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Weakly pseudocompact subsets of nuclear groups

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Abstract

Let G be an Abelian topological group and G^+ the group G endowed with the weak topology induced by continuous characters. We say that G respects compactness (pseudocompactness, countable compactness, functional boundedness) if G and G^+ have the same compact (pseudocompact, countably compact, functionally bounded) sets. The well-known theorem of Glicksberg that LCA groups respect compactness was extended by Trigos-Arrieta to pseudocompactness and functional boundedness. In this paper we generalize these results to arbitrary nuclear groups, a class of Abelian topological groups which contains LCA groups and nuclear locally convex spaces and is closed with respect to subgroups, separated quotients and arbitrary products.

Let G be an Abelian topological group. By a *character* of G we mean a homomorphism of G into the group $\mathbb{R}/\mathbb{Z}$. By the *weak topology* on G we mean the topology induced by the family $G^\frown$ of all continuous characters. It is convenient to denote by G^+ the group G endowed with its weak topology. We say that G respects compactness if G and G^+ have the same compact sets. Similarly, we say that G respects pseudocompactness (countable compactness, functional boundedness) if G and G^+ have the same pseudocompact (countably compact, functionally bounded) sets (a subset X of G is said to be *functionally bounded* in G if every real-valued continuous function on G is bounded on X).

That LCA groups respect compactness is a well-known theorem of Glicksberg. It was proved in [6, 7] that LCA groups respect functional boundedness and pseudo-

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compactness. Next, it was proved in [5] that limits of inverse sequences of LCA groups respect compactness. This result was generalized in [3] to arbitrary nuclear groups, a class of Abelian topological groups introduced in [1]. In the present paper we show that nuclear groups respect pseudocompactness, countable compactness and functional boundedness provided that they are complete.

We have to introduce some notation and terminology. Let X, Y be two symmetric convex subsets of a vector space F (all vector spaces are assumed to be real). Suppose that $X \subset Y$. The *Kolmogorov diameters* of X with respect to Y are defined by

$$d_k(X,Y) = \inf_L \inf\{\, t > 0 : X \subset tY + L\,\}, \quad k = 1,2,\ldots,$$

where the infimum is taken over all linear subspaces L of F with $\dim L < k$.

Let G be an Abelian topological group. By $\mathcal{N}_0(G)$ we denote the family of all neighbourhoods of zero in G. We say that G is a *nuclear group* if it is separated and satisfies the following condition: given arbitrary $U \in \mathcal{N}_0(G)$, $c > 0$ and $m = 1,2,\ldots$, there exist a vector space F, two symmetric and convex subsets X, Y of F with $d_k(X,Y) \leq c\,k^{-m}$ for every k, a subgroup K of F and a homomorphism $\varphi : K \to G$ such that $\varphi(K \cap X) \in \mathcal{N}_0(G)$ and $\varphi(K \cap Y) \subset U$. Motivations for introducing nuclear groups and the theory of such groups are presented in [1]. Let us recall here several basic facts; the proofs can be found in Section 7 of [1].

(F_1) every LCA group is nuclear;

(F_2) every nuclear locally convex space, treated as an additive topological group, is a nuclear group;

(F_3) every subgroup of a nuclear group is nuclear;

(F_4) every separated quotient group of a nuclear group is nuclear;

(F_5) the product of an arbitrary family of nuclear groups is nuclear;

(F_6) the direct sum of a countable family of nuclear groups is nuclear;

(F_7) every group locally isomorphic (in particular, topologically isomorphic) to a nuclear group is nuclear.

Let us also mention that every nuclear group is maximally almost periodic (see Lemma 7 below).

Assertions (F_3)–(F_6) say that the permanence properties of nuclear groups are similar to those of nuclear locally convex spaces. In particular, inverse limits of nuclear groups are nuclear.

In connection with (F_2) it is worth mentioning that if a topological vector space E is a nuclear group, then E is a nuclear locally convex space (Proposition (8.9) of [1]). This means, for instance, that if a Banach space E is a nuclear group, then E must be finite dimensional. On the other hand, every locally convex space over an ultrametric field is a nuclear group (Proposition (7.11) of [1]).

Let F be a vector space and τ a topology on F such that F_τ is an additive topological group. We say that F_τ is a *locally convex vector group* if it is separated and has a base at zero consisting of symmetric convex sets. A locally convex vector group F is called a *nuclear vector group* if to each symmetric convex $U \in \mathcal{N}_0(F)$ there corresponds some symmetric convex $V \in \mathcal{N}_0(F)$ with $d_k(V,U) \le k^{-1}$ for every k. Every nuclear vector group is a nuclear group (Proposition (9.4) of [1]).

Lemma 1. *An Abelian topological group G is nuclear if and only if there exist a nuclear vector group F, a subgroup H of F and a closed subgroup K of H such that G is topologically isomorphic to H/K.*

The "if" part follows from (F_7), (F_3) and (F_4). The "only if" part is the main assertion of Theorem (9.6) of [1].

Lemma 2. *Let $\varphi : G \to H$ be a continuous homomorphism of Abelian topological groups. Then $\varphi : G^+ \to H^+$ is continuous, too.*

This simple fact is a direct consequence of the definitions.

Remark. Theorem 1.2 of [6] says that if G and H are LCA groups, then a homomorphism $\varphi : G \to H$ is continuous if and only if $\varphi : G^+ \to H^+$ is continuous. The analysis of the proof given in Remark 1.8 of [6] shows that it is enough to assume here that the dual group $G^\wedge$ respects compactness and that G is reflexive (i.e. that the canonical mapping $G \to G^{\wedge\wedge}$ is a topological isomorphism). That nuclear groups respect compactness was proved in [3]; see also theorem below. A detailed analysis of reflexivity for nuclear groups is given in Ch. 5 of [1]. In particular, countable products of LCA groups and nuclear Fréchet spaces are reflexive, together with their closed subgroups, Hausdorff quotients and dual groups. In general, however, for a nuclear group G the continuity of $\varphi : G^+ \to H^+$ does not imply the continuity of $\varphi : G \to H$. Here is an example.

Let H be an Abelian topological group with $H^+ \neq H$ (e.g. let $H = \mathbb{R}$). The group $G = H^+$ is nuclear, being a subgroup of the Bohr compactification of H (see (F_1) and (F_3)). Let $\varphi : G \to H$ be the identity homomorphism. Then $\varphi : G^+ \to H^+$ is a topological identity, while $\varphi : G \to H$ is not continuous. It seems to be an interesting question if there exists a similar example with G nuclear and complete.

Let X be a subset of a topological space S. We say that X is *discretely embedded* in S if every real-valued function on X can be extended to a continuous real-valued function on S. It is clear that every discretely embedded and countable subset of a Hausdorff space must be closed and relatively discrete.

Lemma 3. *Let X be an infinite subset of a discrete Abelian group G. Then X contains a denumerable subset which is discretely embedded in G^+.*

This is a direct consequence of Theorem 1.1.3 of [8].

Let A be a subset of a normed space E. The distance of a point $u \in E$ to A is denoted by $d(u,A)$. The closed unit ball of E is denoted by B_E. If $T : E \to F$ is a bounded linear operator acting between normed spaces, then we write $d_k(T) = d_k(T(B_E),B_F)$ for $k = 1,2,\ldots$.

Lemma 4. *Let $T: E \to F$ and $S: F \to G$ be bounded linear operators acting between pre-Hilbert spaces. Suppose that $\sum_{k=1}^{\infty} k\, d_k(T) \leq 1$ and $d_k(S) \to 0$ as $k \to \infty$. Let K be an additive subgroup of E and $(a_n)_{n=1}^{\infty}$ a sequence in E such that*

$$d(ST(a_m - a_n), ST(K)) \geq 1, \quad m \neq n.$$

Then one can choose a subsequence $(a_{n_i})_{i=1}^{\infty}$ of (a_n) satisfying the following condition: to each $u \in E$ there corresponds a bounded linear functional f on E with $f(K) \subset \mathbb{Z}$, such that $\cos 2\pi f(u - a_{n_i}) \leq \sqrt{2}/2$ for almost all i.

This is Lemma 3 of [3]. The completeness of the spaces E, F and G, assumed there, is not essential (cf. the proofs of Lemmas 3 and 4 in [2]).

Let p be a seminorm on a vector space E. We write $B_p = \{u \in E : p(u) \leq 1\}$. The quotient space $E/p^{-1}(0)$ endowed with its canonical norm is denoted by E_p, and the canonical projection of E onto E_p by ψ_p. We say that p is a *pre-Hilbert* seminorm if E_p is a pre-Hilbert space. If $q \leq p$ is another seminorm on E, then the canonical operator from E_p to E_q is denoted by T_{pq}.

Let D be a subgroup of an Abelian topological group E. It is not hard to see that D is weakly closed in E if and only if E/D is maximally almost periodic, i.e. if and only if $(G/H)^+$ is separated. If E is a topological vector space, this holds if and only if H is closed in the weak topology induced on E by continuous linear functionals (see e.g. Proposition (2.5) of [1]). In the proof of Lemma 5 below by the weak topology we shall always mean the topology induced by continuous characters, even when dealing with subgroups and quotient groups of normed spaces.

Lemma 5. *Let X be a subset of a nuclear group G. If X is not totally bounded, then it contains a denumerable subset which is discretely embedded in G^+.*

Proof. By Lemma 1, we may assume that $G = H/K$ where H is a subgroup of some nuclear vector group F, and K is a closed subgroup of H. Consider the canonical diagram

$$\begin{array}{ccc} H & \xrightarrow{\text{id}} & F \\ \downarrow & & \downarrow \gamma \\ H/K & \xrightarrow{\iota} & F/K \end{array}$$

Here $\iota : H/K \to F/K$ is a topological embedding, and Lemma 2 implies that $\iota : (H/K)^+ \to (F/K)^+$ is continuous. Therefore, without loss of generality, we may assume that $G = F/K$.

If X is not totally bounded, then we can find some $U \in \mathcal{N}_0(G)$ and some sequence $(x_n)_{n=1}^{\infty}$ in X such that $x_m - x_n \notin U$ if $m \neq n$. Choose $V \in \mathcal{N}_0(F)$ with $\gamma(V) \subset U$. As in the proof of Theorem 1 in [2], we can find a linear subspace E of F and pre-Hilbert seminorms $p \geq q \geq r \geq s$ on E such that $B_s \subset V$, $B_p \in \mathcal{N}_0(F)$, $\sum_{k=1}^{\infty} k\, d_k(T_{qr}) \leq 1$,

$d_k(T_{pq}) \to 0$ and $d_k(T_{rs}) \to 0$ as $k \to \infty$. We have the canonical diagram

$$\begin{array}{ccccccc} E & \xrightarrow{\text{id}} & E & \xrightarrow{\text{id}} & E & \xrightarrow{\text{id}} & E \\ \downarrow{\scriptstyle \psi_p} & & \downarrow{\scriptstyle \psi_q} & & \downarrow{\scriptstyle \psi_r} & & \downarrow{\scriptstyle \psi_s} \\ E_p & \xrightarrow{T_{pq}} & E_q & \xrightarrow{T_{qr}} & E_r & \xrightarrow{T_{rs}} & E_s \end{array}$$

Since $d_k(T_{pq}) \to 0$ as $k \to \infty$, and T_{pq} maps E_p onto E_q, it follows that E_q is separable. Let $H = E \cap K$ and let D be the weak closure of $\psi_q(H)$ in E_q. Consider the canonical diagram

$$\begin{array}{ccccc} E_q & \xleftarrow{\psi_q} & E & \xrightarrow{\text{id}} & F \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} \\ E_q/D & \xleftarrow{\mu} & E/H & \xrightarrow{\nu} & F/K \end{array}$$

The subspace E spanned over B_p is an open subgroup of F because $B_p \in \mathcal{N}_0(F)$. Hence $A := \gamma(E)$ is an open subgroup of $G = F/K$. Observe that, since E is an open subgroup of F, ν is a topological embedding. By Lemma 2, the canonical projection $\pi : G^+ \to (G/A)^+$ is continuous.

Suppose first that the set $S = \{\pi(x_n)\}_{n=1}^{\infty}$ is infinite. Then, by Lemma 3, it contains a denumerable subset P which is discretely embedded in $(G/A)^+$. For each $p \in P$, choose some n_p such that $\pi(x_{n_p}) = p$. Then it is clear that the set $\{x_{n_p}\}_{p \in P}$ is denumerable and discretely embedded in G^+.

Next, suppose that S is finite. Then we can choose a subsequence $(x'_n)_{n=1}^{\infty}$ of (x_n) such that $\pi(x'_n)$ is constant. Consequently, we can choose a sequence $(a_n)_{n=1}^{\infty}$ in E such that $x'_n - x'_1 = \gamma(a_n)$ for all n. Since $x'_m - x'_n \notin U$ whenever $m \neq n$, we have that

$$d(T_{rs}T_{qr}(\psi_q(a_m) - \psi_q(a_n)),\ T_{rs}T_{qr}(\psi_q(H))) \geq 1.$$

Then it follows directly from Lemma 4 that we can choose a subsequence $(a_{n_i})_{i=1}^{\infty}$ of (a_n) such that the sequence $(\alpha\psi_q(a_{n_i}))_{i=1}^{\infty}$ does not have weak cluster points in E_q/D. Then the set $Z = \{\alpha\psi_q(a_{n_i})\}_{i=1}^{\infty}$ is relatively discrete and closed in $(E_q/D)^+$. Without loss of generality, we may assume that $\alpha\psi_q(a_{n_i}) \neq \alpha\psi_q(a_{n_j})$ if $i \neq j$. To complete the proof, it is enough to show that the set $Y = \{\gamma(a_{n_i})\}_{i=1}^{\infty}$ is discretely embedded in $(F/K)^+$.

Let ξ be an arbitrary real-valued function on Y. Consider the function $\eta : Z \to \mathbb{R}$ given by $\eta(\alpha\psi_q(a_{n_i})) = \xi(\gamma(a_{n_i}))$ for every i. Since E_q was a separable normed space, the group E_q/D is separable and metrizable, hence Lindelöf. The group $(E_q/D)^+$ is a continuous image of E_q/D, therefore it is Lindelöf, too ($(E_q/D)^+$ is separated, hence completely regular, because D was weakly closed in E_q). Thus $(E_q/D)^+$ is a normal space. So, we can extend η to a continuous function $\zeta : (E_q/D)^+ \to \mathbb{R}$.

Being a vector space, E is a divisible group, so that the group $A = \gamma(E) = \nu(E/H)$ is divisible, too. Consequently, the identity homomorphism $A \to A$ can be extended to some homomorphism $\sigma : G \to A$ (see [4], (A.7)). By Lemma 2, the homomorphisms $\sigma : G^+ \to A^+$, $\nu^{-1} : A^+ \to (E/H)^+$ and $\mu : (E/H)^+ \to (E_q/D)^+$ are continuous. Then the function $\zeta\mu\nu^{-1}\sigma : G^+ \to \mathbb{R}$ is a continuous extension of ξ. □

We shall identify $\mathbb{R}/\mathbb{Z}$ with the interval $I = (-\frac{1}{2}, \frac{1}{2}]$ and treat characters as functions with values in I. Let χ be a character of an Abelian group G and let A be a subset of G. We write

$$|\chi(A)| = \sup\{\, |\chi(g)| : g \in A \,\}.$$

By 2χ we denote the character of G given by $(2\chi)(g) = \chi(g+g)$ for $g \in G$. The following simple fact is a direct consequence of definitions.

Lemma 6. *If $|\chi(A)| \le \frac{1}{4}$ and $|(2\chi)(A)| \le \frac{1}{4}$, then $|\chi(A)| \le \frac{1}{8}$.*

Let G be an abelian topological group. The group of all continuous characters of G is denoted by $G^\wedge$. A subset A of G is said to be *quasi-convex* if to each $g \in G \setminus A$ there corresponds some $\chi \in G^\wedge$ with $|\chi(A)| \le \frac{1}{4}$ and $|\chi(g)| > \frac{1}{4}$. We say that G is a *locally quasi-convex* group if it has a base of neighbourhoods of zero consisting of quasi-convex sets. It is clear that every separated locally quasi-convex group is maximally almost periodic.

Lemma 7. *Every nuclear group is locally quasi-convex.*

This is Theorem (8.5) of [1].

Lemma 8. *The completion of a locally quasi-convex group is locally quasi-convex.*

Proof. Let $\widetilde{G}$ be the completion of a locally quasi-convex group G. We may identify G with a dense subgroup of $\widetilde{G}$. Given a subset A of G, by $\overline{A}$ we shall denote the closure of A in $\widetilde{G}$. For each $\chi \in G^\wedge$, let $\widetilde{\chi} \in (\widetilde{G})^\wedge$ be the canonical extension of χ.

Choose an arbitrary $U \in \mathcal{N}_0(\widetilde{G})$. We have to find a quasi-convex neighbourhood of zero in $\widetilde{G}$ contained in U. Since $U \in \mathcal{N}_0(\widetilde{G})$, there is some $V \in \mathcal{N}_0(G)$ with $\overline{V} \subset U$. We may assume that V is a quasi-convex subset of G. Next, we can find some $W \in \mathcal{N}_0(G)$ with $W + W \subset V$. Let us denote

$$V^0 = \{\, \chi \in G^\wedge : |\chi(g)| \le \tfrac{1}{4} \text{ for each } g \in V \,\},$$

$$W^0 = \{\, \chi \in G^\wedge : |\chi(g)| \le \tfrac{1}{4} \text{ for each } g \in W \,\},$$

$$S = \{\, g \in \widetilde{G} : |\widetilde{\chi}(g)| \le \tfrac{1}{4} \text{ for each } \chi \in W^0 \,\}.$$

It is clear that S is a closed and quasi-convex subset of $\widetilde{G}$ containing W. Since $W \in \mathcal{N}_0(G)$, we have $\overline{W} \in \mathcal{N}_0(\widetilde{G})$ and therefore $S \in \mathcal{N}_0(\widetilde{G})$. So, to complete the proof it is enough to show that $S \subset \overline{V}$.

Take an arbitrary $\chi \in V^0$. Then we have

$$|\chi(W)|,\ |(2\chi)(W)| \le |\chi(W+W)| \le |\chi(V)| \le \tfrac{1}{4}.$$

Thus $\chi \in W^0$ and $2\chi \in W^0$. Hence $|\widetilde{\chi}(S)| \le \frac{1}{4}$ and $|(2\widetilde{\chi})(S)| = |(\widetilde{2\chi})(S)| \le \frac{1}{4}$. By Lemma 6, this implies that $|\widetilde{\chi}(S)| \le \frac{1}{8}$. Hence,

$$|\chi((S+S)\cap G)| = |\widetilde{\chi}((S+S)\cap G)| \le |\widetilde{\chi}(S+S)| \le |\widetilde{\chi}(S)| + |\widetilde{\chi}(S)| \le \tfrac{1}{4}.$$

Since this holds for any $\chi \in V^0$, and V is quasi-convex, it follows that

$$(S+S)\cap G \subset V. \qquad (1)$$

For each $P \in \mathcal{N}_0(\widetilde{G})$ one has $S \subset \overline{(S+P)\cap G}$ because G is dense in $\widetilde{G}$. Setting here, in particular, $P = S$ and applying (1), one gets $S \subset \overline{V}$. □

Lemma 9. *Let X be a totally bounded subset of a nuclear group. Then the weak topology on X is equal to the original one.*

Proof. Denote the nuclear group by G. We may identify it with a dense subgroup of its completion $\widetilde{G}$. Being nuclear, G is separated, and therefore so is $\widetilde{G}$. Next, G is locally quasi-convex due to Lemma 7 and therefore, by Lemma 8, $\widetilde{G}$ is locally quasi-convex, too. Thus $\widetilde{G}$ is maximally almost periodic, which means that the canonical homomorphism $\varphi : \widetilde{G} \to \mathbb{T}^{(\widetilde{G})^\wedge}$ given by $\varphi(g)(\chi) = \chi(g)$ for $\chi \in (\widetilde{G})^\wedge$ is injective. Observe that φ is a homeomorphism of $\widetilde{G}^+$ onto its image in $\mathbb{T}^{(\widetilde{G})^\wedge}$. Observe also that the topology of G^+ is equal to the topology induced on G by the embedding of G into $\widetilde{G}^+$.

Since X is totally bounded, its closure $\overline{X}$ in $\widetilde{G}$ is compact. Consequently, $\varphi_{|\overline{X}} : \overline{X} \to \varphi(\overline{X})$ and $\varphi_{|X} : X \to \varphi(X)$ are homeomorphisms. □

Theorem. *Nuclear groups respect compactness, countable compactness and pseudocompactness. Complete nuclear groups respect functional boundedness.*

The authors do not know if the assumption of completeness can be removed.

Proof. Let X be a subset of a nuclear group G. If X is compact, countably compact, pseudocompact or functionally bounded in the weak topology, then X is totally bounded in the original topology, due to Lemma 5. Consequently, by Lemma 9, the weak topology on X is equal to the original one. If, in addition, G is complete, then $\overline{X}$ is compact and hence X is functionally bounded. □

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