / H & \xrightarrow{\nu} & F / K \\
\end{array}
$$

The subspace $E$ spanned over $B \rho$ is an open subgroup of $F$ because $B \rho \in A_\emptyset (F)$. Hence $A := \gamma(E)$ is an open subgroup of $G = F / K$. Observe that, since $E$ is an open subgroup of $F$, $\nu$ is a topological embedding. By Lemma 2, the canonical projection $\pi : G^+ \to (G / A)^+$ is continuous.

Suppose first that the set $S = \{\pi(x_n)\}_{n=1}^{\infty}$ is infinite. Then, by Lemma 3, it contains a denumerable subset $P$ which is discretely embedded in $(G / A)^+$. For each $p \in P$, choose some $n_p$ such that $\pi(x_{n_p}) = p$. Then it is clear that the set $\{x_{n_p}\}_{p \in P}$ is denumerable and discretely embedded in $G^+$.

Next, suppose that $S$ is finite. Then we can choose a subsequence $(x'_n)_{n=1}^{\infty}$ of $(x_n)$ such that $\pi(x'_n)$ is constant. Consequently, we can choose a sequence $(a_n)_{n=1}^{\infty}$ in $E$ such that $x'_n - x'_m = \gamma(a_n)$ for all $n$. Since $x'_m - x'_n \not\in U$ whenever $m \neq n$, we have that

$$
d(T_{rs}, T_{qr}(\psi_q(a_m)) - \psi_q(a_n)), T_{rs}T_{qr}(\psi_q(H))) \geq 1.
$$

Then it follows directly from Lemma 4 that we can choose a subsequence $(a_n)_{n=1}^{\infty}$ of $(a_n)$ such that the sequence $(x\psi_q(a_n))_{n=1}^{\infty}$ does not have weak cluster points in $E \rho / D$. Then the set $Z = \{x\psi_q(a_n)\}_{n=1}^{\infty}$ is relatively discrete and closed in $(E \rho / D)^+$. Without loss of generality, we may assume that $x\psi_q(a_n) \neq x\psi_q(a_j)$ if $i \neq j$. To complete the proof, it is enough to show that the set $Y = \{\gamma(a_n)\}_{n=1}^{\infty}$ is discretely embedded in $(F / K)^+$.

Let $\xi$ be an arbitrary real-valued function on $Y$. Consider the function $\eta : Z \to \mathbb{R}$ given by $\eta(x\psi_q(a_n)) = \xi(\gamma(a_n))$ for every $i$. Since $E \rho$ was a separable normed space, the group $E \rho / D$ is separable and metrizable, hence Lindelöf. The group $(E \rho / D)^+$ is a continuous image of $E \rho / D$, therefore it is Lindelöf, too ($(E \rho / D)^+$ is separated, hence completely regular, because $D$ was weakly closed in $E \rho$). Thus $(E \rho / D)^+$ is a normal space. So, we can extend $\eta$ to a continuous function $\xi : (E \rho / D)^+ \to \mathbb{R}$.

Being a vector space, $E$ is a divisible group, so that the group $A = \gamma(E) = \psi(E / H)$ is divisible, too. Consequently, the identity homomorphism $A \to A$ can be extended to some homomorphism $\sigma : G \to A$ (see [4], (A.7)). By Lemma 2, the homomorphisms $\sigma : G^+ \to A^+$, $\nu^{-1} : A^+ \to (E / H)^+$ and $\mu : (E / H)^+ \to (E \rho / D)^+$ are continuous. Then the function $\xi\mu\nu^{-1}\sigma : G^+ \to \mathbb{R}$ is a continuous extension of $\xi$. \qed
We shall identify $\mathbb{R}/\mathbb{Z}$ with the interval $I = (-\frac{1}{2}, \frac{1}{2}]$ and treat characters as functions with values in $I$. Let $\chi$ be a character of an Abelian group $G$ and let $A$ be a subset of $G$. We write

$$|\chi(A)| = \sup\{|\chi(g)| : g \in A\}.$$  

By $2\chi$ we denote the character of $G$ given by $(2\chi)(g) = \chi(g + g)$ for $g \in G$. The following simple fact is a direct consequence of definitions.

**Lemma 6.** If $|\chi(A)| < \frac{1}{4}$ and $|(2\chi)(A)| < \frac{1}{4}$, then $|\chi(A)| \leq \frac{1}{8}$.

Let $G$ be an abelian topological group. The group of all continuous characters of $G$ is denoted by $G^\wedge$. A subset $A$ of $G$ is said to be quasi-convex if to each $g \in G \setminus A$ there corresponds some $\chi \in G^\wedge$ with $|\chi(A)| \leq \frac{1}{4}$ and $|\chi(g)| > \frac{1}{4}$. We say that $G$ is a locally quasi-convex group if it has a base of neighbourhoods of zero consisting of quasi-convex sets. It is clear that every separated locally quasi-convex group is maximally almost periodic.

**Lemma 7.** Every nuclear group is locally quasi-convex.

This is Theorem (8.5) of [1].

**Lemma 8.** The completion of a locally quasi-convex group is locally quasi-convex.

**Proof.** Let $\widetilde{G}$ be the completion of a locally quasi-convex group $G$. We may identify $G$ with a dense subgroup of $\widetilde{G}$. Given a subset $A$ of $G$, by $\overline{A}$ we shall denote the closure of $A$ in $\widetilde{G}$. For each $\chi \in G^\wedge$, let $\widetilde{\chi} \in (\widetilde{G})^\wedge$ be the canonical extension of $\chi$.

Choose an arbitrary $U \in \mathcal{N}_0(\widetilde{G})$. We have to find a quasi-convex neighbourhood of zero in $\widetilde{G}$ contained in $U$. Since $U \in \mathcal{N}_0(\widetilde{G})$, there is some $V \in \mathcal{N}_0(G)$ with $\overline{V} \subset U$. We may assume that $V$ is a quasi-convex subset of $G$. Next, we can find some $W \in \mathcal{N}_0(G)$ with $W + W \subset V$. Let us denote

$$V^0 = \{ \chi \in G^\wedge : |\chi(g)| \leq \frac{1}{4} \text{ for each } g \in V \},$$

$$W^0 = \{ \chi \in G^\wedge : |\chi(g)| \leq \frac{1}{4} \text{ for each } g \in W \},$$

$$S = \{ g \in \widetilde{G} : |\widetilde{\chi}(g)| \leq \frac{1}{4} \text{ for each } \chi \in W^0 \}.$$  

It is clear that $S$ is a closed and quasi-convex subset of $\widetilde{G}$ containing $W$. Since $W \in \mathcal{N}_0(G)$, we have $\overline{W} \in \mathcal{N}_0(\widetilde{G})$ and therefore $S \in \mathcal{N}_0(\widetilde{G})$. So, to complete the proof it is enough to show that $S \subset \overline{V}$.

Take an arbitrary $\chi \in V^0$. Then we have

$$|\chi(W)|, |(2\chi)(W)| \leq |\chi(W + W)| \leq |\chi(V)| \leq \frac{1}{4}.$$
Thus $\chi \in W^0$ and $2\chi \in W^0$. Hence $|\widetilde{\chi}(S)| \leq \frac{1}{4}$ and $|\widetilde{(2\chi)}(S)| \leq \frac{1}{4}$. By Lemma 6, this implies that $|\widetilde{\chi}(S)| \leq \frac{1}{8}$. Hence,

$$|\chi((S+S) \cap G)| = |\widetilde{\chi}((S+S) \cap G)| \leq |\widetilde{\chi}(S+S)| \leq |\widetilde{\chi}(S)| + |\widetilde{\chi}(S)| \leq \frac{1}{4}.$$ 

Since this holds for any $\chi \in V^0$, and $V$ is quasi-convex, it follows that

$$(S+S) \cap G \subseteq V. \quad (1)$$

For each $P \in A_0(\widetilde{G})$ one has $S \subseteq (S + P) \cap \widetilde{G}$ because $G$ is dense in $\widetilde{G}$. Setting here, in particular, $P = S$ and applying $(1)$, one gets $S \subseteq \overline{V}$. $\Box$

**Lemma 9.** Let $X$ be a totally bounded subset of a nuclear group. Then the weak topology on $X$ is equal to the original one.

**Proof.** Denote the nuclear group by $G$. We may identify it with a dense subgroup of its completion $\widetilde{G}$. Being nuclear, $G$ is separated, and therefore so is $\widetilde{G}$. Next, $G$ is locally quasi-convex due to Lemma 7 and therefore, by Lemma 8, $\widetilde{G}$ is locally quasi-convex, too. Thus $\widetilde{G}$ is maximally almost periodic, which means that the canonical homomorphism $\phi : \widetilde{G} \to \mathbb{T}(\widetilde{G})$ given by $\phi(g)(\chi) = \chi(g)$ for $\chi \in (\widetilde{G})^\wedge$ is injective. Observe that $\phi$ is a homeomorphism of $\widetilde{G}^+$ onto its image in $\mathbb{T}(\widetilde{G})^\wedge$. Observe also that the topology of $G^+$ is equal to the topology induced on $G$ by the embedding of $G$ into $\widetilde{G}^+$.

Since $X$ is totally bounded, its closure $\overline{X}$ in $\widetilde{G}$ is compact. Consequently, $\phi|_{\overline{X}} : \overline{X} \to \mathbb{T}(\overline{X})$ and $\phi|_{\overline{X}} : \overline{X} \to \phi(\overline{X})$ are homeomorphisms. $\Box$

**Theorem.** Nuclear groups respect compactness, countable compactness and pseudo-compactness. Complete nuclear groups respect functional boundedness.

The authors do not know if the assumption of completeness can be removed.

**Proof.** Let $X$ be a subset of a nuclear group $G$. If $X$ is compact, countably compact, pseudocompact or functionally bounded in the weak topology, then $X$ is totally bounded in the original topology, due to Lemma 5. Consequently, by Lemma 9, the weak topology on $X$ is equal to the original one. If, in addition, $G$ is complete, then $\overline{X}$ is compact and hence $X$ is functionally bounded. $\Box$

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