UNIVERSIDAD COMPLUTENSE DE MADRID

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WEAK AND STRONG TOPOLOGIES IN TOPOLOGICAL ABELIAN GROUPS

MEMORIA PARA OPTAR AL GRADO DE DOCTOR

PRESENTADA POR

Lorenzo de Leo

Bajo la dirección de los doctores
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Memoria presentada para optar al grado de Doctor con mención de *Doctor Europeus* en Matemáticas, elaborada bajo la dirección de:

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Resumen

El título de esta Memoria **"Las topologías débiles y fuertes en los grupos topológicos abelianos"** requiere una explicación. En el marco de los espacios vectoriales topológicos nociones como topología débil, topología de Mackey, topología fuerte son primordialmente el objeto de estudio de la teoría de dualidad. La extensión de la teoría de dualidad a la clase más amplia de los grupos topológicos abelianos encuentra serios obstáculos, como por ejemplo la falta de sentido de la noción de convexidad, que es la piedra angular en dicha teoría.

Una primera idea para trasladar la teoría de dualidad a la clase de los grupos topológicos abelianos, es tomar como objeto dualizante el círculo complejo unidad $\mathbb{T}$ en lugar del cuerpo de los reales $\mathbb{R}$, y sustituir las formas lineales continuas por homomorfismos continuos definidos del grupo en cuestión al grupo $\mathbb{T}$. De este modo se obtiene el grupo dual, cuyos elementos se denominan caracteres. La noción de grupo dual de un grupo dado se remonta a Pontryagin hacia los años '30, y uno de los teoremas más bellos y útiles que subyace en los orígenes del análisis armónico es el teorema de dualidad de Pontryagin van-Kampen. La noción equivalente a "convexidad" en el marco de los grupos topológicos, llamada cuasi-convexidad, se inspira en el teorema de Hahn-Banach, válido para los espacios localmente convexos. Fué introducida por Vilenkin (en ruso, en 1951) y de hecho los subconjuntos cuasi-convexos en un espacio vectorial topológico son muy distintos de los convexos. Baste decir que el subconjunto de números reales formado por $\{0, 1, -1\}$ es cuasi-convexo en el grupo $\mathbb{R}$. No es exagerado decir que la teoría de dualidad en grupos topológicos abelianos ha dado lugar a resultados matemáticas de gran complejidad, como iremos desgranando en estas líneas introductorias.

Se define la topología de Bohr (o topología débil) en un grupo topológico abeliano como la topología inicial relativa a sus caracteres continuos. Por tanto es la topología menos fina de todas las que admiten el mismo grupo dual. En sentido amplio es la versión para grupos de la topología débil de un espacio vectorial topológico. Como puede verse en la Bibliografía muchos matemáticos notables se han ocupado de esta topología como Van Dowen, Kunen, Givens, Gladdines, Dikranjan, Hernández, Galindo etc. y los resul-

tados obtenidos inciden en diversos campos de la Matemática.

En el extremo contrario a la topología débil encontramos – en el contexto de los espacios localmente convexos – la llamada topología de Mackey. También cabe definirla para grupos topológicos abelianos, pero ésto se ha hecho muy recientemente. El primero en mencionar algo es Varopoulos (1962) refiriéndose a una clase muy pequeña de grupos topológicos: los localmente precompactos. La clase más natural de grupos topológicos para definir y estudiar la topología de Mackey es la formada por los grupos localmente cuasi-convexos. El primer estudio de este tipo se hace por Chasco, Martín Peinador y Tarieladze en [23]. Este trabajo ha sido el origen de esta Tesis.

A lo largo de la presente Memoria probamos resultados nuevos que permiten ampliar el conocimiento de las topologías débiles y fuertes en los grupos localmente cuasi-convexos. Para lograr nuestro objetivo, es necesario consolidar el conocimiento de la topología de Bohr y de la teoría de los subconjuntos cuasi-convex de un grupo topológico. La Tesis está estructura en tres partes:

1) **Los conjuntos cuasi-convexos**. Sobre la estructura y caracterización de subconjuntos cuasi-convexos. Incluso en grupos elementales como $\mathbb{Z}$ o $\mathbb{T}$ no hay criterios determinantes para dilucidar si un subconjunto es o no cuasi-convexo. Hemos dado luz sobre estos conjuntos en los capítulos 3, 6, 7 y 8.

2) **Nuevos aspectos de la topología de Bohr y otros tipos de topologías "débiles"** corresponde a los capítulos 3, 4 y 5.

3) **La topología de Mackey de un grupo.** De hecho esta definición aparece por primera vez en esta tesis. Corresponde al capítulo 9. El estudio en profundidad de esta topología es la motivacción que subyace en los otros capítulos. Hemos avanzado sobre lo que ya se sabía, a partir de [23], dando resultados nuevos, y estructurando más de fondo la teoría.

Queda de manifiesto a lo largo de nuestro estudio que en los grupos abelianos la teoría de dualidad presenta notables diferencias con la teoria clásica para los espacios localmente convexos y de algún modo tiene una mayor riqueza. No puede considerarse terminado el estudio, de hecho tenemos problemas abiertos que serán motivo de trabajo en los próximos años. En los capítulos 4 y 5 incluímos resultados obtenidos conjuntamente con D. Dikranjan y con M. Tkatchenko, y recogidos en sendos trabajos de próxima publicación (V. [29] y [30]).

La topología de Mackey en los grupos abelianos

El primero en tratar de dualidades de espacios vectoriales reales y de topologías compatibles fué George Mackey ([60, 61, 62, 63]).

En los trabajos [60, 63], Mackey introdujo lo que él llamó *sistema lineal (linear system)* como un par (E, L), donde E es un espacio vectorial real y L es un subespacio vectorial del espacio vectorial de las formas lineales $l : E \to \mathbb{R}$. Además, denominó *regular* a un sistema lineal (E, L) cuando L separa los puntos de E. Estos objetos se conocen hoy día como *par dual* y *par dual separado* o *dualidad (separada).* En [61, 62], se consideró el sistema lineal (E, L) obtenido al fijar una topología localmente convexa $\mathcal{T}$ en E y tomar como espacio L el formado por todos los funcionales lineales $\mathcal{T}$-continuos; se observó asímismo que la correspondencia entre $\mathcal{T}$ y L no es, en general, unívoca.

La existencia de las topologías localmente convexa más débil y más fuerte en E entre todas aquéllas que dan lugar al mismo sistema lineal regular (E, L) fue formulada en [61, Theorem 1], y probada en [62, Theorem 5]. Mackey no fijó ninguna notación para estas topologías, pero en nuestro lenguaje actual son, respectivamente, la topología *débil* (denotada por $\sigma(E, L)$ en [32, 33, 34]) y la *de Mackey* (denotada por $\tau(E, L)$ en [34]) para el par (E, L).

Un espacio localmente convexo E es un *espacio de Mackey* si su topología coincide con $\tau(E, L)$, donde L es el conjunto de todos los funcionales lineales continuos de E. Estos espacios fueron introducidos directamente por Mackey (con otra nomenclatura), que desarrolló una buena parte de la teoría que hoy en día consideramos clásica en el marco de los espacios localmente convexos. Los términos "topología de Mackey" y "espacio de Mackey" aparecieron por primera vez en [18]. En dicho libro podemos encontrar la noción de *topología compatible*: una topología (localmente convexa) $\mathcal{T}$ en E se denomina *compatible con la dualidad* (E, L) si L coincide con el conjunto de todos los funcionales lineales $\mathcal{T}$-continuos de E en $\mathbb{R}$.

Otra demostración de la existencia y la descripción concreta de la topología de Mackey fue alcanzado por R. Arens en [1]. Más precisamente, en [1, Theorem 2] se demuestra que, dado un sistema lineal regular (E, L), la topología κ en E de la convergencia uniforme sobre los subconjuntos $\sigma(L, E)$-compactos y convexos de L es la más fuerte entre todas las topologías localmente convexas t de E tales que "*los elementos de L representan exactamente los funcionales lineales continuous en E^t*". La combinación de [1, Theorem 2] con [61, Theorem 1] se conoce como *el Teorema de Mackey-Arens.*

Casi cuarenta años después del articulo de Mackey [62], Kakol observó en [56] que la convexidad local es esencial para asegurar la validez del Teorema de Mackey-Arens. El probó que, dada una dualidad (E, L), no tiene por qué existir la topología en E más fuerte entre las topologías (no necesariamente localmente convexas) compatibles con (E, L).

Aunque Mackey tenía profundos conocimientos acerca de los grupos topológicos, no formuló el problema en términos de grupos topológicos abelianos. Las dualidades de grupos abstractos y las topologías (de grupo) compatibles fueron consideradas por primera vez por N. T. Varopoulos en [80], donde se desarrolla una teoría de dualidad para la clase de los grupos localmente precompactos. Sin embargo esta clase de grupos resulta demasiado restringida como se demuestra en [23, Proposition 5.5]. Puesto que los grupos localmente cuasi-convexos presentan una buena analogía con los espacios localmente convexos, los autores de [23] abordan el estudio de la topología de Mackey para dicha clase de grupos. La principal pregunta formulada en el trabajo mencionado permanece aún sin respuesta:

si G es un grupo topológico, existe la topología en G más fuerte entre todas las topologías localmente cuasi-convexas que admiten el mismo grupo dual?

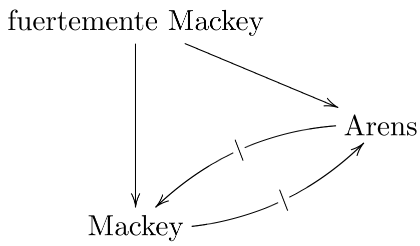
A lo largo de § 8 mejoramos los resultados del articulo antes citado y también de [16] y [13], que se basan en [23], y tratamos otros aspectos de la que se podría llamar *la topología de Mackey de un grupo topológico abeliano.* Para un grupo MAP G y el grupo de sus caracteres $G^\wedge$, hemos definido la topología de Mackey $\tau(G, G^\wedge)$ como la topología localmente cuasi-convexa y compatible de G más fina, siempre y cuando ésta existe (véase Definition 8.6). El estudio de la topología de Mackey se puede restringir a la clase de los grupos localmente cuasi-convexos. Ésto se explica en § 8.1 utilizando la topología débil de un grupo MAP G con respecto a la clase de los grupos localmente cuasi-convexos.
En la misma linea de los espacios localmente convexos, decimos que G es un *grupo de Mackey* si su topología de Mackey $\tau(G, G^\wedge)$ coincide con la topología original de G (véase Definition 8.9). Dicho grupo se caracteriza por la propiedad que si τ es otra topología localmente cuasi-convexa en G que da lugar al mismo grupo dual que (G, ν), entonces $\tau \leq \nu$.

De acuerdo con [23], consideramos la topología localmente cuasi-convexa $\tau_g(G, G^\wedge)$ de un grupo topológico G definida como el extremo superior de la familia de todas las topologías localmente cuasi-convexas en G compatibles con $(G, G^\wedge)$. La existencia de la topología de Mackey en G se caracteriza por la compatibilidad de $\tau_g(G, G^\wedge)$ en el sentido que sigue: *existe la topología de Mackey $\tau(G, G^\wedge)$ en G (y $\tau(G, G^\wedge) = \tau_g(G, G^\wedge)$) si y solo si $\tau_g(G, G^\wedge)$ es una topología compatible en G* (véase Theorem 8.13). Se trata de una caracterización *interna* de la existencia de la topología de Mackey. No obstante, consideramos también la posibilidad de describir la topología de Mackey de un grupo localmente cuasi-convexo G como la topología de la convergencia uniforme sobre una familia de subconjuntos del grupo dual $G^\wedge$, igual que en el contexto de los espacios localmente convexos. Para ello, recordamos y estudiamos la noción de $\mathfrak{S}$-*topología*: dado un grupo topológico G, si $\mathfrak{S}$

es una familia de subconjuntos no vacíos de $G^\wedge$, entonces $\tau_{\mathfrak{S}}(G, G^\wedge)$ es la topología en G de la convergencia uniforme sobre los conjuntos $A \in \mathfrak{S}$. Por ejemplo, la topología de Bohr es la topología de la convergencia uniforme sobre los subconjuntos finitos de $G^\wedge$.

El candidato natural $\mathfrak{S}$ para definir la topología de Mackey de G como $\mathfrak{S}$-topología es la familia $\mathfrak{S}_{qc}$ de todos los subconjuntos $\sigma(G^\wedge, G)$-compactos y cuasi-convexos de $G^\wedge$. De esta manera obtenemos una topología en G –que depende solo del par dual $(G, G^\wedge)$– llamada $\mathfrak{S}_{qc}$-topología y denotada por $\tau_{qc}(G, G^\wedge)$. Dicha topología es más fina que cualquier otra topología localmente cuasi-convexa y compatible en G, y $\sigma(G, G^\wedge) \leq \tau \leq \tau_g(G, G^\wedge) \leq \tau_{qc}(G, G^\wedge)$ siempre que (G, τ) sea un grupo localmente cuasi-convexo (véase Proposition 8.24). Este hecho es muy relevante porque implica que *si* $\tau_{qc}(G, G^\wedge)$ *es compatible* $(G, G^\wedge)$*, entonces la topología de Mackey de* G *existe y coincide con* $\tau_{qc}(G, G^\wedge)$ (véase Corollary 8.25). Hemos llamado *Arens groups* a aquellos grupos para los que $\tau_{qc}(G, G^\wedge)$ es compatible. La relevancia de esta clase de grupos reside en el hecho que es la clase de grupos abelianos donde se cumple fidedignamente lo que podríamos llamar la versión para grupos del Teorema de Mackey-Arens ((véase Remark 8.26). Obsérvese que la propiedad de ser un grupo de Arens depende solo del par dual $(G, G^\wedge)$ en el sentido que sigue: un grupo topológico (G, ν) es Arens si y solo si (G, τ_i) es Arens, para toda topología τ_i compatible con $(G, G^\wedge)$. Un ejemplo en [16] demuestra que la topología $\tau_{qc}(G, G^\wedge)$ no tiene por qué ser compatible. Profundizado en este hecho hemos afirmado que *un grupo de Mackey no tiene por que ser Arens* (véase Theorem 8.61). Por esta razón declaramos que *el Teorema de Mackey Arens no puede imitarse completamente en el marco de los grupos topológicos* dado que la existencia de la topología de Mackey no garantiza que pueda ser descrita como la topología de la convergencia uniforme sobre los subconjuntos $\sigma(G^\wedge, G)$-compactos y cuasi-convexos del grupo dual.

Cuando la topología $\tau_{qc}(G, G^\wedge)$ coincide con la topología original de G, decimos que G es *fuertemente Mackey*. La propiedad de ser fuertemente Mackey es equivalente a la combinación "Arens" y "Mackey". Obsérvese que ninguna de estas propiedades implica individualmente "fuertemente Mackey".



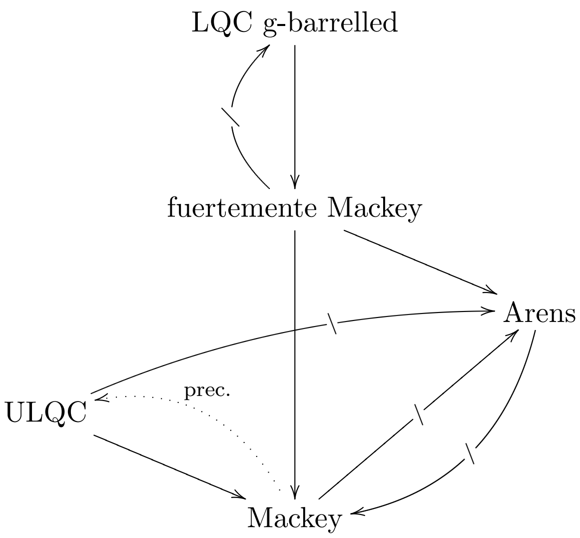
En [23] los autores introdujeron la clase de los grupos *g-barrelled*, destacando que eran grupos que tenían la topología de Mackey, y que además en ellos coincidían las dos posibles definiciones de topología de Mackey. Nosotros percibimos que dicha clase no agotaba la propiedad, y ésto es lo que nos llevó a definir los grupos *fuertemente Mackey*, (V. Theorem 8.34). En Theorem 8.62 probamos la existencia de *una clase de grupos (precompactos) fuertemente Mackey que no son g-barrelled.*

Un resultado interesante de esta tesis es la obtención de una clase de grupos g-barrelled que no se conocía en la literatura. Hemos probado que *todo grupo ω-bounded es g-barrelled* (Theorem 8.37). Un grupo topológico abeliano G se dice que es *ω-bounded* si todo subconjunto numerable de G está contenido en un subgrupo compacto de G (véase Definition 8.41). Tal como observamos en Remark 8.48, la clase de los grupos ω-bounded no compactos no está incluída en las subclases de grupos g-barrelled previamente conocidas, mencionadas en [23], a saber: la clase de *los grupos metrizables hereditariamente Baire*, la de *los grupos Baire separables* y la de *los grupos Čech-completos* (véase también Theorem 8.37).

El hecho de que todo grupo localmente cuasi-convexo g-barrelled es fuertemente Mackey conlleva a lo siguiente: *dado un grupo topológico G, existe a lo sumo una topología localmente cuasi-convexa y compatible τ tal que (G, τ) es g-barrelled* (Theorem 8.50). De aquí deducimos directamente la siguiente propiedad de interés general: *existe a lo sumo una topología localmente cuasi-convexa y compatible perteneciente a la unión de las siguientes clases de grupos topológicos: metrizable hereditariamente Baire, Baire separable, Čech-completo y ω-bounded.* Ésto generaliza el conocido resultado probado por Glicksberg en [49] que afirma que si G es un grupo abeliano localmente compacto, entonces no puede existir otra topología de grupo localmente compacta en G con el mismo grupo dual.

Dejamos como problema abierto el deducir cómo es de grande el conjunto de todas las topologías localmente cuasi-convexas y compatibles de un grupo topológico dado (véase Question 8.92 y Problem8.93). Hasta ahora hemos introducido y estudiado la clase ULQC de los grupo localmente cuasi-convexos que admiten solo una topología localmente cuasi-convexa y compatible, que en ese caso coincide necesariamente con la topología de Bohr. Dichos grupos son precisamente precompactos y Mackey. Demostramos que todo grupo ω-bounded pertenece a dicha clase. Además, consideramos una clase de grupos MAP — que llamamos grupos BTM inspirados en [16] — caracterizados por la propiedad de ser precompactos y de peso estrictamente menor que la cardinalidad del continuo (véase Definition 8.54). Resulta que *todo grupo BTM localmente cuasi-convexo es ULQC* (Corollary 8.60).
En la clase de grupos BTM localmente cuasi-convexos encontramos ejemplos de grupos que son Mackey per no Arens (Theorem 8.61, que generaliza [16, Example 4.2]), y fuertemente Mackey pero no g-barrelled (Theorem 8.62). Gracias a estos resultados, podemos dar una descripción completa de relaciones entre todos los objetos que hemos considerado en el estudio de la topología de Mackey en grupos. Recopilamos la información en el siguiente

diagrama:



En [13], Barr and Kleisly trataron de responder a la pregunta principal de [23] utilizando métodos categóricos. Aunque no alcanzaron del todo su objetivo, el nuevo punto de vista que introdujeron nos proporcionó un nuevo método para atacar dicha pregunta. En § 8.4 preparamos el terreno para tratar la topología de Mackey en diferentes categorías de grupos topológicos. Para toda subcategoría completa $\mathcal{X}$ de la categoría $\mathcal{MAP}$ de los grupos máximamente casi periódicos, definimos la topología $\mathcal{X}$-Mackey $\tau_{\mathcal{X}}(G, G^\wedge)$ de $G \in \mathcal{X}$ como la $\mathcal{X}$-topología de G más fina entre todas las $\mathcal{X}$-topologías que dan lugar al mismo dual que G (véase Definition 8.64). Si $G \in \mathcal{X}$ admite la topología $\mathcal{X}$-Mackey, entonces decimos que G es $\mathcal{X}$-pre-Mackey, mientras que G se llama $\mathcal{X}$-Mackey si su topología original coincide con la topología $\mathcal{X}$-Mackey.
Claramente, la topología de Mackey de un grupo topológico G en los términos descritos en su contexto natural coincide, en este marco más general, con la topología $\mathcal{LQC}$-Mackey, donde $\mathcal{LQC}$ denota la categoría de los grupos localmente cuasi-convexos.

Dado un par $(G, \tau), (H, \gamma)$ de grupos $\mathcal{X}$-pre-Mackey, es natural considerar la siguiente propiedad de *coreflexividad* (o *propiedad CR*), lo cual es esencial en la noción de topología fuerte (véase § 1.3.1): si $f : (G, \tau) \to (H, \gamma)$ es un homomorfismo continuo, entonces el correspondiente homomorfismo $\mu f : (G, \tau_{G^\wedge}) \to (H, \tau_{H^\wedge})$ (que coincide algebraicamente con f) es continuo en el diagrama que sigue a continuación:

$$\begin{array}{ccc} (G, \tau) & \xrightarrow{f} & (H, \gamma) \\ {\scriptstyle id_G}\uparrow & & \uparrow{\scriptstyle id_H} \\ (G, \tau_{\mathcal{X}}(G, G^\wedge)) & \xrightarrow{\mu f} & (H, \tau_{\mathcal{X}}(H, H^\wedge)) \end{array}$$

La subcategoría completa $\mathcal{X}$ formada por todos los grupos $\mathcal{X}$-Mackey es una subcategoría correflexiva de $\mathcal{X}$ en los términos especificados a continuación:

(MS 1) todo grupo en $\mathcal{X}$ es $\mathcal{X}$-pre-Mackey;

(MS 2) todo par (G, τ) y (H, γ) en $\mathcal{X}$ tiene la propiedad CR.

Por lo tanto, la subcategoría de los grupos $\mathcal{X}$-Mackey admite una topología fuerte, la topología de Mackey.

En [13], los autores estudian la posibilidad de caracterizar aquellas categorías que admiten una subcategoría de Mackey. El resultado principal de dicho articulo es el siguiente: $\mathcal{X}$ admite una subcateogría de Mackey si y solo si $\mathbb{T}$ es un objeto inyectivo con respeto a la inclusión en $\mathcal{X}$ (véase Theorem 8.70). Ésto significa que la forma más fuerte (categórica) del problema de la existencia de la topología de Mackey para un grupo topológico $(G, \tau) \in \mathcal{X}$ se traduce completamente en términos categóricos, es decir, el "problema de Mackey" es equivalente a la caracterización de aquellas categorías en las que $\mathbb{T}$ es un objeto inyectivo.

En la presente memoria demostramos una versión más completa de dicho resultado, con el fin de aclarar el sentido en que la hipótesis "inyectividad de $\mathbb{T}$" en la categoría $\mathcal{X}$ asegura la existencia de una subcategoría de Mackey. Nuestra motivación reside en el hecho que $\mathcal{LQC}$ no admite una subcategoría de Mackey ya que se sabe que $\mathbb{T}$ no es inyectivo en $\mathcal{LQC}$. Sin embargo ésto no excluye que todo $G \in \mathcal{LQC}$ sea $\mathcal{LQC}$-pre-Mackey.
Demostramos en Theorem 8.75 que un nivel más débil de inyectividad es suficiente para asegurar la condición (MS 1), en el caso en que $\mathcal{X}$ sea cerrada con respecto a productos arbitrarios y subobjetos. Este resultado se prueba de una manera que recuerda la caracterización "clásica" de la topología de Mackey para los grupos localmente cuasi-convexos. En efecto, definimos la topología $\tau_g^{\mathcal{X}}(G, G^\wedge)$ en un grupo $G \in \mathcal{X}$ como el supremo de todas las $\mathcal{X}$-topologías en G que son compatibles con $(G, G^\wedge)$. Ahora para una categoría $\mathcal{X}$ que sea cerrada con respecto a productos y subgrupos se obtiene que G *es $\mathcal{X}$-pre-Mackey si y solo si $\tau_g^{\mathcal{X}}(G, G^\wedge)$ es compatible* (véase Proposition 8.67). Probamos que la compatibilidad de $\tau_g^{\mathcal{X}}(G, G^\wedge)$ representa una condición que es más débil que la inyectividad de $\mathbb{T}$ en $\mathcal{X}$ (Proposition 8.74). Más precisamente, podemos afirmar que la inyectividad de $\mathbb{T}$ en $\mathcal{X}$ es la unión de las condiciones "compatibilidad de $\tau_g^{\mathcal{X}}(G, G^\wedge)$" — la cual implica la condición (MS 1) — y otra condición (más débil) de inyectividad, que implica (MS 2).

En § 8.5 discutimos ulteriores aspectos de la topología de Mackey. En particular, presentamos brevemente algunos resultados sobre propiedades de estabilidad de dicha topología con respecto a subgrupos, cocientes y productos. Además, inspirados en [45], introducimos la clase de aquellos grupos precompactos que no son la modificación de Bohr de ningún grupo localmente compacto: la clase de los G-grupos. Probamos que *todo grupo ULQC* *es un G-grupo* (Proposition 8.90).

La topología de Bohr

Dado un grupo topológico G, existe un grupo compacto bG y un homomorfismo continuo $r : G \to \mathrm{b}G$ con la siguiente *propiedad universal*: para todo homomorfismo continuo $f : G \to K$ con K compacto, existe un único homomorfismo continuo $\hat{f}$:b$G \to K$ tal que $f = \hat{f} \circ r$. El par (r, bG) se llama *compactificación de Bohr* de G (por el matemático danés Harald Bohr que estudió el caso especial $G = \mathbb{R}$) y es único a menos de isomorfismo topológico. Se tiene que $r(G)$ es denso en $b(G)$ y si r es inyectiva, decimos que G es *maximally almost periodic* (brevemente, MAP). En el marco del Análisis Armónico Abstracto, dichos grupos son muy relevantes y nos restringiremos a esta clase de grupos. Sabemos que todo grupo compacto es MAP (por el Teorema de Peter-Weyl-van Kampen); por lo tanto, para un grupo compacto G tenemos que $G = bG$.

La imagen inversa en G con respeto al homomorfismo $r : G \to bG$ es precisamente la *topología de Bohr* de G, y se utiliza la notación G^+ para denotar el grupo G dotado de su topología de Bohr. Se tiene que la topología de Bohr de un grupo topológico abeliano es la topología inicial con respeto a todos lo homomorfismos continuos $G \to \mathbb{T}$, es decir, coincide con la topología débil $\sigma(G, G^\wedge)$ considerada anteriormente. Un grupo topológico G se dice *totalmente acotado* si y solo si $G = G^+$. En el caso en que G sea un grupo abeliano discreto, denotamos G^+ con $G^\#$, es decir, $G^\#$ es el grupo G dotado de la topología inducida por *todos* los homomorfismo de G al circulo unitario. Por lo tanto obtenemos que la topología de Bohr de un grupo discreto es la *topología totalmente acotada maximal de G*.

La clase de todos los grupos localmente compactos abelianos (LCA) y la de todos los espacios localmente convexos y Hausdorff (LCS) sobre $\mathbb{R}$ o $\mathbb{C}$ representan dos ejemplos fundamentales de grupos MAP. El hecho que LCA está contenido en MAP es un punto clave de la teoría de dualidad de Pontryagin-van Kampen. Por otra parte, los funcionales lineales continuos de un espacio localmente convexo están en correspondencia uno a uno con los homomorfismos al circulo unitario $\mathbb{T}$. Esta sencilla pero importante observación relaciona el estudio de la topología de Bohr con la análisis funcional, y en particular con el concepto de dualidad y de topologías débiles en la clase LCS de los espacios localmente convexos.

La corriente principal en esta dirección es el estudio de la preservación de distintas propiedades topológicas a través del functor de Bohr $G \mapsto G^+$ (entre los muchos autores que trabajan en este tema, mencionamos a Glicksberg, Wu, Comfort, Hernández, Trigos-Arrieta, Remus, Galindo).

Para un grupo abeliano arbitrario G, el grupo precompacto $G^\#$ es trivialmente un grupo pre-Mackey. Uno de los problemas más intrigantes relacionado con grupos de este tipo fue propuesto por van Douwen hace más de quince años (véase [79]):

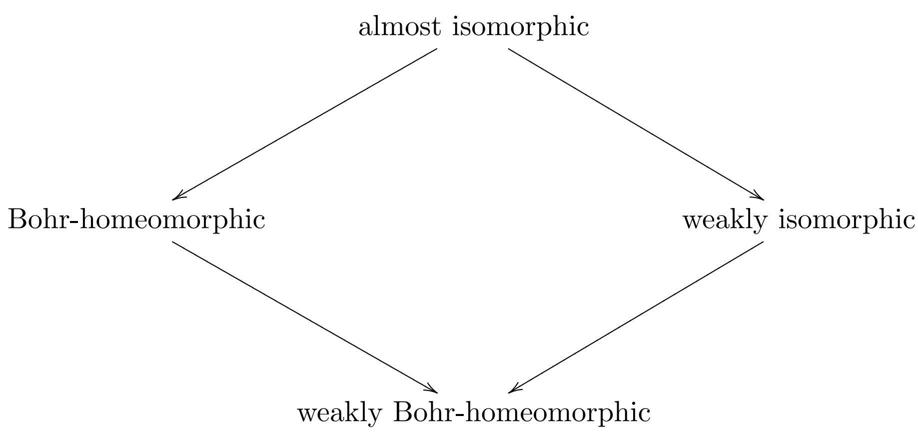
(Question 3.1) *Dados dos grupos abelianos infinitos G, H de la misma cardinalidad, $G^{\#}$ y $H^{\#}$ son homeomorfos como espacios topológicos?*

El problema fue resuelto negativamente por Kunen ([59]) e independientemente por Dikranjan-Watson ([43]). En búsqueda de una solución positiva del problema de van Douwen, diferentes resultados acerca de Bohr-homeomorfismos han sido desarrollados. Véase, por ejemplo, los artículos de Hart y Kunen ([51]) y Comfort, Hernández y Trigos-Arrieta ([25]).

En cada par de grupos que representa una solución negativa al problema de van Douwen, uno de ellos no es ni siquiera Bohr-sumergible en el otro. Esta observación motivó el estudio del problema — más general — del *embedding* en la topología de Bohr. En § 3.0.2 recopilamos y discutimos distintos tipos de posibilidades de embedding para grupos dotados de la topología de Bohr que han sido introducidos en la linea de la Question 3.1. Los definimos a continuación. Dos grupos abelianos G y H son:

- *almost isomorphic* si sus subgrupos de índice finito son isomorfos ([51]);
- *weakly Bohr-homeomorphic* si existen embeddings (de espacios topológicos) $G^{\#} \hookrightarrow H^{\#}$ y $H^{\#} \hookrightarrow G^{\#}$ ([36, 37]);
- *weakly isomorphic* si cada uno de estos grupos tiene un subgrupo de índice finito que es isomorfo a un subgrupo del otro.

A continuación recopilamos las relaciones entre estas nociones:



El resultado principal de § 3 es el siguiente:

(Straightening Theorem 3.10) *toda función continua f entre dos grupos abelianos acotados y dotados de la topología de Bohr coincide con un homomorfismo en un subgrupo infinito del dominio.*

Observemos que, en un cierto sentido, este resultado es el contrapuesto del hecho que *todo homomorfismo entre dos grupos G y H es continuo con respeto a la topología de Bohr*, lo cual expresa nada más que la esencia funtorial de la topología de Bohr.
El Theorem 3.10 extiende los resultados principales de [59, 43] y se basa en interesantes técnicas de partición de funciones definidas sobre el conjunto de n-uplas de ω con valores en $\mathbb{V}_p^\omega$ (véase Theorem 3.29 para más detalles), donde, en general, $\mathbb{V}_m^\kappa$ denota la suma directa de κ copias de $\mathbb{Z}_m$, para todo entero positivo m y todo cardinal κ.

El conjunto de los invariantes de Ulm-Kaplansky determina un grupo (acotado) a menos de isomorfismo, y por lo tanto, de Bohr-homeomorfismo. En § 3.4 aplicamos Theorem 3.10 con el fin de discutir el siguiente problema:

hasta qué punto los Bohr-homeomorfismos preservan los invariantes de Ulm-Kaplansky?

Entre otros resultados, establecemos la equivalencia entre *weak isomorphisms*, *weak Bohr-homeomorphisms* (y otra condición basada en invariantes de tipo algebraico) para dos grupos numerables y acotados G, H (véase Theorem 3.12). Además, presentamos ejemplos de como el Theorem 3.10 puede ser empleado para relacionar el p-rango de dominio y codominio de funciones Bohr-continuas. Pues, probamos lo siguiente:

(see Corollary 3.13) *Los weakly Bohr-homeomorfismos entre grupos acotados preservan la propiedad de poseer infinitos elementos de p-torsión.*

Gracias a esta observación podemos responder negativamente a una pregunta propuesta por Givens y Kunen en [47, §6] sobre la existencia de un embedding topológico de $(\mathbb{V}_2^{\omega_1})^\#$ en $(\mathbb{V}_2^\omega \times \mathbb{V}_3^{\omega_1})^\#$ (véase Corollary 3.14 para más detalles y ulteriores resultados más generales). Además, probamos una versión más general de [47, Theorem 5.1] (que afirma que la propiedad "ser acotado" se preserva por Bohr-homeomorfismos), pues, *los Bohr-homeomorfismos detectan la propiedad de "poseer elementos de p-torsión no triviales, para todo primo p"* (Corollary 3.41).
Consideramos también distintas aplicaciones en la clase de los grupos *almost homogeneous*. Concretamente, probamos que para grupos almost homogeneous y acotados G, H, las propiedades "almost isomorphic", "weakly isomorphic", "Bohr-homeomorphic" y "weakly Bohr-homeomorphic" coinciden (Corollary 3.42).

En § 3.4.3 consideramos aplicaciones del Theorem 3.10 a la teoría de los retractos Bohr-continuos y de las *cross sections*. Probamos una caracterización de los ccs-subgrupos esenciales de los grupos abelianos acotados (Theorem 3.55) y damos una demostración concisa del hecho que *para todo primo p, el subgrupo* $p\mathbb{V}_{p^2}^{\omega} \cong \mathbb{V}_p^{\omega}$ *de* $\mathbb{V}_{p^2}^{\omega}$ *no es un ccs-subgrupo de* $\mathbb{V}_{p^2}^{\omega}$ (véase Example 3.51). Observemos que este resultado fue demostrado en [25] utilizando una prueba bastante más compleja (a la que los autores dedicaron enteramente [25, §5]). Este hecho aporta incluso más interés a nuestras nuevas técnicas introducidas con el Straightening Theorem 3.10 que, en términos más generales, pueden contribuir a solucionar el problema de van Douwen (todavía no resuelto) sobre los subgrupos que son retractos en la topología de Bohr ([77]):

(see Question 3.43) *es cierto que todo subgrupo numerable H de un grupo abeliano G es un retracto de G con respeto a la topología de Bohr?*

En § 4 definimos y estudiamos una nueva topología de grupos que representa una generalización de la topología de Bohr.
En [69] se prueba que existe un *grupo abeliano segundo-numerable universal*, es decir, un grupo abeliano $\mathbb{U}$ que cumple el segundo axioma de numerabilidad y tal que todo grupo segundo-numerable H es topológicamente isomorfo a un subgrupo de $\mathbb{U}$. Además, podemos suponer que $\mathbb{U}$ sea divisible puesto que — de acuerdo con [9, Corollary 3] — dado un grupo abeliano K segundo-numerable, existe un grupo abeliano D segundo-numerable y divisible que contiene a K como subgrupo. Si G es un grupo topológico abeliano G, consideramos la $\mathbb{U}$-topología débil en G, es decir, la topología en G inicial con respeto a $\mathbb{U}$. Está claro que $\mathbb{T} \leq \mathbb{U}$, por lo tanto la $\mathbb{U}$-topología débil refina la topología de Bohr .
Denotamos un grupo topológico G dotado de la $\mathbb{U}$-topología débil por $G^{\ddagger}$. Resulta que *un grupo topológico G es ω-narrow si y solo si $G = G^{\ddagger}$* (la definición original de grupo ω-narrow puede encontrarse en §4). Si el grupo de partida G es discreto, denotamos $G^{\ddagger}$ por $G^{\square}$. En este case, la $\mathbb{U}$-topología débil es la topología inicial con respeto a la familia de todo homomorfismo $G \to \mathbb{U}$, con lo cual es la *topología ω-narrow maximal* en G.
Claramente, $G^{\square}$ y $G^{\#}$ tienen características muy parecidas desde un punto de vista funtorial. Sin embargo, podemos destacar desemejanzas substanciales: a diferencia del caso de la topología de Bohr, *todo subconjunto numerable de $G^{\square}$ es cerrado y discreto en $G^{\square}$* (Corollary 4.11) y también *C-embedded en $G^{\square}$* (Proposition 4.12).

Entre otros resultados, probamos que todo grupo abeliano no numerable G dotado de la topología ω-narrow maximal es un espacio de primera categoría y no es un P-grupo.
Recordemos que un grupo G es $\mathbb{R}$-*factorizable* si toda función continua de

G con valores reales admite una factorización a través de un grupo segundo-numerable K, un homomorfismo continuo $p : G \to K$ y una función continua h en K con valores reales (véase Definition 4.18). La clase de los grupos $\mathbb{R}$-factorizable constituye una subclase propria de la clase de los grupos ω-narrow; sin embargo es muy amplia. Entre otros, contiene todo grupo precompacto, todo grupo Lindelöf, todo subgrupo arbitrario de grupos σ-compact ([73]). Hasta la fecha actual, solo aparecen en la literatura ejemplos esporádicos de grupos que no son $\mathbb{R}$-factorizable y ω-narrow (véase [71, Example 5.14]). En Theorem 4.19 demostramos que todo grupo abeliano no numerable admite una topología de grupo con dicha combinación de propiedades.

Grupos abelianos localmente cuasi-convexos

Después de que Pontryagin introdujera el *grupo dual* de un grupo topológico abeliano, Vilenkin observó que dado un espacio normado considerado como grupo abeliano, el grupo dual y el espacio dual pueden ser identificados. Siguiendo esta linea, introdujo los *conjuntos cuasi-convexos* de un grupo topológico abeliano. Dicha noción se inspira en el Teorema de Hahn-Banach, y es la noción análoga a subconjunto convexo de un espacio vectorial topológico. Gracias a esta nueva importante herramienta, Vilenkin pude definir los grupos localmente cuasi-convexos ([81]). Cuarenta años después, Banaszczyk desarrolló en [10] distintas propiedades de los grupos localmente cuasi-convexos, aunque su principal objetivo era la introducción de los grupos nucleares — una subclase propria de los grupos localmente cuasi-convexos. La clase de los grupos localmente cuasi-convexos incluye LCA y LCS, y es cerrada con respeto a productos arbitrarios y subgrupos.

Las consideraciones anteriores, y también el clásico libro de Banach "Théorie des opérations linéaires", expresan el hecho que el problema de considerar los grupos abelianos topológicos como una clase que abarque los espacios vectoriales topológicos es muy típico en nuestro contexto. Por lo tanto, entran en juego de manera natural los subgrupos de los espacios vectoriales topológicos — como expresa el título de [10] — y se estudia la posibilidad de extender propiedades típicas de los espacios localmente convexos a la clase más amplia de los grupos localmente cuasi-convexos. Muchos autores han estado trabajando en esta dirección (Kye, Hernández, Galindo, Martín-Peinador, Chasco, Tarieladze, entre otros) y distintos teoremas del Análisis Funcional tienen ahora una versión análoga para grupos topológicos abelianos ([20, 23, 52], etc.).

Dado un subconjunto E de G y un subconjunto A de $G^\wedge$, definimos los *polares*

$$E^\triangleright = \{\chi \in G^\wedge \mid \chi(E) \subseteq \mathbb{T}_+\} \text{ y } A^\triangleleft = \{x \in G \mid \chi(x) \in \mathbb{T}_+ ,\forall\, \chi \in A\}.$$

Un subconjunto E de G se dice *cuasi-convexo* si $E = E^{\triangleright\triangleleft}$, es decir, para todo $x \in G \setminus E$ existe $\chi \in E^{\triangleright}$ tal que $\chi(x) \notin \mathbb{T}_+$. En lo que sigue, denotamos por $Q_{(G,\tau)}(E)$ la *envoltura cuasi-convexa* de $E \subseteq G$, que coincide con el subconjunto de (G,τ) más pequeño que contiene a E (escribiremos simplemente $Q_G(E)$ cuando no hay posibilidad de confusión).

Aunque los subconjuntos cuasi-convexos representan la esencia de la teoría de los grupos localmente cuasi-convexos, podemos afirmar que su naturaleza no es del todo conocida. Incluso para grupos elementales como, por ejemplo, los enteros $\mathbb{Z}$ o el círculo unitario $\mathbb{T}$, no existen criterios claros para reconocer los subconjuntos cuasi-convexos.
Uno de los objetivos principales de la presente Tesis consiste en desarrollar la teoría de los conjuntos cuasi-convexos, con especial atención a los casos más desconocidos, es decir, los conjuntos cuasi-convexos *pequeños* (más precisamente, finitos y numerablemente infinitos).

Usualmente la cuasi-convexidad se estudia en la clase de los grupos MAP, pues son grupos en los que los caracteres continuos separan puntos. La razón es clara, de acuerdo con la siguiente equivalencia: *un grupo topológico G es MAP si y solo si $\{0_G\}$ es cuasi-convexos*. Una motivación más profunda se deduce por el hecho que el cálculo de la envoltura cuasi-convexas se puede reducir al caso de topologías precompactas (véase Remark 5.4). En particular, esta observación relaciona la cuasi-convexidad con la noción de precompaccidad y, por lo tanto, con la topología de Bohr. Puesto que $Q_{(G,\tau)}(E) = Q_{(G,\tau^+)}(E)$ para todo $E \subseteq G$ y para toda topología τ en G (véase Fact 5.3), deducimos claramente que un buen conocimiento de la topología de Bohr es esencial en el estudio de la cuasi-convexidad.

Entre los pocos resultados conocidos sobre los conjuntos cuasi-convexos, destacan los que exponemos a continuación.

Theorem 0.1 ([5],[41]) *Sea G un grupo MAP y F un subconjunto finito. Entonces:*

(1) $Q_G(F) \subseteq \langle F \rangle$*;*

(2) $Q_G(F)$ *es finito.*

Obsérvese que ambas propiedades dejan de ser ciertas si F es infinito. Por ejemplo, si F es el conjunto numerablemente infinito $F := \{\pm 2^{-n} \mid n \in \mathbb{N}\} \subseteq \mathbb{T}$, entonces $Q_{\mathbb{T}}(F) = \mathbb{T} \not\subseteq \langle F \rangle \cong \mathbb{Z}(2^\infty)$ (Example 5.24). Motivados por este ejemplo, introducimos en § 5.4 la noción de *subconjunto cc-denso* con el fin de describir aquellos conjuntos $E \subseteq G$ tales que su envoltura cuasi-convexa es lo más grande posible, es decir, $Q_G(E) = G$.

Theorem 0.2 ([5]) *Sea G un grupo abeliano topológico. Entonces G es MAP si y solo si el conjunto $A_x := \{0, \pm x\}$ es cuasi-convexo en G para todo $x \in G$.*

En particular, este ejemplo nos enseña que en algún caso la cuasi-convexidad depende solo aparentemente de la topología.

No se posible generalizar Theorem 0.2 a todo conjunto de la forma $\{0, \pm x, \ldots, \pm kx\}$, incluso en el caso $k = 2$. Pues, $\{0, \pm\frac{1}{12} + \mathbb{Z}, \pm\frac{2}{12} + \mathbb{Z}\} \subseteq \mathbb{T}$ no es cuasi-convexo en $\mathbb{T}$ (véase Example 6.21). Tenemos por lo tanto una motivación concreta para estudiar la cuasi-convexidad de los conjuntos de la forma $E_{x,k} := \{0, \pm x, \ldots, \pm kx\} \subseteq \langle x \rangle$ (para algún $k \geq 1$) y, por otro lado, la siguiente noción:

(Definition 5.32) *Un subconjunto E de un grupo abeliano G es* incondicionalmente cuasi-convexo en G *si E es cuasi-convexo en toda topología MAP de G (obsérvese que es equivalente pedir dicha condición únicamente para las topologías precompactas de G).*

Es la versión análoga a la de *incondicionalmente cerrado* introducida por Markov en [64] (véase también Definition 5.29).
Probamos — utilizando un resultado de [42] — que estas dos nociones coinciden en subgrupos: *si H es un subgrupo de un grupo abeliano infinito G, entonces H es incondicionalmente cerrado en G si y solo si es incondicionalmente cuasi-convexo en G* (Theorem 5.35). En particular, basándonos en la caracterización algebraica de los subgrupos incondicionalmente cerrados probada en [42], podemos afirmar que la dependencia de la cuasi-convexidad a la topología se traduce en restricciones de tipo algebraico.

Inspirados en la noción de conjunto potencialmente denso introducida por Markov (véase § 5.5.2), a continuación proponemos también la noción de subconjunto potencialmente cuasi-convexo:

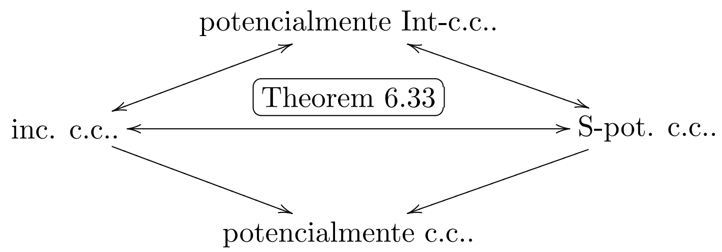
(Definition 5.36) *Un subconjunto E de un grupo abeliano G es* potencialmente cuasi-convexo en G *si existe una topología MAP τ en G tal que E es cuasi-convexo en* (G, τ).

Está claro que éste es el nivel más débil de cuasi-convexidad. Obsérvese que un subconjunto $E \subseteq G$ es potencialmente cuasi-convexo en G si y solo si es cuasi-convexo con respeto a la topología discreta de G (véase Remark 5.38). Tras aplicar Theorem 0.1 *(1)*, podemos deducir que *la cuasi-convexidad potencial y la cuasi-convexidad incondicional coinciden para todo subgrupo finito $H \leq G$* (Remark 5.37); no obstante, no podemos afirmar lo mismo en el caso de *subconjuntos*: pues, *el conjunto $E = \{0, \pm 1, \pm 3\} \subseteq \mathbb{Z}$ es potencialmente cuasi-convexo en $\mathbb{Z}$ pero no es incondicionalmente cuasi-convexo* (Example 5.40). Sin embargo, mostramos una amplia clase de ejemplos de conjuntos incondicionalmente cuasi-convexos en $\mathbb{Z}$: por ejemplo, $E_{1,2} = \{0, \pm 1, \pm 2\}$ y $U_{1,4} = \{0, \pm 1, \pm 4\}$ son incondicionalmente cuasi-convexos en $\mathbb{Z}$ (más en general, véase Example 6.34 and Example 6.35). Además, estos ejemplos se puede extender a cualquier grupo MAP G que tenga un elemento de no torsión: *dado un subconjunto finito E de $\mathbb{Z}$, E es incondicionalmente cuasi-convex en $\mathbb{Z}$ si y solo si $xE = \{xe \mid e \in E\}$ es incondicionalmente cuasi-convexo en cualquier grupo MAP G que contenga un elemento x de no torsión* (se deduce de Theorem 6.36 y Theorem 6.33). De esta manera tenemos una motivación adicional para estudiar los subconjuntos cuasi-convexos y finitos de $\mathbb{Z}$.

El estudio de los conjuntos incondicionalmente cuasi-convexos y finitos está relacionado con otras nociones más fuertes de cuasi-convexidad que hemos introducido en esta Tesis. Se basan en la observación que cuando aplicamos la definición de cuasi-convexidad y separamos un conjunto $E \subseteq \mathbb{Z}$ (no necesariamente finito) de un cierto punto $z \in \mathbb{Z} \setminus E$ por medio de un carácter continuo χ de $\mathbb{Z}$, es extremadamente importante reconocer si la imagen de E a través de χ corta el borde de $\mathbb{T}_+$, formado por $\{-\frac{1}{4}+\mathbb{Z}, \frac{1}{4}+\mathbb{Z}\}$. Por lo tanto distinguimos dos situaciones concretas:
decimos que un subconjunto $E \subseteq \mathbb{Z}$ es

- *potencialmente Int-cuasi-convexo en* $\mathbb{Z}$ si para todo $e \in \mathbb{Z} \setminus E$ existe $\chi : \mathbb{Z} \to \mathbb{T}$ tal que $\chi(E) \subseteq \mathrm{Int}(\mathbb{T}_+) = (-\frac{1}{4}, \frac{1}{4})$ y $\chi(e) \notin \mathbb{T}_+$ (véase también Definition 5.43);
- *S-potencialmente cuasi-convexo en* $\mathbb{Z}$ si para todo $z \in \mathbb{Z} \setminus E$, existe $\chi : \mathbb{Z} \to \mathbb{T}$ tal que vale la siguiente condición:
  - $\chi(E) \subseteq \mathbb{T}_+$ y $\chi(z) \notin \mathbb{T}_+$;
  - si $\chi(e_1) = \frac{1}{4} + \mathbb{Z}$ para algún $0 < e_1 \in E$, entonces $\chi(e) \neq \frac{3}{4} + \mathbb{Z}$ para todo $0 < e \in E$.

Resulta que "potencial. Int-c.c.." $\Longrightarrow$ "S-potencial. c.c.." $\Longrightarrow$ "potencial. c.c.." (véase Lemma 6.30 y Lemma 6.29, respectivamente).
El Theorem 6.33 afirma que las tres nociones "fuertes" de cuasi-convexidad que hemos introducido coinciden en el caso de subconjuntos finitos del los enteros. A continuación presentamos el diagrama completo de relaciones para subconjuntos finitos de $\mathbb{Z}$:



Una herramienta de extrema utilidad a la hora de manejar conjuntos finitos es la siguiente consecuencia de Theorem 0.1 *(2)*: *si $E \subseteq H$ es finito, entonces no hay diferencia a la hora de calcular la envoltura cuasi-convexa de E en G o en $\langle E \rangle$* (Corollary 5.19). Esta sencilla observación es la clave para entender todos los ejemplos de conjuntos cuasi-convexos finitos presentados en §6 como los siguientes:

- $A_{\alpha_1} \cup \ldots \cup A_{\alpha_t} = \{0, \pm\alpha_1, \ldots, \pm\alpha_t\}$ es cuasi-convexo in $\mathbb{T}$ para todo $t \geq 0$ y $\alpha_1, \ldots, \alpha_t \in \mathbb{T}$ linealmente independientes (Corollary 6.2);
- $A_\alpha + \ldots k \text{ veces} \ldots + A_\alpha = E_{\alpha,k} = \{0, \pm\alpha, \ldots, \pm k\alpha\}$ es cuasi-convexo en $\mathbb{T}$ para todo $\alpha \in \mathbb{T} \setminus \mathbb{Q}/\mathbb{Z}$ y $k \geq 0$ (Lemma 6.3); sin embargo, si $\alpha \in \mathbb{Q}/\mathbb{Z}$, la cuasi-convexidad no está garantizada (véase § 6.2.2).

Estos ejemplos se basan en un conocimiento profundo de los conjuntos cuasi-convexos de $\mathbb{Z}$. Realmente no es un hecho sorprendente si tenemos en cuenta la estrecha relación entre $\mathbb{Z}$ y $\mathbb{T}$ manifestada por los isomorfismos topológicos $\mathbb{Z}^\wedge \cong \mathbb{T}$ y $\mathbb{T}^\wedge \cong \mathbb{Z}$. Más precisamente, la clave de nuestras consideraciones se puede encontrar en el estudio de los conjuntos cuasi-convexos *elementales* de $\mathbb{Z}$ que vamos a introducir a continuación.
Dado un grupo topológico, se deduce de la definición que $\chi^{-1}(\mathbb{T}_+)$ es cuasi-convexo para todo $\chi \in G^\wedge$. Utilizaremos el nombre *elementales* para estos conjuntos cuasi-convexos. Ahora, todo conjunto cuasi-convexo se obtiene como intersección de conjuntos cuasi-convexos elementales; más precisamente, la envoltura cuasi-convexa $Q_G(E)$ de cualquier $E \subseteq G$ coincide con la intersección de todos los cuasi-convexos elementales que contienen a E. Ésto explica la importancia de los cuasi-convexos elementales. Los cuasi-convexos elementales de $\mathbb{Z}$ coinciden con la colección de los *conjuntos de Bohr*, que dan lugar a una subbase de $\mathbb{Z}^\#$ (discutimos este hecho en § 5.3). En un cierto sentido, los enteros son probablemente el marco en el que mejor se puede observar la estrecha relación entre la topología de Bohr y la cuasi-convexidad. Ésto nos hizo plantearnos un estudio extensivo de $\mathbb{Z}^\#$ y, más concretamente, de los conjuntos de Bohr, lo cual ha sido desarrollado en § 2.2. El Lemma 2.3 refleja el hecho que la colección de los conjuntos de Bohr posee, en un cierto sentido, un alto nivel de independencia. Por ejemplo, es posible deducir del Lemma 2.3 que

(Corollary 2.7) *Dados dos conjuntos de Bohr $\mathcal{W}_\alpha, \mathcal{W}_\beta \subseteq \mathbb{Z}$, si $\mathcal{W}_\alpha$ está contenido en $\mathcal{W}_\beta$ entonces los dos conjuntos tienen que coincidir.*

Analizamos los conjuntos de Bohr también desde un punto de vista numérico. En el resultado principal de § 2.2.1, concretamente en Theorem 2.9, damos una descripción explícita de un conjunto de Bohr genérico, y mostramos que consiste de bloques (es decir, intervalos) de enteros y "saltos", en términos de su distribución en $\mathbb{Z}$. Además, indicamos descripciones alternativas utilizando las fracciones continuas, y relacionamos los conjuntos de Bohr con una clase más general de conjuntos: la clase de los conjuntos de Hartman y de las sucesiones de Hartman (y, más en general, las sucesiones de Sturmian). De aquí podemos extender las propiedades de los conjuntos de Bohr de $\mathbb{Z}$ a un contexto más amplio: Estocástica, Teoría de Números, Teoría Ergódica y Análisis Armónico.

La estructura en bloques de los conjuntos de Bohr se utiliza en §6.2 para caracterizar aquellos conjuntos cuasi-convexos finitos que están contenidos en algún subgrupo ciclico de $\mathbb{T}$ (véase Lemma 6.8). Como consecuencia, deducimos lo siguiente:

(Theorem 6.9) *Sea* $Q \subsetneq \langle \alpha \rangle$ *un conjunto cuasi-convexo en* $\langle \alpha \rangle \cong \mathbb{Z}$. *Si existe un entero* $m \geq 2$ *tal que* Q *contiene un bloque de longitud* $m + 1$, *entonces:*

(1) la longitud minima de un salto de Q *es* $m - 1$;

(2) $E_{\alpha,r} \subseteq Q$, *donde* $r = \frac{m}{2}$ *si* m *es par y* $r = \frac{m-1}{2}$ *si* m *es impar.*

En particular, los conjuntos cuasi-convexos que no contienen α son *delgados* en el sentido que contienen solo bloques pequeños de longitud 1 ó 2 (Corollary 6.10).
Otra aplicación de Theorem 6.9 es Example 6.14, donde deducimos que la suma de dos conjuntos cuasi-convexos $Q_1, Q_2 \subseteq \langle \alpha \rangle \leq \mathbb{T} \setminus \mathbb{Q}/\mathbb{Z}$ no tiene por qué ser cuasi-convexa (véase también Example 6.6). Por otro lado, probamos en Theorem 6.4 que *si* $Q_i \subseteq \langle \alpha_i \rangle$ *es finito y cuasi-convexo en* $\langle \alpha_i \rangle$ *(o, equivalentemente, en* $\mathbb{T}$*), con* $\alpha_1, \alpha_2, \ldots, \alpha_t \in \mathbb{T}$ *independientes, entonces* $E = Q_1 + Q_2 + \ldots + Q_t$ *es cuasi-convexo en* $\mathbb{T}$. Ésto nos da una herramienta para construir conjuntos cuasi-convexos en $\mathbb{T}$ que no están contenidos en ningún grupo ciclico.

Hablando del grupo $\mathbb{T}$, hay numerosos ejemplos de subconjuntos cuasi-convexos finitos y infinitos no numerables ([5, 4]). En este contexto, Dikranjan propuso la siguiente pregunta ([35]):

Question 0.3 *Es cierto que existe un conjunto cuasi-convexo y numerablemente infinito en* $\mathbb{T}$*?*

En el capitulo 7 contestamos positivamente a esta pregunta. Nuestro resultado se basa en conjuntos cuasi-convexos dados por sucesiones convergentes en $\mathbb{T}$. Por ejemplo:

el subconjunto $\{0\} \cup \{\pm 2^{-2n} \mid n \geq 1\} \subseteq \mathbb{T}$ *es cuasi-convexo en* $\mathbb{T}$.

Más precisamente, probamos lo siguiente (Theorem 7.2):

Dada una sucesión $\underline{a} = (a_n)_n$, *sea*

$$K_{\underline{a}} := \{0\} \cup \{\pm 2^{-(a_n+1)} \mid n \in \mathbb{N}\} \subseteq \mathbb{T}.$$

Si $\underline{a} = (a_n)_n$ *es una sucesión de enteros positivos tales que* $a_{n+1} - a_n > 1$ *para todo* $n \in \mathbb{N}$, *entonces* $K_{\underline{a}}$ *es cuasi-convexo en* $\mathbb{T}$.

Además, $K_{\underline{a}}$ es *hereditariamente cuasi-convexo* en $\mathbb{T}$ en el siguiente sentido: todo subconjunto simétrico y cerrado de $K_{\underline{a}}$ que contiene a $0_{\mathbb{T}}$ es cuasi-convexo (véase Remark 7.45).

Observemos que hemos supuesto $a_0 > 0$ (lo cual es equivalente a $\frac{1}{2} \notin K_{\underline{a}}$). Hay que remarcar que si añadimos el termino $\frac{1}{2}$ a $K_{\underline{a}}$, entonces la cuasi-convexidad de $K_{\underline{a}}$ se pierde: pues, en este caso $Q_{\mathbb{T}}(K_{\underline{a}}) = K_{\underline{a}} \cup (1/2 + K_{\underline{a}})$ (véase Theorem 7.3).

Notemos también que la condición de lacunaridad $a_{n+1} - a_n > 1$ para todo $n \in \mathbb{N}$ no puede se omitida en Theorem 7.2. Pues, ya hemos mencionado Example 5.24 en el que tenemos que si $a_n = n$ para todo $n \in \mathbb{N}$, entonces $K_{\underline{a}}$ es cc-denso en $\mathbb{T}$.

Proponemos una nueva técnica para calcular la envoltura cuasi-convexa $Q_{\mathbb{T}}(K_{\underline{a}})$: se basa en "factorizar" $Q_{\mathbb{T}}(K_{\underline{a}})$ como intersección de "componentes" Q_i que son más prácticas a la hora de calcular. Para definir estas componentes, que son conjuntos cuasi-convexos, utilizamos una partición de la polar de $K_{\underline{a}}$ en subconjuntos más pequeños J_i (véase Notation 7.12). De esta manera, Q_i, al ser la polar de J_i, contiene a $Q_{\mathbb{T}}(K_{\underline{a}})$. La clave de la demostración de Theorem 7.2 reside en el hecho que $Q_{\mathbb{T}}(K_{\underline{a}})$ coincide con la intersección de *únicamente* dos conjuntos de la forma Q_i, concretamente Q_1 y Q_3. Por esta razón, es fundamental caracterizar aquellos $x \in \mathbb{T}$ que están contenidos en Q_1: dicha caracterización se desarrolla en detalle en § 7.2.1 utilizando la representación de $\mathbb{T}$ en términos de "bloques" de potencias negativas de 2. Remarcamos que a la hora de caracterizar Q_1 queda evidente el rol de nuestra hipótesis $a_0 > 0$ en Theorem 7.2 y, por lo tanto, la sutil diferencia entre este resultado y Theorem 7.3.

En § 7.3 presentamos distintas pistas hacía una posible generalización de Theorem 7.2 basada en nuestra nueva técnica de factorización de la envoltura cuasi-convexa.

Introduction

The main topic of this thesis are the weak and strong topologies on abelian groups. The former notion is generally known in the theory of topological abelian groups; the most common example is probably the celebrated Bohr topology. The latter notion is known mainly in the theory of topological vector spaces, as the equally celebrated Mackey topology. This is why, the origin of a "global" study of weak and strong topologies is deeply rooted in the theory of topological vector spaces, where similar notions appeared for the first time (see § for details).
A starting step in the foundation of this kind of study in the framework of topological abelian group was done by Chasco, Martín Peinador and Tarieladze in [23]. In this paper, they show — among other results — that it is natural to restrict to the class of locally quasi-convex groups. Such a class of groups is widely known and used in different instances, but we observed that there is a deep lack of knowledge of the quasi-convex subsets, even in thoroughly studied groups like, for example, the integers or the unitary complex circle.

The main aim of the present thesis is to offer a contribution to the study begun in [23]. This is done by introducing new notions and proving new results that permit to widen the knowledge on the weak and strong topologies in locally quasi-convex groups. In order to develop this line we need a solid background on the Bohr topology and the theory of the quasi-convex subsets of a topological group. The first part of the thesis is dedicated to this trend.

The Mackey topology for abelian groups

The study of dualities of abstract real vector spaces and compatible topologies for them goes back to George Mackey ([60, 61, 62, 63]).
In [60, 63], Mackey introduced what he called a *linear system* as a pair (E, L), where E is a real vector space and L is a vector subspace of the vector space of all linear functionals $l : E \to \mathbb{R}$. Moreover, he called a linear system (E, L) *regular* if L separates the points of E. These objects coincide

with the more recent notion of *(separated) dual pair* or *(separated) duality*. In [61, 62], the linear system (E, L) arising by fixing in E a locally convex topology $\mathcal{T}$ and by taking as L the $\mathcal{T}$-continuous linear functionals is considered, and it is noted that the correspondence between $\mathcal{T}$ and L in general may be the correspondence of many to one.

The existence of the weakest and the strongest locally convex topology in E among all those which give rise to the same regular linear system (E, L) was announced in [61, Theorem 1] and proved in [62, Theorem 5]; nevertheless, Mackey did not fix any notation for these topologies. In our actual language, they are respectively the *weak* (denoted by $\sigma(E, L)$ in [32, 33, 34]) and the *Mackey topology* (denoted by $\tau(E, L)$ in [34]) for the pair (E, L). In [62, p. 524] Mackey noted also that the topologies $\sigma(E, L)$ and $\tau(E, L)$ are *distinct* provided E is an infinite-dimensional normed space and L is the set of all continuous linear functionals on E (still no special notation for the topological dual space of a locally convex space was used).
Mackey also introduced the notion of *Mackey space* (but with another name) as a locally convex space E whose topology coincides with $\tau(E, L)$, whenever L is the set of all continuous linear functionals given on E. In this context, he developed there some of the theory which nowadays we consider classic in the field of locally convex spaces.

Seemingly, the terms “Mackey topology” and “Mackey space” first appeared in [18]. In the same book we can find the notion of *compatible topology*: a (locally convex vector) topology $\mathcal{T}$ in E is said to be *compatible with a duality* (E, L) if L coincides with the set of all $\mathcal{T}$-continuous linear functionals $E \to \mathbb{R}$.

Another proof of the existence and the concrete description of the Mackey topology was achieved R. Arens in [1], in which only [61] is quoted. Namely, for a regular linear system (E, L) it is proved in [1, Theorem 2] that the topology κ in E of uniform convergence on all $\sigma(L, E)$-compact convex subsets of L is the strongest one among the locally convex topologies t for E for which “*the elements of L represent precisely the continuous linear functionals on* E^t”. The combination of [1, Theorem 2] with [61, Theorem 1] is known as *the Mackey Arens Theorem*.

Almost forty years after Mackey's paper [62], Kakol observed in [56] that the local convexity is essential for the validity of the Mackey Arens Theorem. Indeed, he proved that for a duality (E, L) there need not exist the strongest vector topology in E, among all (not necessarily locally convex) vector topologies compatible with (E, L).

Although Mackey also had a deep knowledge of topological groups, he did not set the question in the framework of abelian topological groups. The abstract group dualities and compatible topologies for them were first considered by N. T. Varopoulos in [80]. More precisely, he noticed that a situation similar to vector space dualities in the context of abelian groups can be produced by substituting the reals by the unitary circle group $\mathbb{T}$, and by

considering the group dualities as pairs (G, H), where G is an abelian group and H is a group of homomorphisms $G \to \mathbb{T}$ which separates the points of G. With these new tools, he could introduce the notion of "compatible topology" and "weak topology" in the context of topological groups and group dualities. Moreover, in the same paper he described the locally precompact group topologies on a group G which are compatible with (G, H), for a given dual pair (G, H), and he deduced that $\sigma(G, H)$ is a (precompact) group topology compatible with (G, H).
Varopoulos proved in [80, Proposition 5] that the least upper bound $\upsilon(G, H)$ of all locally precompact group topologies in G which are compatible with (G, H) is also compatible with (G, H). However, the topology $\upsilon(G, H)$ is not a good "group candidate" for the Mackey topology. In fact, a vector space duality (E, L) gives rise in a natural way to the group duality (E, H), where $H = \{\exp(il) \mid l \in L\}$ and — according to [23, Proposition 5.5] — $\upsilon(E, H)$ coincides with the weak topology $\sigma(E, H)$. Therefore, Varopoulos's topology may not coincide with the Mackey topology.
It arises thus the problem of finding the class of compatible group topologies, wider than the class of locally precompact topologies, such that the least upper bound will be again compatible. In [23] it is observed also that the class of MAP compatible topologies, roughly speaking, is too big for these purposes: indeed, its least upper bound need not be compatible.

Since the locally quasi-convex groups are a good group analogue for locally convex spaces, in [23] the authors begun the study of the problem for this class. This paper is the starting point of our work. As a matter of fact, the main question of [23] still remains open, namely:

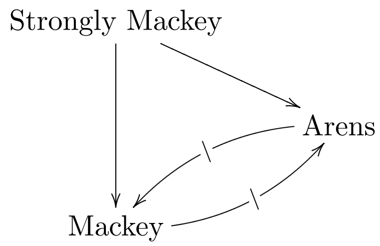
if G is a topological group, is there a strongest topology in G among all those locally quasi-convex topologies that admit the same dual group?

Along § 8 we improve the results of the mentioned paper and also of [16] and [13], which are based upon [23], and we deal with other aspects of what could be called *the Mackey topology for an abelian topological group.*
For a MAP group G and its character group $G^\wedge$, we defined the Mackey topology $\tau(G, G^\wedge)$ as the finest locally quasi-convex topology on G compatible with $(G, G^\wedge)$, whenever it exists (see Definition 8.6). The study of the Mackey topology can be restricted to the class of locally quasi-convex groups. This is explained in § 8.1 using the weak topology on a MAP group G with respect to the class of locally quasi-convex groups.
On the same line as in the case of locally convex spaces, we say that G is a *Mackey group* if the Mackey topology $\tau(G, G^\wedge)$ coincides with the original topology of G (see Definition 8.9). Such a group is characterized by the property that if τ is another locally quasi-convex topology on G with the same dual group as (G, ν), then $\tau \leq \nu$.

Following [23], we consider the locally quasi-convex topology $\tau_g(G, G^\wedge)$ on a topological group G which is defined as the least upper bound of the family of all locally quasi-convex topologies on G compatible with $(G, G^\wedge)$. The existence of the Mackey topology on G is characterized by the compatibility of $\tau_g(G, G^\wedge)$ in the following sense: *there exists the Mackey topology $\tau(G, G^\wedge)$ on G (and $\tau(G, G^\wedge) = \tau_g(G, G^\wedge)$) if and only if $\tau_g(G, G^\wedge)$ is a compatible topology for G* (see Theorem 8.13). This is a sort of *internal* characterization of the existence of the Mackey topology. Nevertheless, we also deal with the possibility of describing the Mackey topology of a locally quasi-convex group G by uniform convergence on a certain family of subsets of the dual $G^\wedge$, as done in the context of locally convex spaces. To this aim, we recall and study the notion of $\mathfrak{S}$*-topology*: given a topological group G, if $\mathfrak{S}$ is a family of non-empty subsets of $G^\wedge$, then $\tau_{\mathfrak{S}}(G, G^\wedge)$ is the topology on G of uniform convergence on the sets $A \in \mathfrak{S}$. For example, the Bohr topology is the topology of uniform convergence on the family of all finite subsets of $G^\wedge$.

The natural candidate $\mathfrak{S}$ to define the Mackey topology of G as an $\mathfrak{S}$-topology is the family $\mathfrak{S}_{qc}$ of all $\sigma(G^\wedge, G)$-compact and quasi-convex subsets of $G^\wedge$. We obtain in this way a topology on G — depending only on the dual pairing $(G, G^\wedge)$ — which is called the $\mathfrak{S}_{qc}$-topology and is denoted by $\tau_{qc}(G, G^\wedge)$. This topology is finer than any other locally quasi-convex compatible topology, and $\sigma(G, G^\wedge) \leq \tau \leq \tau_g(G, G^\wedge) \leq \tau_{qc}(G, G^\wedge)$ whenever (G, τ) is a locally quasi-convex group (see Proposition 8.24). This is a relevant fact since it implies that *if $\tau_{qc}(G, G^\wedge)$ is compatible with $(G, G^\wedge)$, then the Mackey topology of G exists and it coincides with $\tau_{qc}(G, G^\wedge)$* (see Corollary 8.25). This motivates the study of those groups such that $\tau_{qc}(G, G^\wedge)$ is compatible, which we call *Arens groups*. Their importance is given by the fact that they constitute the class of groups for which the counterpart of the Mackey Arens Theorem holds (see Remark 8.26). Note that the property "being an Arens group" depends only on the dual pair $(G, G^\wedge)$ in the following sense: a topological group (G, ν) is Arens if and only if (G, τ_i) is Arens, for every topology τ_i compatible with $(G, G^\wedge)$.
However, an example of [16] alerted us that the topology $\tau_{qc}(G, G^\wedge)$ may not be compatible. A deep study of this fact led us to state that *a Mackey group need not be Arens* (see Theorem 8.61). For this reason we claim that *the Mackey-Arens Theorem cannot be completely imitated in the class of topological groups* since the existence of the Mackey topology does not guarantee that it can be described as the topology of uniform convergence on the $\sigma(G^\wedge, G)$-compact and quasi-convex subsets of the dual group.

If the topology $\tau_{qc}(G, G^\wedge)$ coincides with the original topology of G, then we say that G is *strongly Mackey.* The property of being strongly Mackey is equivalent to the combination of Arens and Mackey. Observe that none of these two properties alone implies strongly Mackey.



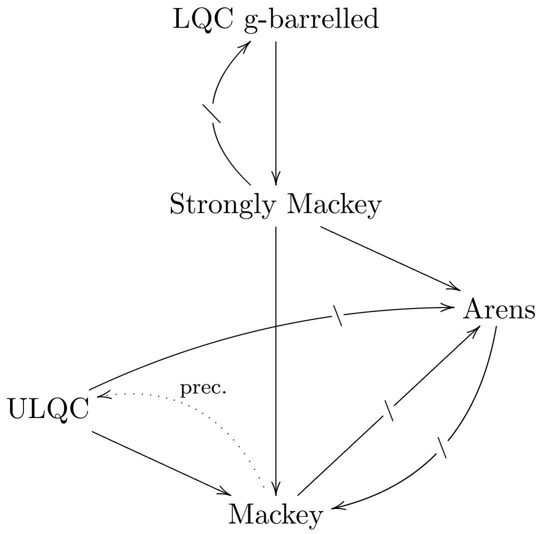
In [23] the authors introduced the class of *g-barrelled* groups and they realized that the counterpart of the Mackey Arens Theorem holds for this class of groups. Actually, it can be proved that *every locally quasi-convex g-barrelled group is strongly Mackey* (Theorem 8.34). We prove however that they do not exhaust the class, showing the existence of *a class of strongly Mackey (precompact) groups which are not g-barrelled* (Theorem 8.62).

Another achievement of this thesis is the determination of another class of g-barrelled groups that was not known in the literature. Recall that an abelian topological group G is said to be *ω-bounded* if every countable subset of G is contained in a compact subgroup of G (see Definition 8.41). Theorem 8.37 states that *every ω-bounded is a g-barrelled group.* In particular, as we observe in Remark 8.48, the class of ω-bounded non-compact groups is not included in the previously known subclasses of g-barrelled groups, that are those mentioned in [23]: the class of *all metrizable hereditarily Baire groups*, of *all separable Baire groups* and of *all Čech-complete groups* (see also Theorem 8.37).

The fact that every locally quasi-convex g-barrelled group is strongly Mackey leads to the following: *given a topological group G, there exists at most one locally quasi-convex compatible topology τ such that (G, τ) is g-barrelled* (Theorem 8.50). This immediately implies the following property of general interest: *there is at most one locally quasi-convex compatible topology which is in the union of the following classes of topological groups: metrizable hereditarily Baire, separable Baire, Čech-complete and ω-bounded.* This generalizes the well known fact proved by Glicksberg in [49] that if G is a locally compact abelian group, there cannot be another locally compact group topology in G with the same dual group.

We leave as an open problem the study of how large is the set of all locally quasi-convex compatible topologies for a given topological group (see Question 8.92 and Problem8.93). So far we have introduced and studied the class ULQC of locally quasi-convex groups that admit only one locally quasi-convex compatible topology, which has to coincide with the Bohr topology. Clearly, such groups are exactly precompact and Mackey. It turns out that every ω-bounded group belongs to this class. Moreover, we consider a class of MAP groups — which we call BTM groups inspired by [16] — characterized by the property to be precompact of weight strictly less than the cardinality of continuum (see Definition 8.54). Then, *every locally*

quasi-convex BTM group is ULQC (Corollary 8.60).
In the class of locally quasi-convex BTM groups we find examples of groups that are Mackey but non-Arens (Theorem 8.61, which generalizes [16, Example 4.2]), and strongly Mackey but non-g-barrelled (Theorem 8.62). By means of these results, we are able to give a complete description of the relations (and non-relations) between all the objects we have considered in the study of the Mackey topology on groups. The situation is resumed in the following diagram:



Barr and Kleisly attempted in [13] to answer the main question of [23] by categorical methods. Although they did not completely achieve their aim, their point of view provided us with a new way to tackling this question. In § 8.4 we settle the natural framework to deal with the Mackey topology in different categories of topological groups.
For every full subcategory $\mathcal{X}$ of the category $\mathcal{MAP}$ of all the maximally almost periodic groups, we define the $\mathcal{X}$-Mackey topology $\tau_{\mathcal{X}}(G, G^{\wedge})$ of $G \in \mathcal{X}$ as the finest $\mathcal{X}$-topology on G among those $\mathcal{X}$-topologies that have the same dual group as G (see Definition 8.64). If $G \in \mathcal{X}$ admits the $\mathcal{X}$-Mackey topology, then it is called a $\mathcal{X}$-pre-Mackey groups, while it is said to be $\mathcal{X}$-Mackey if its original topology coincides with the $\mathcal{X}$-Mackey topology.
Clearly, the Mackey topology of a topological group G as we have described it in its natural context is, in this more general setting, the $\mathcal{LQC}$-Mackey topology, where $\mathcal{LQC}$ is the category of all the locally quasi-convex groups.

Given a pair $(G, \tau), (H, \gamma)$ of $\mathcal{X}$-pre-Mackey groups, it is natural to consider the following property of *coreflectivity* which essentially is the notion of strong topology (see § 1.3.1): whenever $f : (G, \tau) \to (H, \gamma)$ is a continuous homomorphism, then the corresponding homomorphism $\mu f : (G, \tau_{G^{\wedge}}) \to (H, \tau_{H^{\wedge}})$ (algebraically coinciding with f) is continuous in the following diagram:

$$\begin{array}{ccc} (G,\tau) & \xrightarrow{\quad f \quad} & (H,\gamma) \\ {\scriptstyle id_G}\uparrow & & {\scriptstyle id_H}\uparrow \\ (G,\tau_{\mathcal{X}}(G,G^{\wedge})) & \xrightarrow{\mu f} & (H,\tau_{\mathcal{X}}(H,H^{\wedge})) \end{array}$$

The full subcategory of $\mathcal{X}$ having for objects all $\mathcal{X}$-Mackey groups is a coreflective subcategory of $\mathcal{X}$ in the following sense:

(MS 1) every group in $\mathcal{X}$ is a $\mathcal{X}$-pre-Mackey group;

(MS 2) every pair (G,τ) and (H,γ) in $\mathcal{X}$ has the CR-property.

Hence, the subcategory of $\mathcal{X}$-Mackey groups admits a strong topology, namely the Mackey strong topology.

In [13], the authors dealt with the possibility of characterizing those categories admitting a Mackey subcategory. Their main result is the following: $\mathcal{X}$ admits a Mackey subcategory if and only $\mathbb{T}$ is injective with respect to inclusions in $\mathcal{X}$ (see Theorem 8.70). So, the stronger categorical form of the problem of existence of a Mackey topology for a topological group $(G,\tau) \in \mathcal{X}$ is completely translated in categorical terms, i.e. the "Mackey problem" is equivalent to characterize those categories in which $\mathbb{T}$ is an injective object.

We offer a more complete version of this result with the aim of clarifying which is the role of the hypothesis "injectivity of $\mathbb{T}$" on the category $\mathcal{X}$ in order to assure the existence of a Mackey subcategory. We are motivated by the fact that $\mathcal{LQC}$ does not admit a Mackey subcategory since it is well-known that $\mathbb{T}$ is not injective in $\mathcal{LQC}$, but this does not exclude that — at least — every $G \in \mathcal{LQC}$ is $\mathcal{LQC}$-pre-Mackey.
We prove in Theorem 8.75 that a weaker level of injectivity is sufficient to assure condition (MS 1) above, provided $\mathcal{X}$ is closed under arbitrary products and subobjects. This is done in a way that follows the "classical" characterization of the Mackey topology for locally quasi-convex groups. Indeed, we define the topology $\tau_g^{\mathcal{X}}(G,G^{\wedge})$ on a group $G \in \mathcal{X}$ as the supremum of all the $\mathcal{X}$-topologies on G that are compatible with $(G,G^{\wedge})$. Then, for a category $\mathcal{X}$ which is closed under products and subgroups, we have that *G is $\mathcal{X}$-pre-Mackey if and only if $\tau_g^{\mathcal{X}}(G,G^{\wedge})$ is compatible* (see Proposition 8.67). Now, we show that the compatibility of $\tau_g^{\mathcal{X}}(G,G^{\wedge})$ is a condition which is weaker then the injectivity of $\mathbb{T}$ in $\mathcal{X}$ (Proposition 8.74). More precisely, it can be stated that the injectivity of $\mathbb{T}$ in $\mathcal{X}$ is the union of the condition "compatibility of $\tau_g^{\mathcal{X}}(G,G^{\wedge})$" — which yields condition (MS 1) — and another (weaker) instance of injectivity, which yields (MS 2).

In § 8.5 we discuss more aspects of the Mackey topology. In particular, we briefly present some new results concerning properties of permanence of such topology with respect to subgroups, quotients and products. Moreover, inspired by [45], we introduce the class of those precompact groups that are not the Bohr modification of any locally compact group, namely the class of G-groups, and we show that *every ULQC group is a G-group* (Proposition 8.90).

The Bohr topology

Given a topological group G, there exist a compact group bG and a continuous homomorphism $r : G \to \mathrm{b}G$ with the following *universal property*: for any continuous homomorphism $f : G \to K$ with K a compact group, there exists a unique continuous homomorphism $\hat{f}$:b$G \to K$ such that $f = \hat{f} \circ r$. The pair (r, bG) is called the *Bohr compactification* of G (after the Danish mathematician Harald Bohr who studied the special case $G = \mathbb{R}$) and it is unique up to a topological isomorphism. It follows that $r(G)$ is dense in $b(G)$ and when r is an injection, G is said to be *maximally almost periodic* (briefly, *MAP*). In the framework of abstract harmonic analysis these groups are very important and we will restrict ourselves to this class of groups in the sequel. It is known that all compact groups are MAP (Peter-Weyl-van Kampen Theorem). Hence, for a compact group G we have: $G = bG$.

The pre-image topology in G with respect to the homomorphism $r : G \to bG$ is called the *Bohr topology* of G, and the group G endowed with its Bohr topology is frequently denoted by the G^+. It turns out that the Bohr topology of a topological abelian group is the initial topology with respect to all continuous homomorphisms $G \to \mathbb{T}$, i.e. it coincides with the above considered weak topology $\sigma(G, G^\wedge)$. A topological group G is *totally bounded* if and only if $G = G^+$. In case G is a discrete abelian group, G^+ is denoted by $G^\#$, i.e., this is the group G equipped with the "Bohr topology" induced by all homomorphisms to the circle group. Then, the Bohr topology of a discrete abelian group G is the *maximal totally bounded topology on* G.

Two important examples of MAP groups are the classes of locally compact Abelian groups (LCA), and of Hausdorff locally convex spaces (LCS) over $\mathbb{R}$ or $\mathbb{C}$. That LCA is contained in MAP is a fundamental step in the Pontryagin-van Kampen duality theory. On the other hand, the continuous linear functionals of a locally convex space are in one-to-one correspondence with the continuous homomorphisms into the unit circle $\mathbb{T}$. This simple, but important observation connects the study of the Bohr topology with functional analysis, and in particular with the concept of duality and weak topologies on the class LCS of locally convex spaces.

The main trend in this direction is the study of the preservation and reflection of various topological properties by the Bohr functor $G \mapsto G^+$ (among many authors who worked on this topic, we mention Glicksberg, Wu, Comfort, Hernández, Trigos-Arrieta, Remus, Galindo).

For an arbitrary abelian group G the precompact group $G^\#$ is trivially a pre-Mackey group. One of the most challenging problems related with these sort of groups was posed by van Douwen more than fifteen years ago (see [79]):

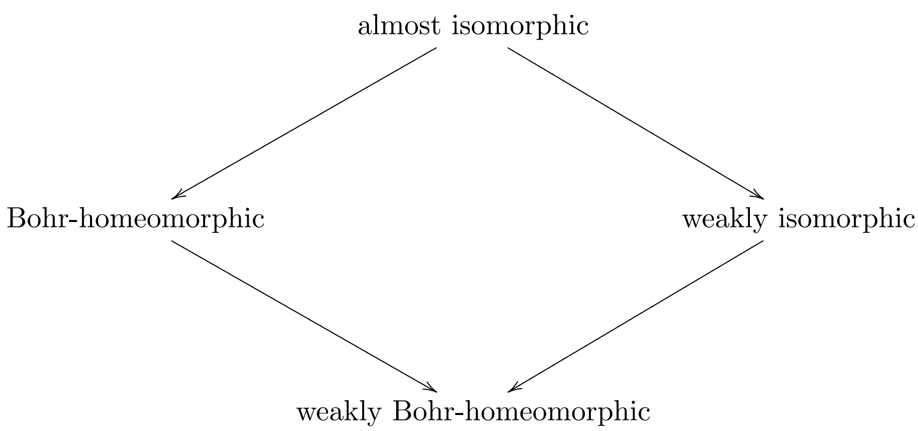
(Question 3.1) *Given two infinite abelian groups G, H of the same cardinality, are $G^{\#}$ and $H^{\#}$ homeomorphic as topological spaces?*

The problem was answered negatively by Kunen ([59]) and independently by Dikranjan-Watson ([43]). Towards the positive direction of van Douwen's problem, results concerning Bohr-homeomorphism have been developed. See, for example, the papers by Hart and Kunen ([51]) and Comfort, Hernández and Trigos-Arrieta ([25]).

In every pair of groups, known to provide a negative solution to van Douwen's homeomorphism problem, one of the groups is not even embeddable into the other under the Bohr topology. This motivates the study of the more general question of *embeddings* in the Bohr topology. In § 3.0.2 we collect and discuss several kinds of possibility of embedding of groups equipped with the Bohr topology that have been introduced in light of Question 3.1. They are defined as follows. Two abelian groups G and H are:

- *almost isomorphic* if they posses isomorphic finite index subgroups ([51]);
- *weakly Bohr-homeomorphic* if there exist topological space embeddings $G^{\#} \hookrightarrow H^{\#}$ and $H^{\#} \hookrightarrow G^{\#}$ ([36, 37]);
- *weakly isomorphic* if each one of these groups has a finite-index subgroup that is isomorphic to a subgroup of the other.

The relation between these notions is given in the next diagram:



The main result of § 3 is the following one:

(Straightening Theorem 3.10) *every continuous function f between two bounded abelian groups equipped with the Bohr topology coincides with a homomorphism when restricted to an infinite subset of the domain.*

Observe that, in some sense, this is the counterpart of the fact that *every homomorphism between groups G and H is continuous with respect to the Bohr topology*, expressing nothing else but the functorial essence of the Bohr topology.
Theorem 3.10 extends the main results of [59, 43] and is based, as well as the counterexamples in [59, 43], on interesting techniques of partition of functions defined over the set of n-tuples of ω into $\mathbb{V}_p^\omega$ (see Theorem 3.29 for details), where, in general, $\mathbb{V}_m^\kappa$ denotes the direct sum of κ copies of $\mathbb{Z}_m$, for every positive integer m and cardinal κ.

The set of Ulm-Kaplansky invariants determines the (bounded) group up to (isomorphism, hence) Bohr-homeomorphism. We apply Theorem 3.10 in § 3.4 with the aim of discussing the following problem:

to what extent do Bohr-homeomorphisms preserve the Ulm-Kaplansky invariants?

Among other results, we establish the equivalence between *weak isomorphisms*, *weak Bohr-homeomorphisms* (and a condition involving algebraic invariants) for two countable bounded groups G, H (see Theorem 3.12). Moreover, we offer examples of how Theorem 3.10 can be used to relate the p-rank of the domain and codomain of Bohr-continuous maps. Indeed we deduce that

(see Corollary 3.13) *Weakly Bohr-homeomorphisms between bounded groups preserve the property of having infinitely many p-torsion elements.*

This observation can be pushed further to answer negatively a question proposed by Givens and Kunen in [47, §6] on the existence of a topological embedding of $(\mathbb{V}_2^{\omega_1})^\#$ into $(\mathbb{V}_2^\omega \times \mathbb{V}_3^{\omega_1})^\#$ (see Corollary 3.14 for details and a more general result). Moreover, we also establish a stronger version of [47, Theorem 5.1] (stating that the property of being bounded is preserved by Bohr-homeomorphisms), namely, *Bohr-homeomorphisms detect the property of having non-trivial p-torsion elements, for every prime p* (Corollary 3.41). We also consider applications in the class of *almost homogeneous groups*. Concretely, we prove that for arbitrary almost homogeneous bounded groups G and H, the properties "almost isomorphic", "weakly isomorphic", "Bohr-homeomorphic" and "weakly Bohr-homeomorphic" are equivalent (Corollary 3.42).

In § 3.4.3 we consider applications of Theorem 3.10 to the theory of Bohr-continuous retracts and cross sections. We give a characterization of the essential ccs-subgroups of bounded abelian groups (Theorem 3.55) and we offer a concise proof of the fact that *for every prime p, the subgroup* $p\mathbb{V}_{p^2}^{\omega} \cong \mathbb{V}_p^{\omega}$ *of* $\mathbb{V}_{p^2}^{\omega}$ *is not a ccs-subgroup of* $\mathbb{V}_{p^2}^{\omega}$ (see Example 3.51). Observe that this was originally proved in [25] with a rather involved proof (consisting of the entire [25, §5]) developing in detail Kunen's approach of normal forms in the case of $\mathbb{V}_{p^2}^{\omega}$. This fact gives an additional interest in the new techniques derived from our Straightening Theorem 3.10 which, more in general, apport a contribution for the solution of the still open van Douwen's problem about retract subgroups in the Bohr topology ([77]):

(see Question 3.43) *is it true that every countable subgroup H of an abelian group G is a retract of G with relation to the Bohr topology?*

The entire § 3 is the object of our publication [28].

In § 4 we define and study a new topology on abelian groups which is a generalization of the Bohr topology.
It is proved in [69] that there exists a *universal second-countable abelian group*, that is, a second-countable topological abelian group $\mathbb{U}$ such that every second-countable topological abelian group H is topologically isomorphic to a subgroup of $\mathbb{U}$. Moreover, we can suppose that $\mathbb{U}$ is divisible since, according to [9, Corollary 3], given a second-countable abelian group K there exists a second-countable divisible abelian group D containing K as a subgroup. So, given a topological abelian group (G, τ), we consider on G the weak $\mathbb{U}$-topology, i.e., the initial topology with respect to $\mathbb{U}$. Now, it is clear that this topology refines the Bohr topology since $\mathbb{T} \leq \mathbb{U}$.
We denote a topological group G equipped with the $\mathbb{U}$-weak topology by $G^{\ddagger}$. Then, *a topological group G is ω-narrow if and only if $G = G^{\ddagger}$* (see §4 for the original definition of ω-narrow group). If the starting group G is discrete, we denote $G^{\ddagger}$ by $G^{\square}$. So, the weak $\mathbb{U}$-topology of a discrete abelian group is the initial topology with respect to the family of all homomorphisms $G \to \mathbb{U}$, hence it is the *maximal ω-narrow topology* on G.
Clearly, $G^{\square}$ and $G^{\#}$ are very close from a functorial point of view. Nevertheless, there are basic differences: unlike the case of the Bohr topology, *every countable subset of $G^{\square}$ is closed and discrete in $G^{\square}$* (Corollary 4.11) and also *C-embedded in $G^{\square}$* (Proposition 4.12).

We prove, among other results, that every uncountable abelian group equipped with the maximal ω-narrow topology is a first category space which is not a P-group.
Recall that a group G is said to be $\mathbb{R}$-*factorizable* if every continuous real-valued function on G admits a decomposition by means of a second-countable group K, a continuous homomorphism $p : G \to K$ and a continuous real-valued function h on K (see Definition 4.18). The class of $\mathbb{R}$-factorizable

groups constitutes a proper subclass of the class of ω-narrow groups, but still it is pretty wide. It contains, among others, all precompact groups, all Lindelöf groups, arbitrary subgroups of σ-compact groups ([73]). During quite a long period of time, only "sporadic" examples of non $\mathbb{R}$-factorizable ω-narrow groups were known (see [71, Example 5.14]). We prove in Theorem 4.19 that every uncountable abelian group admits a group topology with this combination of properties.

The principal results of § 4 are contained in our publication [30].

Locally quasi-convex abelian groups

Soon after Pontryagin's introduction of the *character group* of a topological abelian group, Vilenkin observed that for a normed space considered as an abelian group, the character group and the dual space could be identified (loosely speaking). In this vein, he introduced the *quasi-convex subsets* of a topological abelian group. This notion is inspired by the Hahn-Banach theorem, and it is the counterpart of that of convex subset of a topological vector space. This important tool led him to define the locally quasi-convex groups in [81]. Forty years later, Banaszczyk developed in [10] many properties of locally quasi-convex groups, although the primary objective of that book was the introduction of nuclear groups — a proper subclass of locally quasi-convex groups. The class of Hausdorff locally quasi-convex groups includes LCA and LCS, and it is closed under taking arbitrary products and subgroups.

The above facts, as well as the classical book of Banach "Théorie des opérations linéaires", witness that it is an old project to look at abelian topological groups as a class that embraces the topological vector spaces. Therefore, it is natural to consider the subgroups of topological vector spaces — as expressed in the title of [10] — and also to try to extend properties known to hold for locally convex spaces to the broader class of locally quasi-convex groups. Several authors (Kye, Hernández, Galindo, Martín-Peinador, Chasco, Tarieladze, among others) have worked in this direction and for the time being some big theorems of functional analysis have their counterparts for abelian topological groups ([20, 23, 52], etc.).

For a subset E of G and a subset A of $G^\wedge$, define the *polars*

$$E^\triangleright = \{\chi \in G^\wedge \mid \chi(E) \subseteq \mathbb{T}_+\} \text{ and } A^\triangleleft = \{x \in G \mid \chi(x) \in \mathbb{T}_+ \,, \forall\, \chi \in A\}.$$

A subset E of G is said to be *quasi-convex* if $E = E^{\triangleright\triangleleft}$, i.e., for every $x \in G \setminus E$ there exists $\chi \in E^\triangleright$ such that $\chi(x) \notin \mathbb{T}_+$. In the sequel we denote by $Q_{(G,\tau)}(E)$ the *quasi-convex hull* of $E \subseteq G$, namely the smallest quasi-convex set of (G, τ) containing E (we will simply write $Q_G(E)$ if there is no possibility of confusion).

Although quasi-convex subsets are the corner-stone of the theory of locally quasi-convex groups, we dare to say that their nature is not well-understood. Even for elementary groups like the group of integers $\mathbb{Z}$, or the circle group $\mathbb{T}$, there are no established criteria to recognize quasi-convex subsets.
One of the main aims of this thesis is to develop the theory of quasi-convex sets, with special care to the most unknown cases, namely the *small* (i.e., finite and countably infinite) quasi-convex sets.

Quasi-convexity is usually studied in the class of MAP groups, since they are the groups in which the continuous characters separate points. The reason is clear, according to the following equivalence: *a topological group G is MAP if and only if $\{0_G\}$ is quasi-convex.* A deeper motivation comes from the fact that one can reduce the computation of the quasi-convex hull to the case of precompact topologies (see Remark 5.4). In particular, this also relates quasi-convexity with the notion of precompactness and, consequently, with the Bohr topology. Since $Q_{(G,\tau)}(E) = Q_{(G,\tau^+)}(E)$ for every $E \subseteq G$ and every topology τ on G (see Fact 5.3), it is clear that a good knowledge of the Bohr topology is essential in the study of quasi-convexity.

The following general results are among the few known properties of finite quasi-convex sets:

Theorem 0.4 ([5],[41]) *Let G be a MAP group and F a finite subset. Then:*

(1) $Q_G(F) \subseteq \langle F \rangle$;

(2) $Q_G(F)$ is finite.

Observe that both statements of Theorem 0.4 do not hold in general if F is infinite. For example, if F is the countably infinite set $F := \{\pm 2^{-n} \mid n \in \mathbb{N}\} \subseteq \mathbb{T}$, then $Q_{\mathbb{T}}(F) = \mathbb{T} \nsubseteq \langle F \rangle \cong \mathbb{Z}(2^\infty)$ (this is Example 5.24). Motivated by this example, we introduce in § 5.4 the notion of *qc-dense subset* to describe those sets $E \subseteq G$ such that their quasi-convex hull is the biggest possible, i.e., $Q_G(E) = G$.

Theorem 0.5 ([5]) *Let G be an abelian topological group. Then G is MAP if and only if the set $A_x := \{0, \pm x\}$ is quasi-convex in G for every $x \in G$.*

In particular, this example shows that in some cases the dependence of the quasi-convexity of a set on the topology is only apparent.
Theorem 0.5 cannot be extended to every set of the form $\{0, \pm x, \ldots, \pm kx\}$, even in the case $k = 2$. Indeed, $\{0, \pm\frac{1}{12} + \mathbb{Z}, \pm\frac{2}{12} + \mathbb{Z}\} \subseteq \mathbb{T}$ is not quasi-convex in $\mathbb{T}$ (see Example 6.21). This observation motivates the study of the quasi-convexity of the sets of the form $E_{x,k} := \{0, \pm x, \ldots, \pm kx\} \subseteq \langle x \rangle$ (for some $k \geq 1$) and, on the other hand, the following general definition:

(Definition 5.32) *A subset E of an abelian group G is* unconditionally quasi-convex in G *if E is quasi-convex in every MAP topology on G (this is equivalent to ask it only for precompact group topologies).*

This is the analogue of the *unconditional closedness* introduced by Markov in [64] (see also Definition 5.29). We prove — by means of a result of [42] — that these two notions actually coincide on subgroups: *if H is a subgroup of an infinite abelian group G, then H is unconditionally closed in G if and only if it is unconditionally quasi-convex in G* (Theorem 5.35). In particular, the algebraic characterization of unconditionally closed subgroups given in [42] applies, so the apparent dependence of the quasi-convexity we mentioned above on the topology is translated to algebraic restrictions.

Inspired by the notion of potentially dense set due to Markov (see also § 5.5.2), we also introduce the notion of potentially quasi-convex subset as follows:

(Definition 5.36) *A subset E of an abelian group G is said to be* potentially quasi-convex in G *if there exists a MAP topology τ on G such that E is quasi-convex in (G, τ).*

Clearly, this is the weakest level of quasi-convexity one can expect. Observe that a subset $E \subseteq G$ is potentially quasi-convex in G if and only if it is quasi-convex with respect to the discrete topology of G (see Remark 5.38). Applying Theorem 0.4 *(1)*, it is possible to deduce that *potential quasi-convexity and unconditional quasi-convexity coincide for every finite subgroup $H \leq G$* (Remark 5.37), but yet this does not hold for subsets: indeed, *the set $E = \{0, \pm 1, \pm 3\} \subseteq \mathbb{Z}$ is potentially quasi-convex in $\mathbb{Z}$ but not unconditionally quasi-convex* (Example 5.40). Nevertheless, we show that examples of unconditionally quasi-convex subsets exist in abundance in $\mathbb{Z}$: for instance, $E_{1,2} = \{0, \pm 1, \pm 2\}$ and $U_{1,4} = \{0, \pm 1, \pm 4\}$ are unconditionally quasi-convex in $\mathbb{Z}$ (more in general, see Example 6.34 and Example 6.35). Moreover, these examples can be extended to any MAP group G that possesses a non-torsion element: *given a finite subset E of $\mathbb{Z}$, E is unconditionally quasi-convex in $\mathbb{Z}$ if and only if $xE = \{xe \mid e \in E\}$ is unconditionally quasi-convex in every MAP group G with a non-torsion element x* (this is a consequence of Theorem 6.36 and Theorem 6.33). This motivates the study of the finite quasi-convex subsets of $\mathbb{Z}$.

The study of the finite unconditionally quasi-convex sets is related to other two stronger notions of quasi-convexity that we introduce in this thesis. They are based on the observation that when we apply the definition of quasi-convexity and we separate a (not necessarily finite) given set $E \subseteq \mathbb{Z}$ from a certain $z \in \mathbb{Z} \setminus E$ by means of a continuous character χ of $\mathbb{Z}$, it is extremely important to recognize if the image of E through χ meets the border of $\mathbb{T}_+$, namely $\{-\frac{1}{4}+\mathbb{Z}, \frac{1}{4}+\mathbb{Z}\}$. So we distinguish between two special

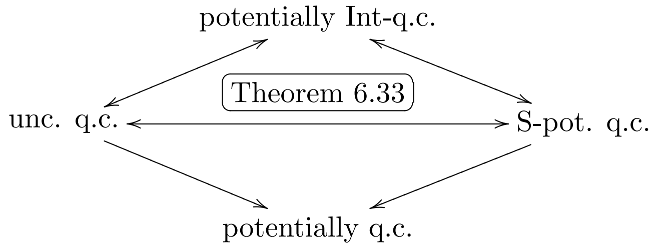
situations:
we say that a subset $E \subseteq \mathbb{Z}$ is

- *potentially Int-quasi-convex in* $\mathbb{Z}$ if for every $e \in \mathbb{Z} \setminus E$ there exists $\chi : \mathbb{Z} \to \mathbb{T}$ such that $\chi(E) \subseteq \mathrm{Int}(\mathbb{T}_+) = (-\frac{1}{4}, \frac{1}{4})$ and $\chi(e) \notin \mathbb{T}_+$ (see also the more general Definition 5.43);

- *S-potentially quasi-convex in* $\mathbb{Z}$ if for every $z \in \mathbb{Z} \setminus E$, there exists $\chi : \mathbb{Z} \to \mathbb{T}$ such that the following conditions hold:

  - $\chi(E) \subseteq \mathbb{T}_+$ and $\chi(z) \notin \mathbb{T}_+$;
  - if $\chi(e_1) = \frac{1}{4} + \mathbb{Z}$ for some $0 < e_1 \in E$, then $\chi(e) \neq \frac{3}{4} + \mathbb{Z}$ for every $0 < e \in E$.

Then, "potential Int-q.c." $\implies$ "S-potential q.c." $\implies$ "potential q.c." (see Lemma 6.30 and Lemma 6.29, respectively).
Theorem 6.33 states that the three strongest notions of quasi-convexity that we have introduced coincide for finite subsets of the integers. Then, the complete diagram of relations for finite subsets of $\mathbb{Z}$ is the following one:



A very useful tool when dealing with finite sets is the following consequence of Theorem 0.4 *(2)*: *if* $E \subseteq H$ *is finite, then there is no difference in calculating the quasi-convex hull of* E *in* G *and in* $\langle E \rangle$ (Corollary 5.19). This simple observation is the key to understand all the examples of finite quasi-convex sets presented in §6 like the following ones:

- $A_{\alpha_1} \cup \ldots \cup A_{\alpha_t} = \{0, \pm\alpha_1, \ldots, \pm\alpha_t\}$ is quasi-convex in $\mathbb{T}$ for every $t \geq 0$ and every linearly independent $\alpha_1, \ldots, \alpha_t \in \mathbb{T}$ (Corollary 6.2);

- $A_\alpha + \ldots k \text{ times} \ldots + A_\alpha = E_{\alpha,k} = \{0, \pm\alpha, \ldots, \pm k\alpha\}$ is quasi-convex in $\mathbb{T}$ for every $\alpha \in \mathbb{T} \setminus \mathbb{Q}/\mathbb{Z}$ and every $k \geq 0$ (Lemma 6.3); however, if $\alpha \in \mathbb{Q}/\mathbb{Z}$ the quasi-convexity is not guaranteed (see § 6.2.2).

These examples are based on a deep knowledge of the quasi-convex sets of $\mathbb{Z}$. This is not surprising, provided the strong relation between $\mathbb{Z}$ and $\mathbb{T}$ given by the topological isomorphisms $\mathbb{Z}^\wedge \cong \mathbb{T}$ and $\mathbb{T}^\wedge \cong \mathbb{Z}$. More precisely, the key of our considerations is given by the elementary quasi-convex subsets of $\mathbb{Z}$ that we introduce as follows.

Given a topological group, it is clear from the definition that $\chi^{-1}(\mathbb{T}_+)$ is quasi-convex for every $\chi \in G^\wedge$. Call these quasi-convex sets *elementary*. Then, every quasi-convex set is an intersection of elementary ones; more precisely, the quasi-convex hull $Q_G(E)$ of any $E \subseteq G$ coincides with the intersection of all the elementary quasi-convex sets containing E. This explains the importance of the elementary quasi-convex sets. The elementary quasi-convex sets of $\mathbb{Z}$ coincide with the collection of the *Bohr sets*, that gives a subbase of $\mathbb{Z}^\#$ (this fact is discussed in § 5.3). As a matter of fact, the integers are probably the best framework in which we can immediately observe how strong is the relation between the Bohr topology and the notion of quasi-convexity. This stimulated a careful study of $\mathbb{Z}^\#$ and the Bohr sets, which has been done in § 2.2. Lemma 2.3 expresses the fact that the collection of Bohr sets has, in some sense, a high level of independency. For example, it can be deduced from Lemma 2.3 that

(Corollary 2.7) *Given two Bohr sets $\mathcal{W}_\alpha, \mathcal{W}_\beta \subseteq \mathbb{Z}$, if $\mathcal{W}_\alpha$ is contained in $\mathcal{W}_\beta$ then they necessarily coincide.*

We also analyze the Bohr sets from a numerical point of view. In the main result of § 2.2.1, namely Theorem 2.9, we give an explicit description of a generical Bohr set, which consists of blocks (i.e., intervals) of integers and "gaps", in terms of its distribution in $\mathbb{Z}$. Moreover, we indicate alternative descriptions by means of continuous fractions, and we also relate the Bohr sets to a more general class of sets, namely the class of Hartman sets and Hartman sequences (and the more general Sturmian sequences). This permits to extend the properties of the Bohr sets in $\mathbb{Z}$ to a much wider context, namely areas as stochastic, number theory, ergodic theory and abstract harmonic analysis.

The block-gap structure of the Bohr sets is employed in §6.2 to characterize those finite quasi-convex sets that are contained in some cyclic subgroup of $\mathbb{T}$ (see Lemma 6.8). As a consequence, we deduce the following:

(Theorem 6.9) *Let $Q \subsetneq \langle\alpha\rangle$ be quasi-convex in $\langle\alpha\rangle \cong \mathbb{Z}$. If there exists an integer $m \geq 2$ such that Q contains a block of length $m+1$, then:*

(1) the minimum length of every gap of Q is $m-1$;

(2) $E_{\alpha,r} \subseteq Q$, where $r = \frac{m}{2}$ if m is even and $r = \frac{m-1}{2}$ if m is odd.

In particular, the quasi-convex sets that do not contain α are *slim* in the sense that they contain only small blocks of length 1 or 2 (Corollary 6.10). Another application of Theorem 6.9 is given in Example 6.14, where we easily deduce that the sum of quasi-convex sets $Q_1, Q_2 \subseteq \langle\alpha\rangle \leq \mathbb{T} \setminus \mathbb{Q}/\mathbb{Z}$ does not need to be quasi-convex (see also Example 6.6). On the other hand, we show in Theorem 6.4 that *if $Q_i \subseteq \langle\alpha_i\rangle$ is finite and quasi-convex in*

$\langle \alpha_i \rangle$ (or, equivalently, in $\mathbb{T}$), for some independent $\alpha_1, \alpha_2, \dots, \alpha_t \in \mathbb{T}$, then $E = Q_1 + Q_2 + \ldots + Q_t$ is quasi-convex in $\mathbb{T}$. This can be also seen as a tool to construct examples of quasi-convex sets of $\mathbb{T}$ that are not contained in a cyclic group.

As far as the group $\mathbb{T}$ is concerned, examples of finite and uncountably infinite quasi-convex sets are well-known ([5, 4]). In this context, Dikranjan asked the following ([35]):

Question 0.6 *Does there exist a countably infinite quasi-convex subset of $\mathbb{T}$?*

In Chapter 7 we answer this question in the positive by means of one of the simplest infinite compact subsets of $\mathbb{T}$, namely a convergent sequence. For instance:

the subset $\{0\} \cup \{\pm 2^{-2n} \mid n \geq 1\} \subseteq \mathbb{T}$ is quasi-convex in $\mathbb{T}$.

More precisely, we prove the following (Theorem 7.2):

For a sequence $\underline{a} = (a_n)_n$, put

$$K_{\underline{a}} := \{0\} \cup \{\pm 2^{-(a_n+1)} \mid n \in \mathbb{N}\} \subseteq \mathbb{T}.$$

If $\underline{a} = (a_n)_n$ is a sequence of positive integers such that $a_{n+1} - a_n > 1$ for every $n \in \mathbb{N}$, then $K_{\underline{a}}$ is quasi-convex in $\mathbb{T}$.

Moreover, $K_{\underline{a}}$ is *hereditarily quasi-convex* in $\mathbb{T}$ in the sense that every symmetric closed subset of $K_{\underline{a}}$ that contains 0_T is still quasi-convex (see Remark 7.45).

Observe that we suppose $a_0 > 0$ (that is, $\frac{1}{2} \notin K_{\underline{a}}$). It is impressing that if we add the term $\frac{1}{2}$ to $K_{\underline{a}}$, then the quasi-convexity of $K_{\underline{a}}$ granted by Theorem 7.2 is lost: indeed, in this case, $Q_{\mathbb{T}}(K_{\underline{a}}) = K_{\underline{a}} \cup (1/2 + K_{\underline{a}})$ (see Theorem 7.3).

Note also that the lacunarity condition $a_{n+1} - a_n > 1$ for every $n \in \mathbb{N}$ cannot be omitted in Theorem 7.2. In fact, we have already mentioned Example 5.24 that states that if $a_n = n$ for every $n \in \mathbb{N}$, then $K_{\underline{a}}$ is qc-dense in $\mathbb{T}$.

We propose a new idea for the computation of the quasi-convex hull $Q_{\mathbb{T}}(K_{\underline{a}})$, namely "factorizing" $Q_{\mathbb{T}}(K_{\underline{a}})$ as an intersection of larger "components" Q_i that are much easier to compute. To define these larger quasi-convex sets Q_i, we use an appropriate partition of the whole polar of $K_{\underline{a}}$ into smaller parts J_i (see Notation 7.12), so that Q_i, being the polar of J_i, contains the $Q_{\mathbb{T}}(K_{\underline{a}})$. The key of the proof of Theorem 7.2 is that the intersection of only *two* sets of the form Q_i, namely Q_1 and Q_3, coincides with $Q_{\mathbb{T}}(K_{\underline{a}})$. For this reason, it is crucial to characterize those $x \in \mathbb{T}$ that are in Q_1; we deal with this in § 7.2.1. Such a characterization is developed gradually by means of the representation of $\mathbb{T}$ in terms of "block" of negative powers of 2. Observe that throughout the characterization of Q_1, it will be clear the role that the assumption $a_0 > 0$ has in Theorem 7.2 and, hence, the intimate difference between this result and Theorem 7.3.

In § 7.3 we present several comments towards a generalization of Theorem 7.2 by means of our techniques of factorization of the quasi-convex hull.

Contents

Chapter 1

Preliminaries

We denote by $\mathbb{R}, \mathbb{Q}, \mathbb{Z}, \mathbb{N}$ respectively the field of real numbers, the field of rationals, the ring of integers and the set of positive integers. The set of all prime numbers is denoted by $\mathbb{P}$. For every $p \in \mathbb{P}$, $\mathbb{F}_p$ is the finite field of order p.
The first infinite cardinal is denoted by ω.

For every $x \in \mathbb{R}$, $[x]$ denotes the integer part of x, and $\{x\}_f := x - [x]$ is the fractional part of x. Then $\{\cdot\}_f : \mathbb{R} \to [0, 1)$.

1.1 Abelian groups

All the groups in this thesis are supposed to be abelian, and we use additive notation. The neutral element of a group G is denoted by 0_G.
Given $g \in G$ and $n \in \mathbb{N}$, the notation ng means $\underbrace{g + g + \cdots + g}_{n \text{ times}}$. Moreover, for a subset $A \subseteq G$, $nA = \{na \mid a \in A\}$.

Given an abelian group G, an element $x \in G$ is said to be *torsion* if there exists an integer $n > 0$ such that $nx = 0$. The *order of* x is the smallest n such that $nx = 0$, and it is denoted by $o(x)$. The group $\mathbb{Z}/m\mathbb{Z}$ is denoted by $\mathbb{Z}_m$, for every $m \geq 2$. Observe that if $p \in \mathbb{P}$, then $\mathbb{Z}_p = \mathbb{F}_p$ as abelian groups.

Given a family $\mathcal{G} = \{G_i\}_{i \in \mathcal{I}}$ of groups, the *(direct) product of* $\mathcal{G}$ is denoted by $\prod_{i \in \mathcal{I}} G_i$. If $\mathcal{I} = \{1, 2, \ldots, n\}$, we also write $G_1 \times G_2 \times \ldots \times G_n$.
Given a cardinal α, the product of α copies of a group G is denoted by G^α, so that $G^\alpha = \prod_{\beta \in \alpha} G_\beta$, where $G_\beta = G$ for each $\beta \in \alpha$.
The *direct sum* of $\mathcal{G}$ is the subgroup of $\prod_{i \in \mathcal{I}} G_i$ of all its elements with finite support. It is denoted by $\bigoplus_{i \in \mathcal{I}} G_i$. Observe that $\bigoplus_{i \in \mathcal{I}} G_i = \prod_{i \in \mathcal{I}} G_i$ whenever $\mathcal{I}$ is finite. Given a cardinal α, the direct sum of α copies of a group G is denoted by $G^{(\alpha)}$.

A non-empty subset M of an abelian group G is said to be *independent* if the equality $\sum_{i=1}^{s} n_i b_i = 0$ for distinct $b_1, \ldots, b_s \in M$ and arbitrary $n_1, \ldots, n_s \in \mathbb{Z}$ implies that $n_1 b_1 = \ldots = n_s b_s = 0$ (for $s > 0$). Equivalently, M is independent if $\langle M \rangle \cong \bigoplus_{m \in M} \langle m \rangle$. We say that M is *maximal independent* if it is not properly contained in any independent subset of G.

A subgroup H of an abelian group G is *essential in* G if for every non-trivial subgroup N of G we have $N \cap H \neq \{0_G\}$. It can be proved that if H is essential in G, then H contains a maximal independent subset of G.
The cardinality of a maximal independent subset of G consisting of elements of infinite order is called *torsion-free rank* of G and is denoted by $r(G)$. Similarly, given a prime number p, the cardinality of a maximal independent subset of G consisting of elements of order p is denoted by $r_p(G)$ and called the *p-rank of G*. Set $G[m] := \{x \in G \mid mx = 0\}$ for every $m \in \mathbb{N}$. Then $r_p(G)$ is the dimension of the vector space $G[p]$ over $\mathbb{F}_p$. Put $Soc(G) := \bigoplus_p G[p]$.

The *p-component of G* (equivalently, *p-primary part of G* or *p-torsion part of G*) is the subgroup G_p of G of all the $g \in G$ such that $p^n g = 0$, for some integer $n \geq 0$.
Put $\mathbb{T} := \mathbb{R}/\mathbb{Z}$. For a prime number p, the Prüfer group $\mathbb{Z}(p^\infty)$ is the p-torsion part of $\mathbb{Q}/\mathbb{Z} \leq \mathbb{T}$. Observe that $\mathbb{T}[m] \cong \mathbb{Z}_m$, for every $m \geq 2$.

A group G is said to be *bounded* if there exists $m \geq 1$ such that $mG = \{0\}$.
For every positive integer m and a cardinal κ, let $\mathbb{V}_m^\kappa := \bigoplus_\kappa \mathbb{Z}_m$. Observe that $\mathbb{V}_p^\kappa$ is the vector space over $\mathbb{F}_p$, for every prime p, and that $G[p] \cong \mathbb{V}_p^{r_p(G)}$. According to Prüfer's theorem (see [44]), every bounded abelian group is a direct sum of cyclic subgroups, so the p-primary part of G can be written as $G_p = \mathbb{V}_p^{\kappa_1} \oplus \ldots \oplus \mathbb{V}_{p^s}^{\kappa_s}$. The cardinals κ_i determine G_p up to isomorphism and are known as *Ulm-Kaplansky invariants* of the group G ([44]). In particular, every infinite bounded group has the form $\bigoplus_{i=1}^{n} \mathbb{V}_{m_i}^{\kappa_i}$, for certain integers $m_i > 0$ and cardinals κ_i.

An abelian group G is *divisible* if for every $x \in G$ and $n \in \mathbb{N}$, there exists $y \in G$ such that $x = ny$. A typical example of divisible group is $\mathbb{T}$. If D is a divisible abelian group with $\alpha = r(D)$ and $\alpha_p = r_p(D)$, then D is isomorphic to the direct sum of α copies of the group $\mathbb{Q}$ of rationals and of α_p copies of the Prüfer group $\mathbb{Z}(p^\infty)$, where p runs over all primes $\mathbb{P}$:

$$D \cong \mathbb{Q}^{(\alpha)} \times \bigoplus_{p \in \mathbb{P}} \mathbb{Z}(p^\infty)^{(\alpha_p)}.$$

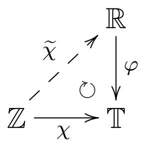
For every abelian group G, there exists a monomorphism $j \colon G \to D$, where D is a divisible abelian group and $j(G)$ is essential in D. The divisible group

D is unique up to isomorphism and is called the *divisible hull* of G. In some sense we can say that D is the smallest divisible abelian group that contains G. The divisible hull of a countable group is still countable.

Proposition 1.1 *Let H be a subgroup of an abelian group G. If f is any homomorphism of H into a divisible abelian group D, then f can be extended to a homomorphism of G into D.*

The following lemma will be used in this thesis.

Lemma 1.2 *Let $\chi : \mathbb{Z} \to \mathbb{T}$. Then there exists $\widetilde{\chi} : \mathbb{R} \to \mathbb{T}$ such that the following diagram is commutative:*



where φ is the natural quotient map $r \mapsto r + \mathbb{Z}$ for every $r \in \mathbb{R}$. Under the additional retrain $\widetilde{\chi}(1) \in [0, 1)$, the lifting $\widetilde{\chi}$ of χ is unique.

Example 1.3 *Since $\mathbb{R}$ does not contain torsion elements, no homomorphism $\chi : \mathbb{Z}_m \to \mathbb{T}$ admits a lifting to $\mathbb{R}$, for every $m \in \mathbb{N}$.*

1.1.1 The group $\mathbb{T}$

The symbol $\mathbb{T}$ usually denotes the unitary circle in the complex plane, and $\mathbb{T}_+ = \{z \in \mathbb{T} \mid \mathrm{Re}(z) \geq 0\}$, where $\mathrm{Re}(z)$ stands for the real part of $z \in \mathbb{C}$. As already mentioned, we identify $\mathbb{T}$ with the quotient group $\mathbb{R}/\mathbb{Z}$ of the reals modulo the integers, through the exponential mapping. Thus, every element of $\mathbb{T}$ has a representative belonging to $[0, 1)$ which is for us the fundamental domain, that is, any $r \in [0, 1)$ is identified with the complex number $e^{2\pi ir}$. However, when it is convenient, the elements $x \in [3/4, 1)$ are replaced by $x - 1$, which obviously are in $[-1/4, 0)$. In some concrete context and when no possibility of confusion may occur, we identify a with $a + \mathbb{Z}$ in order to have a simpler notation.
Via the above identification, $\mathbb{T}_+$ coincides with $[0, 1/4] \cup [3/4, 1)$ or, in the second option, with $[-1/4, 1/4]$.

An element $\alpha \in \mathbb{T}$ is said to be *rational* if $\alpha \in \mathbb{Q}/\mathbb{Z} \leq \mathbb{T}$ and *irrational* if $\alpha \in \mathbb{T} \setminus \mathbb{Q}/\mathbb{Z}$. Put
Observe that if $\alpha_1, \ldots, \alpha_t \in \mathbb{T}$ are linearly independent (for some $t \geq 1$), then $\alpha_1, \ldots, \alpha_t$ are irrational.

1.2 Topological spaces

In this thesis, the discrete topology is denoted by τ_d.

A topological space X is said to be *first-countable* if every $x \in X$ admits a countable base; it is *second-countable* if X has a countable base (i.e., X has countable *weight*: $w(X) \leq \omega$).

A set $A \subset X$ is a G_δ-set if it is a countable intersection of open sets. The space X is a P-space if every G_δ-set is open.

The *pseudocharacter of a point* x in a T_1 space X is the smallest cardinal number of the form $|\mathcal{U}|$, where $\mathcal{U}$ is the family of open subsets of X such that $\bigcap \mathcal{U} = \{x\}$; it is denoted by $\psi(x, X)$. The *pseudocharacter of a* T_1 *space* X is the supremum of all numbers $\psi(x, X)$ for $x \in X$ and it is denoted by $\psi(X)$.

The smallest cardinal number $m \geq \omega$ such that every family of pairwise disjoint non-empty open subsets of X has cardinality $\leq m$ is called *cellularity of* X (or *Souslin number of* X), and it is denoted by $c(X)$.

The *tightness of a point* $x \in X$ is the smallest cardinal number $m \geq \omega$ with the property that if $x \in \overline{C}$, then there exists $C_0 \subseteq C$ such that $|C_0| \leq m$ and $x \in \overline{C_0}$; it is denoted by $t(x, X)$. The *tightness of* X is the supremum of all numbers $t(x, X)$ for $x \in X$ and it denoted by $t(X)$.

A topological space X is said to be a *first category space* if X is the countable union of closed sets with empty interior, and a *second category space* (or *Baire*) if it is not a first category space.

A subset Y of a topological space X is C-*embedded* in X if every continuous real-valued function on Y extends to a continuous function over X.

1.3 Topological groups

A *topological group* is a triplet $(G, \cdot, \tau)$, where $(G, \cdot)$ is an abstract group and τ is a topology defined on G such that the map

$$\begin{aligned} G \times G &\longrightarrow G \\ (x, y) &\longmapsto x \cdot y^{-1} \end{aligned}$$

is continuous. Since we adopt additive notation, we omit the symbol $\cdot$ and we will simply write (G, τ). We refer to such a τ as *group topology*.

All groups are Hausdorff in this thesis.

In a topological group G, the translations are homeomorphisms, therefore every neighborhood V of $g \in G$ is of the form $g + V'$, for some neighborhood V' of 0_G. Given (G, τ), we denote by $\mathcal{N}_{(G,\tau)}(0)$ or simply by $\mathcal{N}_G(0)$ the filter of neighborhoods of 0_G.

We observe that

Lemma 1.4 *Every discrete subgroup of a topological group is closed.*

We consider on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ the quotient topology. This group is important since it is the "dualizing" object for the duality theory of topological abelian groups.
For $x \in \mathbb{T}$, $||x||$ is the distance to the nearest integer, so $0 \leq ||x|| \leq 1/2$. In particular, $\mathbb{T}_+ = \{x \in \mathbb{T} : ||x|| \leq 1/4\}$.

1.3.1 Functorial topologies

The idea that certain group topologies are naturally associated with any abelian group (such as e.g. the p-adic topology) leads to the concept of a functorial topology introduced by B. Charles. It was later the subject of study by D. Boyer, A. Mader, and R. Mines.

A functor $U : \mathbf{A} \to \mathbf{B}$ is forgetful, whenever $Uf = Ug$ for two morphisms $f, g : X \to Y$ in the category $\mathbf{A}$ implies $f = g$. For example, if $\mathbf{A}$ is the category **AGr** (of abelian groups and homomorphisms) or **VS** (of vector spaces and linear maps), then the usual forgetful functor is $U : A \to \mathbf{Set}$, where **Set** is the category of sets, and $U(X)$ is the underlying set of, respectively, the group or vector space. Clearly, this example can be naturally extended to the categories **TAGr** (of topological abelian groups and continuous homomorphisms), **Top** (of topological spaces and continuous maps) or **TVS** (of topological vector spaces and continuous linear maps). Nevertheless, the category **TAGr** admits two other forgetful functors:

- $V : \mathbf{TAGr} \to \mathbf{Top}$, that assigns to every topological group its underlying topological space;
- $V_g : \mathbf{TAGr} \to \mathbf{Grp}$, that assigns to every topological group its underlying abstract group.

Definition 1.5 *A* functorial topology *is a functor* $T \colon \mathbf{AGr} \to \mathbf{TAGr}$ *with* $V_g T = 1_{\mathbf{AGr}}$. *Moreover,*

- *if T sends epimorphisms to open maps, then it is said to be* ideal*;*
- *T is Hausdorff if $T(L)$ is Hausdorff, for every L in* **AGr**.

Proposition 1.6 *Let T be a functorial topology on* **AGr**. *Then:*

(a) $T(G_1 \times G_2) = T(G_1) \times T(G_2)$, *for arbitrary abelian groups* G_1, G_2;

(b) $T(G) \geq \prod_{i\in I} T(G_i)$, *for arbitrary family* $\{G_i\}_{i\in I}$ *of abelian groups and* $G = \prod_{i\in I} G_i$;

(c) $T(H) \geq T(G) \restriction_H$, *for every group abelian G and every subgroup* $H \leq G$;

(d) for every abelian group G and every subgroup N of G, the quotient topology of G/N is finer than $T(G/N)$; equality holds for every G and N if and only if T is an ideal functorial topology.

Given G, the pro-finite topology on G can be also characterized as the initial topology on G with respect to the family of all homomorphism $G \to F$, for every finite group F. In other words, the pro-finite topology belongs also to another wide class of "functorial topologies", namely the class of *initial topologies* or *weak topologies*, i.e., the topologies induced by (continuous) homomorphisms in topological groups. To give a formal definition of initial topology, let us note first that **AGr** can be considered in a natural way as a full subcategory of **TAGr** (identifying each abelian group G with the discrete group (G, τ_d)).

Initial topologies are defined in the larger category **TAGr** as follows. Let $\mathbf{E}$ be a subclass of **TAGr**. Then we consider the functor $T_{\mathbf{E}} : \mathbf{TAGr} \to \mathbf{TAGr}$ that assigns to every topological abelian group G the group $T_{\mathbf{E}}(G)$ with underlying group G and topology the initial topology of the family $\{\mathcal{C}\mathrm{Hom}(G, E) \mid E \in \mathbf{E}\}$. Clearly, one can restrict the functor $T_{\mathbf{E}}$ to the subcategory **AGr**. This will produce a functorial topology in **AGr**.

In what follows, given G in **TAGr** we denote $T_{\mathbf{E}}(G)$ by G° and we call the topology of $T_{\mathbf{E}}(G)$ the **E**-*initial topology* of G, or also the *weak* **E**-*topology*. Moreover, keeping the same terminology as in the case of functorial topologies, we say that:

- the weak **E**-topology is *ideal*, if the functor $T_{\mathbf{E}}$ preserves quotients;
- $T_{\mathbf{E}}$ is *Hausdorff*, whenever $T_{\mathbf{E}}(G)$ is Hausdorff for every Hausdorff group G.

Observe that initial topologies have the following fundamental property: *if $G \in$ **TAGr** and $K \in \mathbf{E}$, then a homomorphism $f : G \to K$ is continuous if and only if $f : G^\circ \to K$ is continuous.* Moreover,

Proposition 1.7 *Let $G, H \in$ **TAGr**. Then:*

1) *A function* $f : G \to H^{\circ}$ *is continuous if and only if* $g \circ f : G \to K$ *is continuous for every* $K \in \mathbf{E}$ *and every continuous homomorphism* $g : H \to K$.

$$\begin{array}{ccc} G & \xrightarrow{f} & H \\ & \searrow^{g\circ f} & \downarrow g \\ & & K \end{array}$$

2) *If* $f : G \to H$ *is a continuous homomorphism, then* $f : G^{\circ} \to H^{\circ}$ *is continuous.*

Corollary 1.8 *Let* G, H *be in* $\mathbf{AGr}$. *Then:*

1) *A function* $f\colon G^{\circ} \to H^{\circ}$ *is continuous if and only if* $g \circ f\colon G^{\circ} \to K$ *is continuous for every* $K \in \mathbf{E}$ *and every homomorphism* $g\colon H \to K$.

2) *If* $f\colon G \to H$ *is a homomorphism, then* $f\colon G^{\circ} \to H^{\circ}$ *is continuous.*

Proposition 1.9 *Fix* $H \leq G$ *in* $\mathbf{TAGr}$. *Then:*

(a) *Suppose that* $\mathbf{E}$ *verifies the following: for every* $A \in \mathbf{E}$, *there exists a divisible* $A' \in \mathbf{E}$ *that contains* A. *Then* $T_{\mathbf{E}}(H) = T_{\mathbf{E}}(G)\restriction_H$;

(b) *if* $T_{\mathbf{E}}$ *is Hausdorff, then* H *is closed in* G°.

In general, the group $T_{\mathbf{E}}(G)$ need not belong to the class $\mathbf{E}$. Nevertheless, this is the case when $\mathcal{C}\mathrm{Hom}(G, E)$ separates the points of G and the class $\mathbf{E}$ is stable under taking isomorphisms, direct products and subgroups (for example, consider the class LQC of locally quasi-convex groups).

An important and simple instance of weak $\mathbf{E}$-topologies is obtained when the class $\mathbf{E} = \{E_i \mid i \in I\}$ is a set, so that one can consider the group $E = \prod_{i\in I} E_i$. Then the weak $\mathbf{E}$-topology and the weak $\{E\}$-topology coincide. In other words, it is not restrictive to take a singleton $\mathbf{E} = \{E\}$. In such a case we say briefly *weak E-topology* instead of weak $\{E\}$-topology.

The *Bohr topology* represents the fundamental example of ideal initial topology. It is the initial topology given by $\mathbf{E} = \{\mathbb{T}\}$. In particular, $T_{\mathbb{T}}\restriction_{\mathbf{AGr}}\colon \mathbf{AGr} \to \mathbf{TAGr}$ is the $\mathrm{Hom}(\ \ ,\mathbb{T})$-topology.
The group (G, τ) equipped with the Bohr topology will be denoted by (G, τ^{+}) or simply by G^{+}. Following van Douwen ([77]), we denote G^{+} by $G^{\#}$ when G is discrete. We say that a topological group G is *Minimally Almost Periodic* (briefly, *MinAP*), if τ^{+} is the indiscrete topology of G.

A topological group (G, τ) is *totally bounded* if it is a fixed point of the Bohr functor, i.e., $T_{\mathbb{T}}((G, \tau)) = (G, \tau)$. Note that a totally bounded group need not be Hausdorff, actually every indiscrete group is totally bounded (while a MinAP group is totally bounded only if it is indiscrete).

A totally bounded group which is Hausdorff is called *precompact*. Observe that G is precompact if and only if $G \hookrightarrow \mathbb{T}^{\mathcal{C}\mathrm{Hom}(G,\mathbb{T})}$, i.e., G is topologically isomorphic to a subgroup of a product of $\mathbb{T}$. In particular, *every totally bounded group is a subgroup of a compact group.*

A *strong topology* is a coreflector in the category of topological abelian group, i.e., a functor $T : \mathbf{TAGr} \to \mathbf{TAGr}$ that assigns every (G, τ) to (G, τ_s) with $\tau_s \geq \tau$.
A typical example is given by the coreflector T_d that sends any topological group G to the (abstract) group (G, τ_d). Let us show an example of a special coreflector in the category of topological vector spaces. Our interest in this example is related to § 8.4.

Example 1.10 *Let us consider the subcategory* $\mathbf{LCVS} \subseteq \mathbf{TVS}$ *of all locally convex vector spaces. Recall that by the Mackey Arens Theorem, for every* $(E, \tau) \in \mathbf{LCVS}$ *there exists the Mackey topology* $\tau(E, E^*)$ *and* $\tau(E, E^*) \geq \tau$*. Moreover, according to [67, Chapter IV, 7.2], if* $f : E \to F$ *is a continuous linear map between two locally convex vector spaces, then also* $f : (E, \tau(E, E^*)) \to (F, \tau(F, F^*))$ *is continuous. Therefore the Mackey functor* $T_M : \mathbf{TVS} \to \mathbf{TVS}$ *that sends a locally convex vector space* E *to the Mackey space* $(E, \tau(E, E^*))$ *is a corflector, i.e., the Mackey topology of a locally convex vector space is a strong topology in this category.*

1.3.2 MAP groups and duality

The following corollary of Proposition 1.1 states that $\mathrm{Hom}(G, \mathbb{T})$ separates the points of G, for every group G.

Corollary 1.11 *If* G *is an abelian group. Then for any distinct* $g, h \in G$ *there exists a homomorphism* $\Phi : G \to \mathbb{T}$ *such that* $\Phi(g) \neq \Phi(h)$*, i.e.,* $\mathrm{Hom}(G, \mathbb{T})$ *separates the points of* G*.*

In particular, $G^{\#}$ is always Hausdorff whenever G is abelian and, hence, the Bohr topology of a discrete group G is precisely the maximal precompact topology on G.

Definition 1.12 *Given a topological group* G*, we say that* G *is* Maximally Almost Periodic *(briefly,* MAP*) if* G^+ *is Hausdorff.*

The class of abelian MAP groups includes, among other, the class of precompact groups, of locally compact abelian (briefly, LCA) groups (according to Peter-Weyl Theorem) and of additive groups of locally convex vector spaces (by Hahn-Banach Theorem).

Observe that we can give a characterization of MAP groups in terms of the separation property of Corollary 1.11.

Remark 1.13 A group G is MAP if and only if for every $0 \neq g \in G$ there exists a continuous character $\chi : G \to \mathbb{T}$ such that $\chi(g) \neq 0$, i.e., $\mathcal{C}Hom(G, \mathbb{T})$ separates the points of G.

For a topological abelian group G, we denote by $G^\wedge$ the dual group of G, that is the group $\mathcal{C}\mathrm{Hom}(G, \mathbb{T})$ equipped with the compact-open topology τ_{co}.

Example 1.14 *The following topological isomorphisms are well-known:* $\mathbb{T}^\wedge = \mathbb{Z}$, $\mathbb{Z}^\wedge = \mathbb{T}$.

A group G is said to be *reflexive* if the canonical map $\alpha_G : G \to G^{\wedge\wedge}$ defined by $x \to (\chi \to \chi(x))$ is a topological isomorphism. The celebrated Pontryagin-van Kampen theorem states that *every locally compact abelian group is reflexive* ([65, 78]). In particular, every discrete group and every compact group is reflexive.

Fact 1.15 *The following properties are equivalent:*

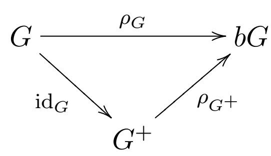
- *G is MAP;*
- *the canonical homomorphism α is injective.*

Given a group G and a subgroup H of $\mathrm{Hom}(G, \mathbb{T})$, we denote by τ_H the initial topology on G with respect to H. Then, τ_H is totally bounded for every $H \leq \mathrm{Hom}(G, \mathbb{T})$. More precisely, the following characterization holds:

Theorem 1.16 ([27]) *A MAP topology τ on a group G is precompact if and only if $\tau = \tau_H$ for some $H \leq \mathrm{Hom}(G, \mathbb{T})$.*

1.3.3 The Bohr compactification

According to Definition 1.12, given a MAP group (G, τ), (G, τ^+) is Hausdorff, so we can consider the completion bG of (G, τ^+). The *Bohr compactification of* (G, τ^+) or, equivalently, *of* (G, τ) is the pair (ρ_G, bG), where $\rho_G : G \to bG$ is a continuous injective homomorphism such that $\rho_{G^+} : G^+ \to bG$ is the immersion of G^+ into bG.

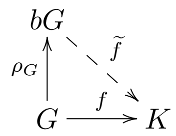


Fact 1.17 *The following properties hold:*

(1) $\rho_G(G)$ *is dense in* bG*;*

(2) bG is a compact group;

(3) for every compact group K and every continuous homomorphism $f : G \to K$ there exists a unique continuous homomorphism $\widetilde{f} : bG \to K$ such that $f = \widetilde{f} \cdot \rho_G$.



Indeed, (1) is immediate, (2) follows from (1) and the fact that every precompact group is a subgroup of a compact group, and (3) holds since such a K is complete, therefore $f : G \to K$ admits an extension to the completions $\widetilde{f} : bG \to K$.

The Bohr compactification of G is uniquely determined by *(1)*, *(2)* and *(3)*, and the Bohr topology of G is the topology induced by its Bohr compactification.

For the following theorem, see [53, Prop. 26.19].

Theorem 1.18 ***Kronecker's Approximation Theorem***
Let G be a MAP group and $\chi : G \to \mathbb{T}$ a (possibly discontinuous) character. For every finite number of elements $g_1, \dots, g_t \in G$ and for every $\varepsilon > 0$ there exists a continuous *character $\xi \in G^\wedge$ such that*

$$||\chi(g_i) - \xi(g_i)|| < \varepsilon, \quad \text{for every } i = 1, \dots, t.$$

This result simply expresses the fact that a topological group G is dense in its Bohr compactification. For reader's convenience, we give a sketch of the proof in the case when G is discrete.

Proof. We want to show that $G^\wedge$ is dense in $\mathrm{Hom}(G, \mathbb{T})$ with the topology of pointwise convergence. Indeed, otherwise $\overline{G^\wedge}$ is a proper closed subgroup in the compact group $\mathrm{Hom}(G, \mathbb{T}) \subseteq \mathbb{T}^G$. Thus there exists a non-trivial character $\chi : \mathrm{Hom}(G, \mathbb{T}) \to \mathbb{T}$ such that $\chi(\overline{G^\wedge}) = 0$. Now, $(G, \tau_d)^{\wedge\wedge} \cong G$ by the Pontryagin duality Theorem (see § 1.3.2), then χ can be identified with a non-null element of G; write $\chi \in G$ with abuse of notation. Now, $\xi(\chi) = 0$ for every $\xi \in G^\wedge$ (i.e., $G^\wedge$ does not separate χ and 0), and this contradicts the fact that G is MAP. So the theorem is proved. QED

We will often make use of the following interpretations of Theorem 1.18.

Corollary 1.19 *Let E be a finite subset of (G, τ). Fix $z \in G \setminus E$, and suppose that there exists a (non-necessarily continuous) character $\chi : G \to \mathbb{T}$ such that $\chi(E) \subseteq Int(\mathbb{T}_+)$ and $\chi(z) \in \mathbb{T} \setminus \mathbb{T}_+$. Then there exists a continuous $\chi' \in E^\triangleright$ such that $\chi'(E) \subseteq Int(\mathbb{T}_+)$ and $\chi'(z) \in \mathbb{T} \setminus \mathbb{T}_+$.*

Corollary 1.20 *Let $\mathcal{I}_1, \ldots, \mathcal{I}_t \subseteq \mathbb{T}$ be open intervals. For every independent (over $\mathbb{Z}$) irrational $\alpha_1, \ldots, \alpha_t \in \mathbb{T}$ there exists an integer n such that $n\alpha_i \in \mathcal{I}_i$ for every $i = 1, \ldots, t$.*

Chapter 2

The Bohr topology

In this chapter we study the Bohr topology in two concrete instances. In § 2.1 we focus on the class of bounded groups; the considerations that we present here will be used in § 3. In § 2.2 we study the case of the group of integers. In particular, we deduce from Lemma 2.3 several results that will be applied in § 5 and § 6 for the study of quasi-convexity.

2.1 The Bohr topology on bounded groups

Recall that every infinite bounded abelian group has the form $\bigoplus_{i=1}^{n} \mathbb{V}_{m_i}^{\kappa_i}$, for certain integers $m_i > 0$ and cardinals κ_i (see § 1.1). For this reason, the study of the Bohr topology of the bounded abelian groups can be restricted on the groups $\mathbb{V}_m^{\kappa}$.
It is clear that the homomorphisms $\mathbb{V}_m^{\kappa} \longrightarrow \mathbb{Z}_m$ suffice to describe the Bohr topology of $\mathbb{V}_m^{\kappa}$, and a typical neighborhood of 0 in $(\mathbb{V}_m^{\kappa})^{\#}$ is a finite-index subgroup of $\mathbb{V}_m^{\kappa}$. So, in this case the Bohr topology coincides with the pro-finite topology. See [38, 40, 59, 43] for more details on $(\mathbb{V}_m^{\kappa})^{\#}$.

In § 3 we study several aspects concerning Bohr-homeomorphisms and Bohr-embeddings, with special care on bounded abelian groups.

2.2 The topological group $\mathbb{Z}^{\#}$

By definition of the Bohr topology, a typical member of a base of $\mathbb{Z}^{\#}$ is of the form

$$\mathcal{U}(\alpha_1, \dots, \alpha_t; \varepsilon) = \{n \in \mathbb{Z} : \|n\alpha_1\| \leq \varepsilon, \dots, \|n\alpha_t\| \leq \varepsilon\}.$$

Such a set is called a *Bohr set*. In particular, a subbase of $\mathbb{Z}^{\#}$ is given by the collection of Bohr sets $\{\mathcal{W}_\alpha\}_{\alpha \in \mathbb{T}}$, where

$$\mathcal{W}_\alpha = \mathcal{U}(\alpha; 1/4) = \{n \in \mathbb{Z} \mid n\alpha \in \mathbb{T}_+\}$$

for every $\alpha \in \mathbb{T}$. Equivalently, $\mathcal{W}_\alpha = \chi_\alpha^{-1}(\mathbb{T}_+)$, where χ_α is the character of $\mathbb{Z}$ defined by $\chi : 1 \to \alpha \in \mathbb{T}$.

Observe that a finite intersection of $\mathcal{W}_\alpha$'s gives arbitrary small $\mathcal{U}(\alpha, \varepsilon)$. Indeed,

Lemma 2.1 *Fix $m \geq 1$ and $\alpha \in \mathbb{T}$. Then $\mathcal{U}(\alpha; 1/4m) = \bigcap_{k=1}^m \mathcal{W}_{k\alpha}$.*

This is a trivial consequence of the following

Fact 2.2 (Lemma 6.3,[5]) *For every $x \in \mathbb{T}$, $||x|| \leq 1/4m$ if and only if $kx \in \mathbb{T}_+$ for every $k = 1, \ldots, m$.*

The following lemma expresses the fact that the $\mathcal{W}_\alpha$'s posses, roughly speaking, a high level of independence. It is Lemma 2.1 in [15]; nevertheless, we offer a shorter and more complete proof.

Lemma 2.3 ([15], Lemma 2.1) *Let $\alpha_1, \ldots, \alpha_t \in \mathbb{T}$ be independent (over $\mathbb{Z}$), and let $m \in \mathbb{N}$. If*

$$\bigcap_{i=1}^{t} \bigcap_{k=1}^{m} \mathcal{W}_{k\alpha_i} = \mathcal{U}(\alpha_1, \ldots, \alpha_t; 1/4m) \subseteq \mathcal{W}_\beta \tag{2.1}$$

for some $\beta \in \mathbb{T}$, then $\beta = \sum_{i=1}^t k_i \alpha_i$ with $k_i \in \mathbb{Z}$ such that $\sum_{i=1}^t |k_i| \leq m$.

Proof. Let $H = \langle \alpha_1, \ldots, \alpha_t \rangle$. We show first that $\beta \in H$. Assume for a contradiction that $\beta \notin H$. Choose $z \in \mathbb{T} \setminus \mathbb{T}_+$ such that:

- $z = 1/2 + \mathbb{Z}$, if $\langle \beta \rangle \cap H = 0$;
- $tz = 0$, if t is the smallest positive integer such that $t\beta \in H$.

There exists a (discontinuous) character $\psi : \mathbb{T} \to \mathbb{T}$ such that $\psi(\alpha_i) = 0$ for $i = 1, 2, \ldots, t$ and $\psi(\beta) = z$. By Corollary 1.19, there exists a continuous character $n \in \mathbb{Z} = \mathbb{T}^\wedge$ such that $nk\alpha_i \in \mathbb{T}_+$ for $i = 1, 2, \ldots, t$ and $k = 1, 2, \ldots, m$, and also $n\beta \notin \mathbb{T}_+$. Then $n \in \mathcal{U}(\alpha_1, \ldots, \alpha_t; 1/4m)$, but $n \notin \mathcal{W}_\beta$, a contradiction. Hence $\beta \in H$.

So $\beta = \sum_{i=1}^t k_i \alpha_i$ with $k_i \in \mathbb{Z}$. To prove that $\sum_{i=1}^t |k_i| \leq m$, argue by contradiction. Let $\mathbb{T}_m := \{x \in \mathbb{T} : \|x\| \leq 1/4m\}$ (see § 5.3) and $C = \mathbb{T}_m \times \ldots \times \mathbb{T}_m$ (t-times). Consider the continuous homomorphism $h : \mathbb{T}^t \to \mathbb{T}$ defined by $h(x_1, \ldots, x_t) = \sum_{i=1}^t k_i x_i$. We intend to show that $h(\mathrm{Int}(C)) \not\subseteq \mathbb{T}_+$. Indeed, if $h(\mathrm{Int}(C)) \subseteq \mathbb{T}_+$ then by closedness of $\mathbb{T}_+$ also $h(C) \subseteq \mathbb{T}_+$. Now, for every $1 \leq i \leq t$, take $z_i = \frac{1}{4m} + \mathbb{Z}$ if $k_i \geq 0$ and $z_i = -\frac{1}{4m} + \mathbb{Z}$ in the other case. Then $(z_1, \ldots, z_t) \in C$ but $h(z_1, \ldots, z_t) \notin \mathbb{T}_+$, a contradiction. This claim produces a point $(z_1, \ldots, z_t) \in \mathrm{Int}(C)$ such that $h(z_1, \ldots, z_t) \notin \mathbb{T}_+$. Consider the (discontinuous) character $\psi : \mathbb{T} \to \mathbb{T}$ such that $\psi(\alpha_i) = z_i$ (hence, $\psi(k\alpha_i) = kz_i \in \mathrm{Int}(\mathbb{T}_+)$ for $k = 1, 2, \ldots, m$) and also $\psi(\beta) = \sum_i k_i z_i \in \mathbb{T} \setminus \mathbb{T}_+$. By Corollary 1.19 there exists $n \in \mathbb{Z}$ such that $nk\alpha_i \in \mathbb{T}_+$ for $k = 1, 2, \ldots, m$ and $i = 1, 2, \ldots, t$, while $n\beta \in \mathbb{T} \setminus \mathbb{T}_+$. This contradicts our assumption (2.1). QED

Corollary 2.4 *If $\alpha_1 \ldots \alpha_t \in \mathbb{T}$ are independent, then for $\beta \in \mathbb{T}$*

$$\bigcap_{i=1}^{t} \mathcal{W}_{\alpha_i} \subseteq \mathcal{W}_\beta \iff \beta = \pm\alpha_i, \text{ for some } i \in \{1, 2, \ldots, t\}.$$

Another corollary of Lemma 2.3:

Corollary 2.5 *If $\alpha \in \mathbb{T}$ is irrational, then for $m \in \mathbb{N}$ and $\beta \in \mathbb{T}$*

$$\bigcap_{k=1}^{m} \mathcal{W}_{k\alpha} \subseteq \mathcal{W}_\beta \iff \beta = \pm k\alpha, \text{ for some } k \in \{1, 2, \ldots, m\}.$$

We observe here that the previous corollary may fail if we consider an intersection of $\mathcal{W}_{\alpha_i}$'s where the α_i's are not all consecutive multiples of one of them. En example is the following one. Consider $\alpha \in \mathbb{T}$ such that α is irrational. Then $\mathcal{W}_{2\alpha} \cap \mathcal{W}_{3\alpha} \cap \mathcal{W}_{4\alpha} \subseteq \mathcal{W}_\alpha$ (this can be deduced from Example 6.11 and Lemma 5.20) but $\alpha \notin \{\pm 2\alpha, \pm 3\alpha, \pm 4\alpha\}$.

Remark 2.6 Note that this is a strong property of *independence*: if an elementary quasi-convex set $\mathcal{W}_\beta$ contains a finite intersection $\bigcap_{i=1}^m \mathcal{W}_{\alpha_i}$ of elementary quasi-convex sets, then it contains (actually coincides with) one of them. More generally, if $\bigcap_{i=1}^m \mathcal{W}_{\alpha_i} \subseteq \bigcap_{j=1}^s \mathcal{W}_{\beta_j}$, then $\{\beta_1, \ldots, \beta_s\} \subseteq \{\pm\alpha_1, \ldots, \pm\alpha_m\}$. This holds true when the elements $\alpha_1, \ldots, \alpha_t$ are either independent, or all consecutive multiples of one of them (i.e., $\{\alpha_1, \ldots, \alpha_t\} = \{\alpha_1, 2\alpha_1, \ldots, t\alpha_1\}$).

The following can be deduced from Corollary 2.5:

Corollary 2.7 *For $\alpha, \beta \in \mathbb{T} \setminus \{0\}$ the following properties are equivalent:*

(a) $\beta = \pm\alpha$;

(b) $\mathcal{W}_\alpha \subseteq \mathcal{W}_\beta$;

(c) $\mathcal{W}_\beta \subseteq \mathcal{W}_\alpha$;

(d) $\mathcal{W}_\alpha = \mathcal{W}_\beta$.

Proof. According to the Corollary 2.5, *(b)* is equivalent to *(a)* when $\alpha \notin \mathbb{Q}/\mathbb{Z}$. Now assume that $\alpha \in \mathbb{Q}/\mathbb{Z}$. Suppose *(b)* holds. Write $\alpha = \frac{b}{a} + \mathbb{Z}$, with a, b coprime positive integers, $0 < b < a$. Then $a\mathbb{Z} \subseteq \mathcal{W}_\alpha$, hence $a\mathbb{Z} \subseteq \mathcal{W}_\beta$ by our assumption *(b)*. Since $\mathbb{T}_+$ contains no non-trivial subgroups, $a\beta = 0$.

Since α has period precisely a, this implies $\beta = k\alpha$ for some $k \in \mathbb{Z}$. If $k \neq \pm 1$, choose an element $z \in \mathbb{T} \setminus \mathbb{T}_+$ such that $az = 0$. Now define a (discontinuous) character $\psi : \mathbb{T} \to \mathbb{T}$ such that $\psi(\alpha) = 0$ and $\psi(\beta) = z$. Apply Corollary 1.19 and find a continuous character $n \in \mathbb{Z} = \mathbb{T}^\wedge$ such that $n\alpha \in \mathbb{T}_+$ and $n\beta \notin \mathbb{T}_+$. Therefore, $n \in \mathcal{W}_\alpha$ and $n \notin \mathcal{W}_\beta$, a contradiction. Hence $k = \pm 1$. This proves that *(b)* is equivalent to *(a)*. Analogously one can prove that *(a)* is equivalent to *(c)* as well. This shows that *(a)* $\Leftrightarrow$ *(b)* $\Leftrightarrow$ *(c)* $\Leftrightarrow$ *(d)* as *(d)* coincides with the conjunction of *(b)* and *(c)*. QED

2.2.1 Description of $\mathcal{W}_\alpha$

First, consider the case in which α is rational. It is easy to see that *the set $\mathcal{W}_\alpha$ is periodic if and only if α is rational.* Let us consider some examples.

Example 2.8 *We identify $\mathcal{W}_{1/n}$ with $\mathcal{W}_{1/n+\mathbb{Z}}$. Then:*

- $\mathcal{W}_{1/2} = 2\mathbb{Z}$;
- $\mathcal{W}_{1/3} = 3\mathbb{Z}$;
- $\mathcal{W}_{1/4} = \{0, 1, 3\} + 4\mathbb{Z}$;
- $\mathcal{W}_{1/5} = \{0, 1, 4\} + 5\mathbb{Z}$;
- $\dots$

In the general case, $\mathcal{W}_\alpha$ can be described as follows. Let $\alpha \in \mathbb{T} \setminus \{0\}$. Define $a_\alpha = a := \left[\frac{1}{2||\alpha||}\right] \geq 1$ and, for every integer n, $k_n := \left[\frac{n+\frac{1}{4}}{||\alpha||}\right]$. Then Theorem 2.9 states that a controls the structure of $\mathcal{W}_\alpha$ in the following sense: $\mathcal{W}_\alpha$ consists of blocks of integers J_n with cardinality a or $a+1$, and the length of every gap between two blocks J_n and J_{n+1} can be $a-1$, a or $a+1$.

Theorem 2.9 *Let $\alpha \in \mathbb{T} \setminus \{0\}$. Then $\mathcal{W}_\alpha = \bigcup_{n\in\mathbb{Z}} J_n$, where, for every $n \in \mathbb{Z}$,*

$$J_n = \begin{cases} \{k_n - a, k_n - a + 1, \dots, k_n\}, & \text{if } (k_n - a)||\alpha|| - n + \frac{1}{4} \geq 0; \\ \{k_n - a + 1, k_n - a + 2, \dots, k_n\}, & \text{otherwise.} \end{cases}$$

Moreover, if L_n is the gap between J_n and J_{n+1}, then $|L_n| \in \{a-1, a, a+1\}$.

Proof. Fix $\alpha \neq 0$ in $\mathbb{T}$ and consider the character $\chi_\alpha : \mathbb{Z} \longrightarrow \mathbb{T}$ which takes 1 to α. By definition,

$$\mathcal{W}_\alpha = \chi_\alpha^{-1}(\mathbb{T}_+) = \chi_\alpha^{-1}\left(\left[-\frac{1}{4}, \frac{1}{4}\right] + \mathbb{Z}\right) = \bigcup_{n\in\mathbb{Z}}\left(\left[\frac{n-\frac{1}{4}}{||\alpha||}, \frac{n+\frac{1}{4}}{||\alpha||}\right] \cap \mathbb{Z}\right).$$

Define $I_n := \left[\frac{n-\frac{1}{4}}{||\alpha||}, \frac{n+\frac{1}{4}}{||\alpha||}\right] \cap \mathbb{Z}$ for every integer n. We want to show that $I_n = J_n$ for every $n \in \mathbb{Z}$.
Fix n. Start noting that $k_n \in I_n$ (indeed, $k_n \geq \frac{n+\frac{1}{4}}{||\alpha||} - 1 = \frac{n+\frac{1}{4}-||\alpha||}{||\alpha||} \geq \frac{n-\frac{1}{4}}{||\alpha||}$). Moreover, $k_n - a + 1$ belongs to I_n too (and, consequently, $\{k_n - a + 1, k_n - a + 2, \ldots, k_n\} \subseteq I_n$). To see it, it is sufficient to show that $(k_n - a + 1)$ is close enough to $\frac{n+\frac{1}{4}}{||\alpha||}$, and this is true according to the following calculation:

$$\frac{n+\frac{1}{4}}{||\alpha||} - (k_n - a + 1) = \frac{n+\frac{1}{4}}{||\alpha||} - k_n + k_n - (k_n - a + 1) <$$

$$< 1 + k_n - (k_n - a + 1) = 1 + a - 1 = a \leq \frac{1}{2||\alpha||} = \frac{n+\frac{1}{4}}{||\alpha||} - \frac{n-\frac{1}{4}}{||\alpha||}.$$

Now, $k_n - a \in I_n$ if and only if $k_n - a \geq \frac{n-\frac{1}{4}}{||\alpha||}$, that is to say $(k_n - a)||\alpha|| - n + \frac{1}{4} \geq 0$. This proves that $J_n \subseteq I_n$. For the converse, let $b \in I_n$. Of course, $b \leq k_n$ by definition of k_n. Now, $k_n - a - 1$ does not belong to I_n (because $\frac{n+\frac{1}{4}}{||\alpha||} - (k_n - a - 1) \geq \frac{n+\frac{1}{4}}{||\alpha||} - \frac{n+\frac{1}{4}}{||\alpha||} + a + 1 > \frac{1}{2||\alpha||}$), therefore $b \geq k_n - a$. If $b \geq k_n - a + 1$ then $b \in J_n$. If $b = k_n - a$, then $k_n - a \geq \frac{n-\frac{1}{4}}{||\alpha||}$ (because $b \in I_n$), thus $(k_n - a)||\alpha|| - n + \frac{1}{4} \geq 0$ and $b \in J_n$.
To conclude, note that $L_n = \left(\frac{n+\frac{1}{4}}{||\alpha||}, \frac{n+\frac{3}{4}}{||\alpha||}\right) \cap \mathbb{Z} = \left(\frac{1}{2||\alpha||} + \left(\frac{n-\frac{1}{4}}{||\alpha||}, \frac{n+\frac{1}{4}}{||\alpha||}\right)\right) \cap \mathbb{Z}$. Therefore $|L_n| \leq |J_n|$. Moreover, $\{k_n + 1, \ldots, k_n + a - 1\} \subseteq L_n$ (it follows from the fact that $k_n + 1 \in L_n$ by definition of k_n, and $k_n + a - 1 \leq \frac{n+\frac{1}{4}}{||\alpha||} + \frac{1}{2||\alpha||} - 1 = \frac{n+\frac{3}{4}}{||\alpha||} - 1 < \frac{n+\frac{3}{4}}{||\alpha||}$), then $|L_n| \geq a - 1$. QED

The following example shows that $a - 1$ is actually a possible value for $|L_n|$ (and therefore there is no "symmetry" between the cardinality of the *blocks* J_n and of the *gaps* L_n).

Example 2.10 *Take α such that $||\alpha|| = \frac{1}{4m}$ for a certain integer m. Then $\langle\alpha\rangle = \mathbb{Z}_{4m} \leq \mathbb{T}$, $a = 2m$, $|J_n| = a + 1$ and $|L_n| = a - 1$ for every n.*

In the more complex case, namely when $\alpha \in \mathbb{T}$ is irrational, there are other interesting ways to describe $\mathcal{W}_\alpha$, for example using continuous fractions.
First, let us note that there is a natural relation between the expansion in continuous fractions of an irrational α and $\mathcal{W}_\alpha$ (observe that we identify $\mathcal{W}_\alpha$ with $\mathcal{W}_{\alpha+\mathbb{Z}}$). Recall that for $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, its expansion in continued fractions is given by integers $a_0, a_1 \ldots$ such that

$$\alpha = [a_0; a_1, a_2, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ldots}}$$

where $a_0 \geq 0$ and $a_i > 0$ for every $i \geq 1$. Define two increasing sequences $(p_n)_n$ and $(q_n)_n$ in the following recursive way:

$$p_{-2} = 0, \quad p_{-1} = 1, \quad p_i = a_i p_{i-1} + p_{i-2} \quad \text{for} \quad i \geq 0;$$

$$p_{-2} = 1, \quad q_{-1} = 0, \quad q_i = a_i q_{i-1} + q_{i-2} \quad \text{for} \quad i \geq 0.$$

Then $\left(\frac{p_n}{q_n}\right)_n$ is a sequence of rational numbers — called *the sequence of convergence of* α — such that

$$[a_0; a_1, \dots, a_n] = \frac{p_n}{q_n}.$$

Now, it is well-known (for example, see [57]) that for every $n \geq 0$ we have $|\alpha q_n - p_n| < 1/q_n$. Now, recalling that $(q_n)_n$ is increasing, take n_0 such that $1/q_{n_0} < 1/4$. Then $\{q_n \mid n_0 \leq n\} \subseteq \mathcal{W}_\alpha$. Furthermore, it has been proved in [15, Theorem 1*] that there exists m_0 such that the set $\{rq_n \mid 1 \leq r \leq a_{n+1}, m_0 \leq n\}$ is contained in $\mathcal{W}_\alpha$.

Continued fractions are also involved in more sophisticated descriptions of $\mathcal{W}_\alpha$. We are not going to enter into details, but we point out that this is related to a more general class of sets that includes the $\mathcal{W}_\alpha$'s, namely the class of *Hartman sets*. Here we give a brief description.
Let C be a compact monothetic group. A subset $M \subseteq C$ is said to be a *continuity set* if its topological boundary has Haar measure equal to 0. For example, every open subset of $\mathbb{T}$ is a continuity set for being a countable union of disjoint open intervals.
Now, let $g \in C$ be such that $\overline{\langle g \rangle} = C$ and define the translation T by $T : x \mapsto x + g$, for every $x \in C$ (observe that the hypothesis $\overline{\langle g \rangle} = C$ is equivalent to say that T is an *ergodic* translation). Then $H = \{k \in \mathbb{Z} \mid T^k(0_C) \in M\}$ is a Hartman set. Of course, to see that any elementary quasi-convex set of $\mathbb{Z}$ is a Hartman set just take $C = \mathbb{T}$, $M = \mathbb{T}_+$ and $T : x \mapsto x + \alpha$.
Given a Hartman set H, we can define a *Hartman sequence* $\mathbf{a}_H = (a_k)_{k=-\infty}^{\infty} : \mathbb{Z} \to \{0, 1\}$ setting $a_k = 1$ if $k \in H$ and $a_k = 0$ otherwise. Hartman sets and Hartman sequences (that are a generalization of Sturmian sequences) are connected to various areas as stochastic, number theory, ergodic theory and abstract harmonic analysis. For more details on Hartman sequences, see [68, 70, 82].

Chapter 3

Homeomorphisms in the Bohr topology

A pair of abelian groups G and H are said to be *Bohr-homeomorphic* if $G^{\#}$ and $H^{\#}$ are homeomorphic as topological spaces ([36, 37]). Clearly, Bohr-homeomorphic groups have the same cardinality, and isomorphic abelian groups are always Bohr-homeomorphic.

The following natural question was proposed by E. K. van Douwen (Question 80, [79]):

Question 3.1 *Are abelian groups of the same cardinality always Bohr-homeomorphic?*

The problem was answered negatively by Kunen ([59]) and independently by Dikranjan-Watson ([43]). The counterexamples will be discussed below.

Towards the positive direction of van Douwen's problem, Kunen and Hart established the following

Fact 3.2 ([51], Lemma 3.3.3) *Every infinite abelian group is Bohr-homeomorphic to its subgroups of finite index.*

They introduced the notion of almost isomorphism as follows: two bounded abelian groups G and H are *almost isomorphic* if they have isomorphic finite index subgroups. Obviously, Fact 3.2 implies that

Corollary 3.3 *Almost isomorphic abelian groups are always Bohr-homeomorphic.*

The following example by Comfort, Hernández and Trigos-Arrieta shows that Bohr-homeomorphic groups need not be almost isomorphic.

Example 3.4 ([25]) $\mathbb{Q}$ *and* $\mathbb{Q}/\mathbb{Z} \times \mathbb{Z}$ *are Bohr-homeomorphic.*

3.0.2 Embeddings in the Bohr topology

In every pair of groups, known to provide a negative solution to van Douwen's homeomorphism problem, one of the groups is not even embeddable into the other under the Bohr topology. This motivates the study of the more general question of *embeddings* in the Bohr topology.

An important step in the embedding problem for the Bohr topology was achieved by Givens and Kunen ([47]). Making use of chromatic numbers of hypergraphs, they proved the following theorem characterizing those abelian groups that admit a topological embedding into the group $(\mathbb{V}_p^\kappa)^\#$, for an infinite cardinal κ and a prime number p:

Theorem 3.5 ([47], Corollary 1.4) *Fix a cardinal $\kappa \geq \omega$, let G be an abelian group of cardinality κ and let p be a prime number. Then the following properties are equivalent:*

1. *$G^\#$ is homeomorphic to a subset of $(\mathbb{V}_p^\kappa)^\#$;*

2. *G and $\mathbb{V}_p^\kappa$ are Bohr-homeomorphic;*

3. *G and $\mathbb{V}_p^\kappa$ are almost isomorphic.*

In the same paper it is also proved that *if there exists a topological space embedding $G^\# \hookrightarrow H^\#$ and H is a bounded abelian group, then also G must be bounded* ([47], Theorem 5.1).

The above results compared to the original van Douwen's homeomorphism problem justify the following notion ([36, 37]):

Definition 3.6 *Two abelian groups G, H are said to be* weakly Bohr-homeomorphic *if there exist topological space embeddings $G^\# \hookrightarrow H^\#$ and $H^\# \hookrightarrow G^\#$.*

In order to provide instances when two groups are weakly Bohr-homeomorphic, we need also the following:

Definition 3.7 *We say that two abelian groups G and H are* weakly isomorphic *if each one of these groups has a finite-index subgroup that is isomorphic to a subgroup of the other.*

The next lemma trivially follows from Fact 3.2 and from the fact that the Bohr topology is functorial.

Lemma 3.8 *Weakly isomorphic abelian groups are weakly Bohr-homeomorphic.*

According to Example 3.4, the converse implication fails ($\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z} \times \mathbb{Z}$ are Bohr-homeomorphic, and yet these groups are not weakly isomorphic).

For a bounded group G, we denote by $exp(G)$ the exponent of G (i.e., the smallest positive integer k with $kG = 0$). The *essential order* $\mathrm{eo}(G)$ of G is the smallest positive integer m with mG finite ([47]). Then, $G = F \times H$, with $\mathrm{eo}(H) = exp(H) = m$ and F finite.
The following result is Theorem 5.3 in [47]. It gave a complete solution of the embedding problem in the case of countable bounded abelian groups:

Theorem 3.9 [47] *For countably infinite bounded abelian groups G and H, there exists an embedding $G^\# \longrightarrow H^\#$ if and only if* $\mathrm{eo}(G)|\mathrm{eo}(H)$.

Observe that in § 3.4.2 we deduce this theorem from a more general result.

It is not clear whether a similar simple criterion for the existence of a continuous injective map $G^\# \longrightarrow H^\#$ can be formulated in terms of $\mathrm{eo}(G)$ and $\mathrm{eo}(H)$ (see Question 3.66).

3.0.3 Main results

The non-homeomorphism results from [59, 43] are based on interesting techniques of partition of functions defined over the set of n-tuples of ω into $\mathbb{V}_p^\omega$ (see Theorem 3.29 for details).

In [43] the authors present a theorem concerning partition of functions defined over the set of four-tuples of a sufficiently large cardinal κ into $\mathbb{V}_3^\kappa$. This was pushed further on in [38] and [40] to a more general situation; in this framework the first idea of the "Straightening Theorem" (Theorem 3.10) was born (see also [36, 37]). We need the following definitions before we formulate our main theorem.

If κ is a cardinal and $A \subseteq \kappa$, we denote by $[A]^n$ the set of all the subsets of A with n elements, where n is a positive integer. For every $m \in \omega$ greater than 1, denote by $\mathcal{B}_m^\kappa$ the canonical base of the group $\mathbb{V}_m^\kappa$. We will often consider κ naturally embedded in $\mathbb{V}_m^\kappa$ via the map which enumerates the elements of $\mathcal{B}_m^\kappa$. Also $[\kappa]^n$ embeds in $\mathbb{V}_m^\kappa$, for every positive integer n: to see it, fix n and write an element $a \in [\kappa]^n$ as $a = (a_1, \ldots, a_n)$ (where $a_1 < a_2 < \ldots < a_n$), and consider the embedding $\iota_{\kappa,m}^{(n)} : [\kappa]^n \longrightarrow \mathbb{V}_m^\kappa$ defined by $\iota_{\kappa,m}^{(n)}(a) = a_1 + a_2 + \ldots + a_n$.

Briefly, we prove that *every continuous function f between two bounded abelian groups $G^\#$ and $H^\#$ coincides with a homomorphism when restricted to an infinite subset of the domain*:

Theorem 3.10 (Straightening Theorem) *Let J be a bounded abelian group, $m > 1$ and $\pi : (\mathbb{V}_m^\omega)^\# \longrightarrow J^\#$ a continuous function with $\pi(0) = 0$. Then there exist an infinite subset $A \subseteq \omega$ and a homomorphism $\ell : \mathbb{V}_m^\omega \longrightarrow J$ such that*

$$\pi \restriction_{[A]^m} = \ell \restriction_{[A]^m}$$

and, consequently,

$$\pi([A]^m) \subseteq J[m].$$

Moreover, if π is an embedding then ℓ can be chosen to be a monomorphism.

As an immediate corollary of Theorem 3.10, one can see that there exists no injective continuous functions from $(\mathbb{V}_p^\omega)^\#$ into $(\mathbb{V}_q^\omega)^\#$ when p and q are distinct prime numbers (see Corollary 3.17 for a more precise result). This fact, established first in [59], answers negatively Question 3.1.

It must be emphasized that Theorem 3.10 depends strongly on the domain $\mathbb{V}_m^\omega$ of the function we want to "straighten". Indeed, according to Example 3.4, there exists a topological embedding $j : (\mathbb{Q}/\mathbb{Z})^\# \longrightarrow \mathbb{Q}^\#$, even if it is easily seen that it can coincide with the restriction of a homomorphism $\ell : \mathbb{Q}/\mathbb{Z} \longrightarrow \mathbb{Z}$ to a non-empty subset A of $\mathbb{Q}/\mathbb{Z}$ only if A is a singleton (as j is injective while ℓ is necessarily the zero homomorphism).

In § 3.1 we list some definitions and basic properties that will be useful further. In § 3.2 we introduce and study *(homogeneous) derived forms* and *normal forms*. In Proposition 3.27 we study the continuity of functions in normal form with respect to the Bohr topology, and in Proposition 3.28 we characterize the continuous homogeneous derived 1-ary forms. We also make use of the following fundamental result by Kunen ([59], see Theorem 3.29): *given a function π defined over the l-tuples of ω into $\mathbb{V}_p^\omega$, there exists a restriction of the domain on which π is in normal form.* These results are essential ingredients of the proof of Theorem 3.10, which is the object of § 3.3.

In § 3.4 we give several applications. According to Theorem 3.9, two countably infinite bounded abelian groups G and H are weakly Bohr-homeomorphic if and only if $\mathrm{eo}(G) = \mathrm{eo}(H)$ (see also Theorem 3.12 for further comments). This motivates the following algebraic condition for bounded abelian groups G, H presenting a further weakening of the notion of "weak isomorphism":

(B) $\mathrm{eo}(G) = \mathrm{eo}(H)$ and $r_p(G) = r_p(H)$ for all p with $\max\{r_p(G), r_p(H)\} \geq \omega$.

The relation between (B) and the previously introduced notions is the following:

almost isomorphic $\Rightarrow$ *weakly isomorphic* $\Rightarrow$ *weakly Bohr-homeomorphic* $\Rightarrow$

$$\Rightarrow \text{(B)} \Rightarrow \mathrm{eo}(G) = \mathrm{eo}(H).$$

The first and the last implications are trivial, the second is Lemma 3.8, the third one is proved in Theorem 3.16. Note that the last implication is not invertible in the uncountable case (e.g., the pair $G = \mathbb{V}_2^{\omega} \times \mathbb{V}_3^{\omega_1}$, $H = \mathbb{V}_3^{\omega} \times \mathbb{V}_2^{\omega_1}$ satisfies $\mathrm{eo}(G) = \mathrm{eo}(H)$ but does not satisfy (B)).

Our aim is to establish conditions that imply the validity of the converses of some of the above implications. That will lead, in particular, to a complete classification (up to (weak) Bohr homeomorphism) of the bounded abelian groups in terms of their essential order in the case of countable bounded abelian groups (see Lemma 3.11 and Theorem 3.12) and the groups of the form $\mathbb{V}_m^{\kappa}$ (see Corollary 3.17).

The next lemma entails that the invariant $\mathrm{eo}(G)$ alone allows for a complete classification (up to weak isomorphism) of all countable bounded abelian groups (the proof can be found in §3.4).

Lemma 3.11 *Let G and H be countable bounded abelian groups. Then G is weakly isomorphic to H if and only if* $\mathrm{eo}(G) = \mathrm{eo}(H)$.

This lemma gives the equivalence (a) $\iff$ (c) in the next theorem, providing a more complete result:

Theorem 3.12 *For countable bounded abelian groups G and H, the following properties are equivalent:*

(a) G and H are weakly isomorphic;

(b) G and H are weakly Bohr-homeomorphic;

(c) $\mathrm{eo}(G) = \mathrm{eo}(H)$.

The implication (a) $\Longrightarrow$ (b) follows from Lemma 3.8; (b) $\Longrightarrow$ (c) follows from Theorem 3.9.

The following result (see § 3.4.2 for the proof) shows how Theorem 3.10 can be used to relate the p-rank of the domain and codomain of Bohr-continuous functions.

Corollary 3.13 *Let $f : G^{\#} \longrightarrow H^{\#}$ be a continuous injective function between abelian groups. If H is bounded and $r_p(G)$ is infinite for a certain prime p, then $r_p(H) \geq r_p(G)$.*

As an immediate corollary we obtain

Corollary 3.14 *If p and q are distinct prime numbers, then there is no continuous injective map $(\mathbb{V}_p^{\omega_1})^{\#} \to (\mathbb{V}_p^{\omega} \times \mathbb{V}_q^{\omega_1})^{\#}$.*

This answers negatively the question proposed by Givens and Kunen ([47, § 6]) on the existence of a topological embedding of $(\mathbb{V}_2^{\omega_1})^{\#}$ into $(\mathbb{V}_2^{\omega} \times \mathbb{V}_3^{\omega_1})^{\#}$.

Here is another immediate consequence of corollary 3.13:

Corollary 3.15 *If G and H are bounded abelian groups and there exist continuous injective maps $G^{\#} \longrightarrow H^{\#}$ and $H^{\#} \longrightarrow G^{\#}$, then $r_p(G) = r_p(H)$ for every prime p such that at least one of the cardinals $r_p(G), r_p(H)$ is infinite.*

Theorem 3.16 *If there exists an embedding $G^{\#} \to H^{\#}$, then $\mathrm{eo}(G)|\mathrm{eo}(H)$ and $r_p(G) \leq r_p(H)$ if $r_p(G)$ is infinite. In particular, weakly Bohr-homeomorphic abelian groups G and H satisfy condition* (B)*.*

The proof of this theorem can be found in § 3.4.2.

Theorem 3.10 implies a stronger version of [47, Theorem 5.1], namely, *Bohr-homeomorphisms preserve the property of having non-trivial p-torsion elements, for every prime p* (Corollary 3.41).

We show that $\mathbb{V}_n^\kappa$ and $\mathbb{V}_m^\kappa$ are not weakly Bohr-homeomorphic where κ is any infinite cardinal and $n \neq m$ (see Corollary 3.17). This implies that the invariant $\mathrm{eo}(G)$, along with the cardinality $|G|$, allows for a complete classification, up to Bohr-homeomorphism, of all infinite abelian groups of the form $\mathbb{V}_m^\kappa$:

Corollary 3.17 *The following properties are equivalent:*

(1) $\mathbb{V}_n^\kappa$ *and* $\mathbb{V}_m^\kappa$ *are Bohr-homeomorphic;*

(2) $\mathbb{V}_n^\kappa$ *and* $\mathbb{V}_m^\kappa$ *are weakly Bohr-homeomorphic;*

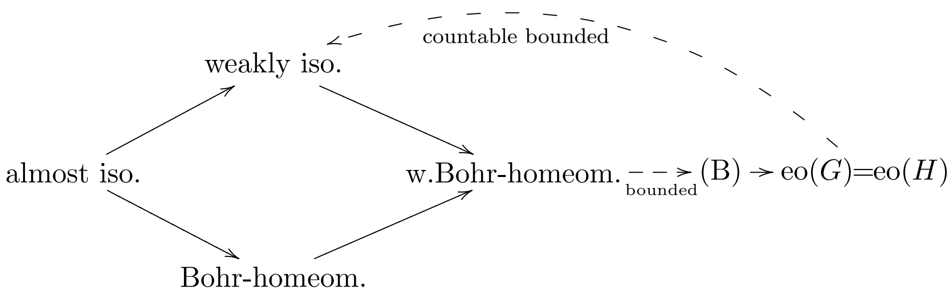
(3) $\mathbb{V}_n^\kappa$ *and* $\mathbb{V}_m^\kappa$ *are isomorphic as topological groups;*

(4) $n = m$.

The implications (4) $\to$ (3) $\to$ (1) $\to$ (2) are trivial. The implication (2) $\to$ (4) follows from Lemma 3.37 (or can be directly obtained from Theorem 3.12).

The groups $\mathbb{V}_n^\kappa$ are *almost homogeneous* (see §3.4 for the definition) of a very specific form. Nevertheless, Corollary 3.17 remains true also for the larger class of all almost homogeneous groups (see Corollary 3.42).

In the next diagram we collect all the implications – those with solid arrows are always valid, the ones with dotted arrows are valid in some specific class of abelian groups indicated *explicitly* on the diagram ((countable) bounded groups). All items but "$\mathrm{eo}(G) = \mathrm{eo}(H)$" are equivalent for almost homogeneous bounded groups (Corollary 3.42).



We also discuss applications of Theorem 3.10 to the theory of Bohr-continuous retracts and cross sections, giving a characterization of the essential ccs-subgroups of bounded abelian groups (Theorem 3.55) and offering a contribution for the solution of van Douwen's question about retract subgroups in the Bohr topology.

The last section of this chapter is dedicated to questions and open problems related mainly to the possibility of inverting the implications in the above diagram.

3.1 Preliminaries

3.1.1 Independence

We start by defining independence for various objects related to an abelian group H: subsets, families of subgroups, (families of) maps into H.

Definition 3.18 *Let H be an abelian group.*

(a) A family $\{H_i\}_{i\in I}$ of subgroups of H is independent *if their sum is direct.*

(b) A subset M of H is independent *if the family $\{\langle x\rangle \mid x \in M\}$ of (cyclic) subgroups of H is independent.*

(c) Let $f : I \longrightarrow H$ be an injective map, where I is a non-empty set, and $m \in \mathbb{N}$.

- *f is said to be* independent *if the set $\{f(s) \mid s \in I\} \subseteq H$ is independent;*
- *f is said to be* normalized of period m *(briefly, m-*normalized*) if $o(f(i)) = m$ for every $i \in I$.*

When f is normalized of period m we write shortly $o(f) = m$.

One can extend the idea of independency also to the case of a finite family $\mathcal{F} = \{f_j\}_{j\in J}$ of injective functions $f_j : I_j \longrightarrow H$, $j \in J$, as follows:

- $\mathcal{F}$ is *independent* if the set $\{f_j(s) \mid s \in I_j,\ j \in J\}$ is independent;
- $\mathcal{F}$ is *weakly independent* if the family of subgroups $\{\langle f_j(I_j)\rangle \mid j \in J\}$ is independent.

In particular, the functions f_j of an independent family $\mathcal{F}$ are independent and pairwise distinct, while both these properties may fail if the family $\mathcal{F}$ is only weakly independent.

Remark 3.19 We observe the following.

(a) Let $\mathcal{H} = \{h_1, \dots, h_t\}$ be a family of functions $h_k : I \longrightarrow H$. Then:

- if $h_1, \dots, h_t$ are independent, then $\mathcal{H}$ is independent if and only if the family $\{\langle h_k(I)\rangle \mid k = 1, \dots, t\}$ of subgroups of H is independent;
- a linear combination $h = c_1h_1 + \ldots + c_mh_m : I \longrightarrow H$ is independent whenever $\mathcal{H}$ is independent and not all c_ih_i are zero. Moreover, if h_i is m_i-normalized for every i, then for every $a \in I$ the period of $h(a)$ is the least common multiple n of the periods $o(c_kh_k(a))$'s and thus it does not depend on a; therefore h is n-normalized.

(b) Let $h_1, h_2 : I \to \mathbb{V}^{\kappa}_{p^m}$ be maps such that h_1 is p^m-normalized and independent while $h_2(I) \subseteq p\mathbb{V}^{\kappa}_{p^m}$. Then $h_1 + h_2$ is p^m-normalized and independent.

3.1.2 The spaces $\mathcal{D}^{(n)}_{A,m}$

Throughout this chapter, for $m|n$ and $A \subseteq \kappa$ with κ a certain infinite cardinal, the set $[A]^n \cup \{0\}$ will be equipped with a topology depending on m. The space obtained in this way will be denoted by $\mathcal{D}^{(n)}_{A,m}$. All points of $[A]^n$ are isolated, while the filter of neighborhoods of 0 in $\mathcal{D}^{(n)}_{A,m}$ has as a base the collection of the sets

$$\mathcal{V}_m(\mathcal{I}) = \{0\} \cup \{(a_1, \dots, a_n) \in [A]^n : |I_j \cap (a_1, \dots, a_n)| \equiv_m 0 \quad \forall\, j = 1, \dots, t\},$$

where $\mathcal{I} = \{I_1, \dots, I_t\}$ runs over all finite partitions of A.

It turn out that this topology on $[A]^n \cup \{0\}$ is induced by the canonical embedding $\iota^{(n)}_{A,m} : [A]^n \cup \{0\} \longrightarrow (\mathbb{V}^{|A|}_m)^\#$. Actually, for any independent m-normalized map $h : A \longrightarrow \mathbb{V}^{|A|}_m$, the map $\lambda_h : [A]^n \cup \{0\} \longrightarrow (\mathbb{V}^{|A|}_m)^\#$, defined by $\lambda_h(0) = 0$ and

$$\lambda_h(a) = h(a_1) + \ldots + h(a_n)$$

for every $a = (a_1, \dots, a_n) \in [A]^n$, induces an embedding of $\mathcal{D}^{(n)}_{A,m}$ in $(\mathbb{V}^{|A|}_m)^\#$. For further use, extend the definition of λ_h for $h = 0$ putting $\lambda_0 = 0$.

See [47] for a nice alternative definition of a class of topologies of $[A]^n \cup \{0\}$ which makes use of "chromatic numbers". The above topology corresponds, in the notation of [47], to the vector $b = (1, 1, \dots, 1) \in \mathbb{Z}(m)^n$.

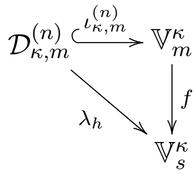
More details on $\mathcal{D}^{(n)}_{A,m}$, as well as many examples, can be found in [38]. For the sake of completeness we include two topological properties of the map λ_h which were presented in a similar way in [47, Lemma 3.3, Lemma 3.5] in the case $\kappa = \omega$.

Lemma 3.20 *Let $h : \kappa \longrightarrow \mathbb{V}_s^\kappa$ be an independent s-normalized function and let m, n be positive integers such that $m|n$. Then:*

(1) $\lambda_h : \mathcal{D}_{\kappa,m}^{(n)} \longrightarrow (\mathbb{V}_s^\kappa)^\#$ is continuous $\iff$ $s|m$;

(2) λ_h is an embedding $\iff$ $m = s$.

Proof. Observe that, since h is independent and s-normalized, the image $h(\kappa)$ generates a subgroup H of $\mathbb{V}_s^\kappa$ which is isomorphic to $\mathbb{V}_s^\kappa$. By *(a)* of Proposition 1.9, the subgroup topology of $H \cong \mathbb{V}_s^\kappa$ coincides with its Bohr topology. Therefore, we can replace H by the whole group $\mathbb{V}_s^\kappa$. Furthermore, we can suppose — up to an automorphism of $\mathbb{V}_s^\kappa$ — that the image of h is precisely the canonical base $\mathcal{B}_m^\kappa$ of $\mathbb{V}_s^\kappa$. So h coincides with the embedding $\iota_{\kappa,s}^{(1)}$ of κ into $\mathbb{V}_s^\kappa$. Therefore λ_h is the immersion $\iota_{\kappa,s}^{(n)}$.
(1) Assume that $s|m$ and note that the map $f : \mathbb{V}_m^\kappa \longrightarrow \mathbb{V}_s^\kappa$, defined by $f(x) = (m/s)x$ is a homomorphism, hence it is continuous in the Bohr topology. Therefore, the continuity of λ_h follows easily from the definition of the topology of $\mathcal{D}_{\kappa,m}^{(n)}$ and the following commutative diagram:



Let us suppose now that $\lambda_h : \mathcal{D}_{\kappa,m}^{(n)} \longrightarrow (\mathbb{V}_s^\kappa)^\#$ is continuous and check that $s|m$. Assume $n > m$. Choose a partition of κ into two infinite sets $\kappa = I_1 \cup I_2$ and pick a net $x_d = (a_1, \dots, a_m) \in [I_1]^m$ such that $x_d \longrightarrow 0$ in $\mathcal{D}_{\kappa,m}^{(m)}$. Analogously, take another net $y_d = (a_{m+1}, \dots, a_n) \in [I_2]^{n-m}$ which converges to 0 in $\mathcal{D}_{\kappa,m}^{(n-m)}$. So we have a net $z_d = x_d + y_d$ converging to 0 in $\mathcal{D}_{\kappa,m}^{(n)}$, and by the continuity of λ_h also $\lambda_h(z_d)$ converges to 0 in $\mathbb{V}_s^\kappa$. Using the fact that h is independent, build a character $\chi : \langle h(I_1) \cup h(I_2) \rangle \longrightarrow \mathbb{Z}_s$ defined by

$$\chi = \begin{cases} 1, & \text{on } h(I_1); \\ 0, & \text{on } h(I_2). \end{cases}$$

Now, $\chi(\lambda_h(z_d)) = \chi(\lambda_h(x_d)) = m$ for every d, and from the continuity of χ it turns out that $m \longrightarrow 0$ in $\mathbb{Z}_s$. Hence $s|m$.
If $n = m$ then repeat the same proof with $I_1 = \kappa$ ($I_2 = \emptyset$) and $x_d = z_d$.
(2) Suppose that $\lambda_h : \mathcal{D}_{\kappa,m}^{(n)} \longrightarrow (\mathbb{V}_s^\kappa)^\#$ is an embedding. Since $s|m$ and $m|n$, we have $s|n$, so by the definition of the topology of $\mathcal{D}_{\kappa,s}^{(n)}$ we can consider it as naturally embedded in $\mathbb{V}_s^\kappa$ with $\mathcal{D}_{\kappa,s}^{(n)} = \lambda_h(\mathcal{D}_{\kappa,m}^{(n)})$. Hence we can consider the composition $g : \mathcal{D}_{\kappa,s}^{(n)} \to (\mathbb{V}_m^\kappa)^\#$ of the inverse map $\lambda_h^{-1} : \mathcal{D}_{\kappa,s}^{(n)} \to \mathcal{D}_{\kappa,m}^{(n)}$ and the embedding $\iota_{\kappa,m}^{(n)} : \mathcal{D}_{\kappa,m}^{(n)} \hookrightarrow (\mathbb{V}_m^\kappa)^\#$. Its continuity, granted by the

fact that λ_h is an embedding, yields $m|s$ by item *(1)*. Therefore, $m = s$. QED

3.2 Normal forms and their continuity

The definition of normal forms follows the line of [59]. For the sake of completeness we give here all details.

Let A be a subset of an infinite cardinal κ and let $l \geq r$ be positive integers. For every $I = \{i_1, \ldots, i_r\} \in [l]^r$ define the *restriction* p_I:

$$p_I : [A]^l \longrightarrow [A]^r; \quad p_I(\alpha) = \alpha \restriction_I = (a_{i_1}, \ldots, a_{i_r})$$

$$\text{for any } \alpha = (a_1, \ldots, a_l) \in [A]^l.$$

Fix $I \in [l]^r$. For a function $w : [A]^r \longrightarrow H$ defined from $[A]^r$ into an abelian group H, we define the *simple r-ary derived form of w (related to I)* as follows:

$$\widetilde{w}^{(I)} : [A]^l \longrightarrow H,$$

$$\widetilde{w}^{(I)} = w \circ p_I.$$

In other words, we set $\widetilde{w}^I(\alpha) = w(\alpha \restriction_I)$ for every $\alpha \in [A]^l$. Note that, apart from trivial cases, w and $\widetilde{w}^{(I)}$ have different domains (the arity of $\widetilde{w}^{(I)}$ is always bigger or equal to the arity of w) but the images $\widetilde{w}^{(I)}([A]^l)$ and $w([A]^r)$ always coincide.

A *homogeneous derived r-ary form of w* is a linear combination of simple derived r-ary forms of w. More precisely, put $s = \binom{l}{r}$ and let $[l]^r = \{I_1, \ldots, I_s\}$ list the family of all r-element subsets of l. If $\underline{c} = (c_1, \ldots, c_s) \in \mathbb{Z}^s$, we denote by $\widetilde{w}_{\underline{c}}$ the homogeneous derived r-ary form of w related to $\underline{c}$:

$$\widetilde{w}_{\underline{c}} = \sum_{j=1}^{s} c_j \widetilde{w}^{(I_j)} : [A]^l \longrightarrow H.$$

Example 3.21 *If $\underline{c} = (0, \ldots, 0, 1, 0, \ldots, 0)$, where 1 appears at the jth position, $\widetilde{w}_{\underline{c}}$ coincides with the simple derived form $\widetilde{w}^{(I_j)}$.*

Definition 3.22 *For a finite family $\mathcal{W} = \{w_1, \ldots, w_n\}$ of functions, where $w_i : [A]^{r_i} \longrightarrow H$ for every $i = 1, 2, \ldots, n$, and for $l \geq \max\{r_i : i = 1, 2, \ldots, n\}$, a* derived form of $\mathcal{W}$ *is a sum of homogeneous derived forms of w_i:*

$$\widetilde{\mathcal{W}} = \sum_{i=1}^{n} \widetilde{w_i}_{\underline{c}^i} : [A]^l \longrightarrow H,$$

where $\underline{c}^i \in \mathbb{Z}^{\binom{l}{r_i}}$.

Adopting the terminology from [59], we say that a map $f : [A]^l \to H$ is in *normal form* if f coincides with $\widetilde{\mathcal{W}}$ for some independent family of functions $\mathcal{W} = \{w_1, \ldots, w_n\}$ and coefficients $\underline{c^i} \in \mathbb{Z}^{\binom{l}{r_i}}$ as in Definition 3.22. We call the forms $\widetilde{w_i}_{\underline{c^i}}$ *homogeneous components* of f (*relative to* w_i).

Remark 3.23 If $\mathcal{W} = \{w_1, \ldots, w_n\}$ is an independent family of functions, where $w_i : [A]^{r_i} \longrightarrow H$, then for every $\underline{c^i} \in \mathbb{Z}^{\binom{l}{r_i}}$ the family $\{\widetilde{w_1}_{\underline{c^1}}, \ldots, \widetilde{w_n}_{\underline{c^n}}\}$ is weakly independent. In particular, if $f : [A]^l \longrightarrow H$ is in normal form with respect to an independent family, then f is the sum of the members of a weakly independent family of functions.

3.2.1 Continuity

In order to be able to discuss continuity of r-forms, we extend them by sending 0 to 0 (since their original domain $[A]^r$ is discrete). The extended in this way forms will be called *extended forms*, and we shall keep the same notation for them to avoid heavy notations.

Example 3.24 *Let $h : A \longrightarrow H$ be a map (i.e., a 1-ary form) and $\underline{c} = \underbrace{(1, \ldots, 1)}_{l}$, for some $l \geq 1$. Then the extended derived form $\widetilde{h}_{\underline{c}} : \{0\} \cup [A]^l \longrightarrow H$ coincides with λ_h.*

The following technical facts will be fundamental in proving Proposition 3.27 and Proposition 3.28.

Lemma 3.25 *Let X be a topological space, let H be an abelian group and let $\{w_1, \ldots, w_n\}$ be a weakly independent family of functions $w_i : X \to H$. Then $w = w_1 + \ldots + w_n : X \to H^\#$ is continuous if and only if every $w_i : X \to H^\#$ is continuous.*

Proof. It suffices to prove that every $w_i : X \to H^\#$ is continuous whenever $w : X \to H^\#$ is continuous. Denote by H_i the subgroup of H generated by $w_i(X)$. Then our hypothesis implies that the sum $H_0 = H_1 + \ldots + H_n$ is direct. Let $p_i : H_0 \to H_i$ be the canonical projection. Then $p_i : H_0^\# \to H_i^\#$ is continuous and the composition $p_i \circ w$ coincides with w_i. Since $H_0^\#$ and $H_i^\#$ are topological subgroups of $H^\#$, this proves that also $w_i : X \to H^\#$ is continuous. QED

Corollary 3.26 *Let H be an abelian group and let H_0, H_1 be subgroups of H with $H_0 \cap H_1 = \{0\}$. If, for $\nu = 0, 1$, $\{h_d^\nu : d \in I\}$ is a net in H_ν such that the net $\{h_d^0 + h_d^1 : d \in I\}$ converges to 0 in $H^\#$, then also h_d^ν converges to 0 in $H^\#$ for $\nu = 0, 1$.*

Proof. The hypothesis $H_0 \cap H_1 = \{0\}$ implies that $\{h_d^0, h_d^1\}$ is weakly independent. Now Lemma 3.25 applies. QED

The following proposition follows from Remark 3.23 and Lemma 3.25.

Proposition 3.27 *Let $m|l$, $m > 1$ and let $A \subseteq \omega$ be an infinite set. Let $f : \mathcal{D}^{(l)}_{A,m} \longrightarrow H^{\#}$ be in normal form w.r.t. the independent family of functions $\mathcal{W} = \{w_1, \ldots, w_n\}$, where $w_i : [A]^{r_i} \to H$, $i = 1, 2, \ldots, n$, and $\max\{r_i : i = 1, 2, \ldots, n\} \leq l$. Then the following properties are equivalent:*

1. *$f : \mathcal{D}^{(l)}_{A,m} \longrightarrow H^{\#}$ is continuous;*
2. *for every $i = 1, \ldots, n$, the homogeneous component $\widetilde{w_i}_{\underline{c}^i} : \mathcal{D}^{(l)}_{A,m} \longrightarrow H^{\#}$ of f is continuous.*

The next proposition characterizes the continuous extended homogeneous derived 1-ary forms. It is shown that they are of the form λ_h when restricted to an appropriate subset.

Proposition 3.28 *Let H be a bounded abelian group and let $\tau : \omega \longrightarrow H$ be an independent normalized function. Let l, m be positive integers with $m|l$ and $1 < m < l$. If the extended homogeneous derived form $\widetilde{\tau}_{\underline{c}} : \mathcal{D}^{(l)}_{\omega,m} \longrightarrow H^{\#}$ is continuous for some $\underline{c} = (c_1, \ldots, c_l) \in \mathbb{Z}^l$, then there exist an infinite subset $A \subseteq \omega$ and a normalized independent function $h : A \longrightarrow H[m]$ such that $\widetilde{\tau}_{\underline{c}} \restriction_{[A]^l} = \lambda_h \restriction_{[A]^l}$.*

Proof. Let $c = \sum_i c_i$. We prove first the following

Claim 1. There exists an infinite set $A \subseteq \omega$ such that $c\tau(\alpha) = 0$ for every $\alpha \in A$.

Proof of Claim 1. Let t be the period of τ. If t divides $c = \sum_i c_i$, then there is nothing to prove. Assume now that t does not divide c. It suffices to show that the set $A' = \{\gamma \in \omega \mid c_1\tau(\gamma) + \ldots + c_l\tau(\gamma) \neq 0\}$ is finite. Assume A' is infinite.
We can build a character $\xi' : \langle \tau(\gamma) \mid \gamma \in A' \rangle \longrightarrow \mathbb{T}$ with the property $\xi'(\tau(\gamma)) = a \neq 0$ for a certain $a \in \mathbb{T}$ of period t and for all $\gamma \in A'$. Our hypothesis yields $ca \neq 0$. Note that if $\gamma_1 < \ldots < \gamma_l$ in A' then

$$\xi'(\widetilde{\tau}_{\underline{c}}(\gamma_1, \ldots, \gamma_l)) = \xi'(c_1\tau(\gamma_1)) + \ldots + \xi'(c_l\tau(\gamma_l)) = ca \neq 0$$

in $\mathbb{T}$. Extend ξ' to a character $\xi : H \longrightarrow \mathbb{T}$. Now, if $(\gamma_1, \ldots, \gamma_l)_d = (\gamma_1, \ldots, \gamma_l)$ is a net in $\mathcal{D}^{(l)}_{A',m}$ which converges to 0 in $(\mathbb{V}^{\omega}_m)^{\#}$, then we have that $\widetilde{\tau}_{\underline{c}}(\gamma_1, \ldots, \gamma_l) \longrightarrow 0$ in $H^{\#}$ and $ca = \xi(\widetilde{\tau}_{\underline{c}}(\gamma_1, \ldots, \gamma_l)) \longrightarrow 0$ in $\mathbb{T}$ (by the continuity of ξ). This leads to a contradiction. The claim is proved.

Now we continue the proof of the proposition. Let $M_1, \ldots, M_q$ be all the $q = \binom{l}{m}$ subsets of $\{1, \ldots, l\}$ of cardinality m and let $N_i = \{1, \ldots, l\} \setminus M_i$ for $i = 1, \ldots, q$. We are going to show that there exists an infinite subset $A \subseteq \omega$ such that

$$\left(\sum_{j \in M_i} c_j\right) \tau(\gamma) = 0 \qquad \text{for all } i = 1, \ldots, q \quad \text{and for all } \gamma \in A. \tag{3.1}$$

For the initial step $i = 1$, let us consider a partition $A' \cup A''$ of ω into infinite sets. Let y_d be a net in $\mathcal{D}^{(m)}_{A',m}$ such that $y_d \longrightarrow 0$, and take $z_d \longrightarrow 0$ in $\mathcal{D}^{(l-m)}_{A'',m}$. Then $x_d := y_d + z_d$ is a net $\mathcal{D}^{(l)}_{\omega,m}$ that converges to 0 and, by continuity, $\widetilde{\tau}_{\underline{c}}(x_d) = \sum_{j \in M_1} c_j \tau(\gamma_j) + \sum_{j \in N_1} c_j \tau(\gamma_j) \longrightarrow 0$ in $H^{\#}$. Now, use Corollary 3.26 to get $\sum_{j \in M_1} c_j \tau(\gamma_j) \longrightarrow 0$. Since we chose the net y_d arbitrarily, we deduce that the map $\sum_{j \in M_1} c_j \tau$ is continuous and Claim 1 shows that $\sum_{j \in M_1} c_j \tau(\gamma) = 0$ for every γ in a certain infinite subset $A_1 \subseteq A'$. Suppose now we have a set $A_r \subseteq \omega$ such that $\sum_{j \in M_u} c_j \tau(\gamma) = 0$ for every $\gamma \in A_r$, where $u = 1, \ldots, r$ and r is fixed between 1 and $q - 1$ (start picking $A_2 \subseteq A_1$ infinite and go on choosing A_u contained in A_{u-1}). The case $r + 1$ follows considering a partition $A'_r \cup A''_r$ of A_r in a similar way as the case $i = 1$. This concludes the induction and proves (3.1).

To $c_j \tau(\gamma)$ ($\gamma \in A$ and $j = 1, \ldots, l$) apply the following claim to conclude that $h = c_1 \tau = \ldots = c_l \tau$ and $mh(\gamma) = 0$ for every $\gamma \in A$.

Claim 2. *Let H be an abelian group and m, l positive integers with $m|l$ and $m < l$. For $x_1, \ldots, x_l$ in H such that $\sum_{i \in I} x_i = 0$ for every subset $I \subseteq \{1, \ldots, l\}$ with $|I| = m$, there exists $a \in H[m]$ such that $x_1 = \ldots = x_l = a$.*

So we have proved that $o(h(\gamma))$ divides m (not necessarily equals m). We can further restrict, since there are only finitely many divisors of m, to get an infinite subset A such that $o(h(\gamma))$ is constant.

QED

3.3 Proof of the Straightening Theorem

We start with a partition result of Kunen:

Theorem 3.29 (Lemma 3.3, [59]) *Let p be a prime number and $l \geq 1$. If $\pi : [\omega]^l \longrightarrow \mathbb{V}_p^\omega$, then there exists an infinite subset A of ω such that $\pi \restriction_{[A]^l}$ is in normal form.*

The next step consists in imposing continuity on the extended function π. According to Proposition 3.27, $\pi \restriction_{[A]^l} = \sum_{i=1}^n \widetilde{w_{i}}_{\underline{c}^i}$ from Theorem 3.29 is continuous if and only if each factor $\widetilde{w_{i}}_{\underline{c}^i}$ is continuous. By Proposition

3.30, for every $i \in \{1, \ldots, n\}$ such that w_i has arity strictly greater than 1 the corresponding homogeneous derived form $\widetilde{w_i}_{\underline{c}^i}$ vanishes when further restricted to a smaller subset.

Proposition 3.30 *Let r be an integer strictly greater than 1, and let $\tau : [\omega]^r \longrightarrow \mathbb{V}_p^\omega$ be a normalized independent function, where p is prime. Let l, m be positive integers with $m > 1$, $l \geq r$, $m|l$ and let $s = \binom{l}{r}$. For any $\underline{c} = (c_1, \ldots, c_s) \in \mathbb{Z}^s$, the extended homogeneous derived form $\widetilde{\tau}_{\underline{c}} : \mathcal{D}_{\omega,m}^{(l)} \longrightarrow (\mathbb{V}_p^\omega)^\#$ is continuous if and only if $\widetilde{\tau}_{\underline{c}} = 0$.*

Proof. Assume that $\widetilde{\tau}_{\underline{c}}$ is continuous. Let $S = \{I_j \in [l]^r \mid c_j \tau \neq 0\}$ and suppose $S \neq \emptyset$. From a direct application of Lemma 4.3 of [59], we get a set $\mathcal{D} \subseteq [\omega]^r$ such that for every finite partition $\mathcal{K}$ of ω there exists $K_t \in \mathcal{K}$ which contains $\gamma_1 < \ldots < \gamma_l$ with the property $(\gamma_{t_1}, \ldots, \gamma_{t_r}) \in \mathcal{D}$ for precisely one element $(t_1, \ldots, t_r) \in S \subseteq [l]^r$.
Since the period of $\tau(\beta)$ does not depend on β, choose an element $a \neq 0$ on the circle such that $o(a) = o(\tau(\beta))$; consequently, $c_j a \neq 0$ in $\mathbb{T}$ for every j such that $I_j \in S$. Moreover, recalling that the set $\{\tau(\beta) \mid \beta \in [\omega]^r\}$ is independent, we can find a character $f : \mathbb{V}_p^\omega \longrightarrow \mathbb{T}$ such that $f(\tau(\beta)) = a$ for every $\beta \in \mathcal{D}$ and $f(\tau(\beta)) = 0$ if $\beta \notin \mathcal{D}$.
Let us consider the map

$$h = f \circ \widetilde{\tau}_{\underline{c}} : \mathcal{D}_{\omega,m}^{(l)} \longrightarrow \mathbb{T}.$$

The map h is continuous as a composition of continuous functions. By definition,

$$h(\alpha) = f(\widetilde{\tau}_{\underline{c}}(\alpha)) = f\left(\sum_{j=1}^{s} c_j \tau(\alpha \restriction_{I_j})\right) = \sum_{I_j \in S} c_j f(\tau(\alpha \restriction_{I_j})).$$

Since $|S| < \infty$ and f takes only values 0 and a on the image of τ, it follows that the image of h in $\mathbb{T}$ is finite. In particular, $\{0\}$ is an open set in the image of h. Therefore there exists a partition $\mathcal{K}_0$ of ω such that the neighborhood $\mathcal{V}_m(\mathcal{K}_0)$ is sent to 0 by h. Choose now $K_t \in \mathcal{K}_0$ such that for some $\gamma = (\gamma_1, \ldots, \gamma_l)$ with $\gamma_1 < \ldots < \gamma_l$ in K_t there exists precisely one element $I_{j_0} = (t_1, \ldots, t_r) \in S$ with $(\gamma_{t_1}, \ldots, \gamma_{t_r}) \in \mathcal{D}$. Now,

$$h(\gamma) = f(\widetilde{\tau}_{\underline{c}}(\gamma)) = f\left(\sum_{j=1}^{s} c_j \tau(\gamma \restriction_{I_j})\right) = c_{j_0} a$$

by definition of f. On the other hand, $\gamma \in \mathcal{V}_m(\mathcal{K}_0)$ as $m|l$, thus $c_{j_0} a = h(\gamma) = 0$. This contradicts the choice of $a \in \mathbb{T}$. QED

Note that here one can have $m = l$ (compare with Proposition 3.28).

In the following theorem we employ all the tools developed until now to prove that λ_h is a *typical* continuous map $\pi : \mathcal{D}^{(2m)}_{\omega,m} \longrightarrow J^{\#}$. Here 2 can be replaced by any integer $l > 1$.

Theorem 3.31 *Let $m > 1$. For every continuous function $\pi : \mathcal{D}^{(2m)}_{\omega,m} \longrightarrow J^{\#}$ with $\pi(0) = 0$ and J any bounded abelian group there exists an infinite subset $A \subseteq \omega$ such that*

$$\pi \restriction_{[A]^{2m}} = \lambda_h \restriction_{[A]^{2m}}$$

where $h : A \longrightarrow J$ is an s-normalized independent function with $s|m$ (so, $\pi([A]^{2m}) \subseteq J[s]$) or $h = 0$.
If π is an embedding then $m = s$.

The proof closely follows the proof of [47, Lemma 4.10]. Our choice to present a complete proof is motivated by the fact that the topology of $\mathcal{D}^{(2m)}_{\omega,m}$ we consider here, as well as the "typical" form λ_h, are different from their counterparts in [47].

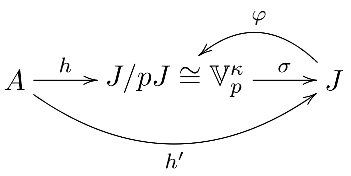
Proof. Let us consider first the case in which $J \cong \mathbb{V}^{\kappa}_{p^r}$ for a prime p, a fixed $r \in \mathbb{N}$ and a cardinal $\kappa \geq \omega$. We proceed inductively on r. Note that the image of π is countable, so actually we can suppose without loss of generality that $J \cong \mathbb{V}^{\omega}_{p^r}$.

If $r = 1$ we apply Theorem 3.29 and we express π — restricted to some infinite set $[A']^{2m} \subseteq [\omega]^{2m}$ — in normal form w.r.t. a certain independent family of functions $\mathcal{W} = \{w_1, \ldots, w_n\}$. Combining Proposition 3.27 and Proposition 3.30, and taking into account the hypothesis $\pi(0) = 0$, we get that the only non-zero summands in the expression of $\pi \restriction_{[A']^{2m}}$ are the homogeneous derived form of the w_i's with arity 1, say $\sigma_1, \ldots, \sigma_u$ (where $u \leq n$). Thus we have that:

$$\pi \restriction_{[A']^{2m}} = \sigma_1 + \ldots + \sigma_u.$$

Now, Proposition 3.28 applies to each summand σ_i to get an infinite subset A_i of A' such that $\sigma_i \restriction_{D^{(2m)}_{A_i,m}} = \lambda_{h_i}$, where $h_i : A_i \longrightarrow \mathbb{V}^{\omega}_p$ is independent p-normalized (or identically zero). Without loss of generality, we can choose $A' \supseteq A_0 \supseteq A_1 \supseteq \ldots \supseteq A_u = A$. Thus the map π coincides with $\sum_{i=1}^{u} \lambda_{h_i} = \lambda_h$ on $\mathcal{D}^{(2m)}_{A,m}$, where we set $h = \sum_{i=1}^{n} h_i : A \longrightarrow \mathbb{V}^{\omega}_p$. By virtue of Remark 3.19, h is either independent p-normalized or zero. Let us note that $\langle h(A) \rangle \cong \mathbb{V}^{|A|}_p \cong \mathbb{V}^{\omega}_p$ in the non-trivial case.

Let us suppose now that the theorem is true when the codomain is $\mathbb{V}^{\omega}_{p^t}$ for $t = 1, \ldots, r$ and let us check it in the case $J \cong \mathbb{V}^{\omega}_{p^{r+1}}$. We denote by φ the canonical homomorphism $J \longrightarrow J/pJ$. Following the case $r = 1$ there exists an infinite set $A \subseteq \omega$ and a function $h : A \longrightarrow J/pJ \cong \mathbb{V}^{\omega}_p$ such that the restriction of $\varphi \circ \pi : \mathcal{D}^{(2m)}_{\omega,m} \longrightarrow J/pJ$ to $[A]^{2m}$ coincides with λ_h. Let us note that if $\sigma : J/pJ \longrightarrow J$ is a section of φ (i.e., $\varphi \circ \sigma = id \restriction_{J/pJ}$), then $h' = \sigma \circ h : A \longrightarrow J$ is independent and p^{r+1}-normalized (it follows from the fact that $J[p^r] = pJ$).



Consider now $\lambda_{h'} : \mathcal{D}^{(2m)}_{A,m} \longrightarrow J$. Then $\varphi \circ \lambda_{h'} = \lambda_h$, and we get $(\varphi \circ \pi) \restriction_{\mathcal{D}^{(2m)}_{A,m}} = (\varphi \circ \lambda_{h'}) \restriction_{\mathcal{D}^{(2m)}_{A,m}}$, therefore $\varphi(\pi - \lambda_{h'}) \restriction_{\mathcal{D}^{(2m)}_{A,m}} = 0$; in other words, the image of $\mathcal{D}^{(2m)}_{A,m}$ under $\pi_1 = \pi - \lambda_{h'}$ is contained in $pJ \cong \mathbb{V}^\omega_{p^r}$. Applying our inductive hypothesis to π_1 we get an infinite $A_1 \subseteq A$ such that $\pi_1 \restriction_{[A_1]^{2m}} = \lambda_{h_1} \restriction_{[A_1]^{2m}}$, where $h_1 : A_1 \longrightarrow pJ$ is normalized independent. Thus π coincides with $\lambda_{h_1} + \lambda_{h'} = \lambda_{h_1+h'}$ on $\mathcal{D}^{(2m)}_{A_1,m}$. It follows from Remark 3.19 that $h_1 + h'$ is p^{r+1}-normalized and independent.

In the general case, J can be expressed as a finite direct sum of p-groups being a bounded abelian group. More precisely (see §1.1), $J = \bigoplus_{i=1}^n J_i$, where each J_i is of the form $\mathbb{V}^\kappa_{p^k}$ for certain p, k and κ (which depend on i). The function $\pi : \mathcal{D}^{(2m)}_{\omega,m} \longrightarrow J$ is the sum of n functions $\pi_i = p_i \circ \pi : \mathcal{D}^{(2m)}_{\omega,m} \longrightarrow J_i$ where $p_i : J \longrightarrow J_i$ is the canonical projection. Each π_i is continuous as a composition of continuous functions. Arguing as before we get an infinite subset $A_i \subseteq A$ such that $\pi_i \restriction_{[A_i]^{2m}} = 0$ or $\pi_i \restriction_{[A_i]^{2m}} = \lambda_{h_i} \restriction_{[A_i]^{2m}}$ with $h_i : A_i \longrightarrow J_i$ independent for every $i = 1, \ldots, n$. Moreover, we can choose the sets A_i in order to have $A_1 \supseteq A_2 \supseteq \ldots \supseteq A_n$. Setting $A = A_n$ we obtain an infinite subset of ω such that the properties of the n functions π_i hold simultaneously. The function $h = h_1 + \ldots + h_n : A \longrightarrow J$ is normalized and independent by Remark 3.19. With the set A and the function h we are done as $\lambda_h = \lambda_{h_1+\ldots+h_n}$. To check that $s = o(h)$ divides m just note that the range of π is $\langle h(A) \rangle \cong \mathbb{V}^\omega_s$ and that $\lambda_h \restriction_{\mathcal{D}^{(2m)}_{A,m}} : \mathcal{D}^{(2m)}_{A,m} \longrightarrow \mathbb{V}^\omega_s$ is continuous since it coincides with $\pi \restriction_{\mathcal{D}^{(2m)}_{A,m}}$, so Lemma 3.20 implies that $s|m$. Again Lemma 3.20 assures that if π is an embedding then $m = s$. QED

A simple lemma before proving our main result.

Lemma 3.32 *In $(\mathbb{V}^\kappa_m)^\#$, with $m > 1$, we have that $[\kappa]^l \subseteq \overline{[\kappa]^{l+m}}$ for every positive integer l.*

Proof. Just observe that for every $x = (a_1, \ldots, a_l) \in [\kappa]^l$ and for every net $(b_1, \ldots, b_m)_d = (b_1, \ldots, b_m)$ converging to 0 in $[\kappa]^m$ with $a_l < b_1$, we have that $(a_1, \ldots, a_l, b_1, \ldots, b_m) \longrightarrow x$. QED

Now we derive Theorem 3.10 from Theorem 3.31.

Proof of Theorem 3.10. We first show that there exist an infinite $A \subseteq \omega$ and a homomorphism $\ell : \langle A \rangle \longrightarrow J$ such that $\pi \restriction_{[A]^{2m}} = \ell \restriction_{[A]^{2m}}$. As already noted in Lemma 3.20, the set $\mathcal{D}^{(2m)}_{\omega,m}$ embeds into the group $\mathbb{V}^{\omega}_{m}$. For this reason we can apply Theorem 3.31 to the restriction of π to $\mathcal{D}^{(2m)}_{\omega,m}$ and we find an infinite subset $A \subseteq \omega$ and a map $h : A \longrightarrow J$ independent s-normalized or zero such that $\pi \restriction_{\mathcal{D}^{(2m)}_{A,m}} = \lambda_h \restriction_{\mathcal{D}^{(2m)}_{A,m}}$ (so $\pi(\mathcal{D}^{(2m)}_{A,m}) \subseteq J[s]$). In the case $h = 0$ just choose ℓ to be the null homomorphism. Suppose h is not identically zero. Then $h(A)$ generates a subgroup $H \leq J$ isomorphic to $\mathbb{V}^{|A|}_{s}$. Now, since h is independent, we can take the homomorphism $\ell_1 : \langle A \rangle \longrightarrow J$ simply as the unique extension of h (and λ_h)

$$\ell_1 \restriction_{[A]^{2m}} = \lambda_h \restriction_{[A]^{2m}} = \pi \restriction_{[A]^{2m}} . \tag{3.2}$$

Since $\langle A \rangle$ is a direct summand of $\mathbb{V}^{\omega}_{m}$ we can extend ℓ_1 to $\ell : \mathbb{V}^{\omega}_{m} \longrightarrow J$. Note that ℓ is Bohr-continuous being a homomorphism. Clearly ℓ coincides with ℓ_1 on $[A]^{2m} \subseteq \mathbb{V}^{\omega}_{m}$, therefore $\ell \restriction_{[A]^{2m}} = \pi \restriction_{[A]^{2m}}$ by (3.2). Since the closure of $[A]^{2m}$ in $(\mathbb{V}^{\omega}_{m})^{\#}$ contains $[A]^{m}$ by Lemma 3.32 and since both π and ℓ are continuous, we conclude that they coincide also on $[A]^{m}$.

If π is an embedding then $s = m$ by Theorem 3.31. Hence $h(A)$ is an independent set consisting of elements of order $s = m$. Thus $\ell_1 : \langle A \rangle \rightarrow H \cong \mathbb{V}^{\omega}_{m}$, defined as above, is an isomorphism. If H has infinite index in $J[m]$, then we find a subgroup $L \cong H$ of $J[m]$ with $L \cap H = \{0\}$. This allows us to build an injective extension ℓ of ℓ_1 by sending a complement of the subgroup $\langle A \rangle$ into L. If H has finite index in $J[m]$, we can first replace A by a smaller subset A_1 of ω with infinite complement. Then the respective subgroup $H_1 = \langle h(A_1) \rangle$ of J has infinite index as a subgroup of $J[m]$ and we continue the argument as in the previous case. QED

3.4 Applications

As already pointed out, the interest in Theorem 3.10 is related to Question 3.1 proposed by van Douwen. Kunen's counterexample ([59]), which is essentially based on Theorem 3.29, shows that there exists no injective continuous functions from $(\mathbb{V}^{\omega}_{p})^{\#}$ into $(\mathbb{V}^{\omega}_{q})^{\#}$ in the special case in which p and q are distinct prime numbers. On the other hand, Dikranjan and Watson ([43]) worked with an extra hypothesis on the cardinality of the domain to get their counterexample.

3.4.1 Cardinal invariants and weak Bohr-homeomorphisms

Recall that a bounded abelian group G is *homogeneous* if for every prime p its p-primary subgroup G_p has the form $\mathbb{V}^{\kappa}_{p^m}$, for some $m \in \mathbb{N}$ and some cardinal κ. It is easy to see that *every bounded group of square-free order is homogeneous.*

A general approach in set-theoretic topology is to study appropriate invariants (most often, cardinal ones) that allow for an easy solution of the homeomorphism problem. Sometimes certain collections of appropriate cardinal invariants permit a complete solution of the homeomorphism problem, in other cases one obtains only necessary conditions. In the sequel we are interested in (weak) Bohr-homeomorphisms. An easy complete set of cardinal invariants giving a *sufficient* condition for Bohr-homeomorphism is the set of all Ulm-Kaplansky invariants. Recall that, according to Prüfer's theorem, the set of Ulm-Kaplansky invariants determines any bounded abelian group up to algebraic isomorphism (see §1.1), hence up to Bohr-homeomorphism. Our aim is to find less rigid sufficient conditions that turn out to be also necessary in certain cases. (For example, the finite Ulm-Kaplansky invariants obviously play no role, since the set of infinite Ulm-Kaplansky invariants determines the group up to almost isomorphism (so Bohr-homeomorphism).)

Definition 3.33 *A bounded abelian group G is* almost homogeneous *if, for every prime p such that G_p is non-trivial, at most one of the Ulm-Kaplansky invariants of G_p is infinite.*

It is clear that G is almost homogeneous, whenever G is almost isomorphic to a homogeneous group.

Remark 3.34 The following holds:

(a) It is easy to see that every bounded group of square-free essential order is almost homogeneous.

(b) For bounded group G of square-free order, the Ulm-Kaplansky invariants of G are precisely the cardinals $r_p(G)$.

(c) Every bounded abelian group G is almost isomorphic to a bounded one in which every non-zero Ulm-Kaplansky invariant is infinite.

(d) A bounded group G with $\mathrm{eo}(G) = m$ contains a subgroup isomorphic to $\mathbb{V}_m^\omega$.

In the next theorem we give a convenient (from algebraic point of view) characterization of weak isomorphisms.

Theorem 3.35 *Two bounded abelian groups G, H are weakly isomorphic if and only if they satisfy the following condition:*

(A) $|mG| = |mH|$ *whenever* $m \in \mathbb{N}$ *and* $\max\{|mG|, |mH|\} \geq \omega$.

Proof. Let G_1 and H_1 be finite-index subgroups of G and H respectively, such that G_1 is isomorphic to a subgroup of H and H_1 is isomorphic to a subgroup of G. If mG is infinite for some m, then also mG_1 is infinite and

$|mG_1| = |mG|$. Since mG_1 is isomorphic to a subgroup of mH, we conclude that also mH is infinite and $|mG| = |mG_1| \leq |mH|$. Analogously, we get $|mH| \leq |mG|$. Hence, $|mH| = |mG|$ whenever at least one of these cardinals is infinite. This proves that (A) holds.

Now assume that (A) holds. Since we need to prove that each one of these groups has a finite-index subgroup that is isomorphic to a subgroup of the other, it is not restrictive to assume, passing to a finite index subgroup, that $exp(G) = \mathrm{eo}(G)$ and $exp(H) = \mathrm{eo}(H)$. Consider first the case when both groups are bounded p-groups and let $G = \mathbb{V}_p^{\kappa_1} \oplus \mathbb{V}_{p^2}^{\kappa_2} \oplus \ldots \oplus \mathbb{V}_{p^n}^{\kappa_n}$. Our first aim is to prove that the leading Ulm-Kaplansky invariants of G and H coincide. Indeed, $p^n = \mathrm{eo}(G)$ and $\kappa_n = |p^{n-1}G|$, so (A) yields $\mathrm{eo}(H) = p^n$ as well. Hence from $|p^{n-1}H| = |p^{n-1}G|$ we deduce that

$$H = \mathbb{V}_p^{\lambda_1} \oplus \ldots \oplus \mathbb{V}_{p^{n-1}}^{\lambda_{n-1}} \oplus \mathbb{V}_{p^n}^{\lambda_n} \text{ with } \lambda_n = |p^{n-1}H| = \kappa_n. \qquad (3.3)$$

Similarly, since it is clear that $|p^iG| = \max\{\kappa_{i+1}, \ldots \kappa_n\}$ for every $i = 0, 1, 2, \ldots, n-1$, we have

$$\max\{\kappa_{i+1}, \ldots \kappa_n\} = |p^iG| = |p^iH| = \max\{\lambda_{i+1}, \ldots \lambda_n\}. \qquad (3.4)$$

We shall assume without loss of generality that all Ulm-Kaplansky invariants of G and H are infinite. For $i = 0, 1, 2, \ldots, n-1$ let

$$G_i = \mathbb{V}_{p^{n-i}}^{\kappa_{n-i}} \oplus \ldots \oplus \mathbb{V}_{p^n}^{\kappa_n} \text{ and } H_i = \mathbb{V}_{p^{n-i}}^{\lambda_{n-i}} \oplus \ldots \oplus \mathbb{V}_{p^n}^{\lambda_n}.$$

Since $G = G_{n-1}$ and $H = H_{n-1}$, it suffices to prove (arguing by induction on i) that G_i embeds in H_i and H_i embeds in G_i. For $i = 0$ this follows from (3.3). Assume $i < n$ and G_i embeds in H_i. Then

$$G_{i+1} = \mathbb{V}_{p^{n-i-1}}^{\kappa_{n-i-1}} \oplus G_i \text{ and } H_{i+1} = \mathbb{V}_{p^{n-i-1}}^{\lambda_{n-i-1}} \oplus H_i. \qquad (3.5)$$

If $\kappa_{n-i-1} \leq \max\{\kappa_{n-i}, \ldots, \kappa_n\} = \max\{\lambda_{n-i}, \ldots, \lambda_n\}$, then $\mathbb{V}_{p^{n-i-1}}^{\kappa_{n-i-1}}$ is isomorphic to a subgroup of G_i that embeds in H_i by hypothesis. Since all Ulm-Kaplansky invariants are infinite, $H_i \cong H_i \oplus H_i$. Therefore, G_{i+1} embeds actually in the subgroup H_i of H_{i+1}.
If $\kappa_{n-i-1} > \max\{\kappa_{n-i}, \ldots, \kappa_n\} = \max\{\lambda_{n-i}, \ldots, \lambda_n\}$, then also $\lambda_{n-i-1} > \max\{\lambda_{n-i}, \ldots, \lambda_n\}$ by (3.4). So (3.4) yields $\kappa_{n-i-1} = \lambda_{n-i-1}$. Now (3.5) implies that G_{i+1} embeds into H_{i+1}.

The general case easily follows from the local one by taking appropriate subgroups of the form $G' = mG$, where $exp(G) = mp^k$, m is coprime to p and p is prime. Then the subgroup G' coincides with the p-primary part of G, so the above argument applies to G' and mH. QED

Clearly, *almost isomorphism implies weak isomorphism.* Now we shall see in more detail the precise relation between these two notions and (B).

Lemma 3.36 *For bounded abelian groups G and H:*

(a) if G and H are weakly isomorphic, then they satisfy condition (B)*;*

(b) if G and H are countable, then they are weakly isomorphic if and only if they satisfy condition (B)*;*

(c) if G and H are almost homogeneous, then the following properties are equivalent:

- *G and H are weakly isomorphic;*
- *G and H satisfy condition* (B)*;*
- *G and H are almost isomorphic.*

Proof. *(a)* Assume G and $H$are weakly isomorphic. Then the equality $\mathrm{eo}(G) = \mathrm{eo}(H)$ follows directly from the definition of essential order. Fix a prime p such that $r_p(G)$ is infinite. Write $\mathrm{eo}(G) = \mathrm{eo}(H) = p^n m$, where m is coprime to p. Then the subgroup mG (mH) coincides with the p-torsion part of G (respectively, H). Therefore, $r_p(G) = r_p(mG) = |mG| = |mH|$ is infinite. Hence,

$$r_p(H) = r_p(mH) = |mH| = |mG| = r_p(G).$$

This proves (B).

(b) Assume now that G and H are countable and satisfy (B). To prove that they are weakly isomorphic we shall assume that all non-zero Ulm-Kaplansky invariants of these groups are infinite (see remark 3.34). Fix a prime p and write $\mathrm{eo}(G) = \mathrm{eo}(H) = p^n m$, where m is coprime to p. Since the p-torsion part of G (H) coincides with mG (resp., with mH), it follows that the leading Ulm-Kaplansky invariants of mG and mH coincide with ω, i.e., they contain both $V = \mathbb{V}_{p^n}^{\omega}$ as a direct summand. Since $V \cong V^n$, it follows that mG and mH are weakly isomorphic. Since these are the p-torsion parts of G and H for an arbitrary prime p, we deduce that G and H are weakly isomorphic.

(c) Now assume that G and H are almost homogeneous groups satisfying (B). In order to prove *(c)*, it suffices to check that G and H are almost isomorphic. Since almost isomorphism is a transitive property, and since almost homogeneous groups are almost isomorphic to homogeneous ones, we can assume that G and H are homogeneous. By Remark 3.34, we can assume that the Ulm-Kaplansky invariants of both groups are either zero or infinite. Since our hypothesis gives $r_p(G) = r_p(H) = \kappa_p$ for every prime p, this entails that the p-primary subgroups of G and H are isomorphic to some $\mathbb{V}_{p^k}^{\kappa_p}$. Hence, G and H are almost isomorphic. QED

The implication proved in (a) should be compared to the following chain of implications

$$\text{w. isomorphic} \implies \text{w. Bohr-homeomorphic} \implies \text{(B)}$$

that holds according to Theorem 3.12 (in the case of *countable* bounded groups) and Theorem 3.16.

Now we are in position to prove Lemma 3.11:

Proof of Lemma 3.11. We have to prove that if G and H are countable bounded abelian groups, then G is weakly isomorphic to H if and only if $\mathrm{eo}(G) = \mathrm{eo}(H)$. Since $\mathrm{eo}(G) = \mathrm{eo}(H)$ is a part of (B), it suffices to see that this condition alone implies (B), so that (b) of the above lemma applies. Indeed, it is easy to see that $r_p(G)$ is infinite iff $p|\mathrm{eo}(G)$. Since G, H are countable, $\mathrm{eo}(G) = \mathrm{eo}(H)$ yields that $r_p(G) = r_p(H)$ whenever at least one of these cardinals is infinite. QED

3.4.2 Classification up to (weak) Bohr-homeomorphisms

Here we involve the Bohr topology.

Lemma 3.37 *If there exists an embedding* $(\mathbb{V}_m^\omega)^\# \to H^\#$*, then* $m|\mathrm{eo}(H)$.

Proof. By Theorem 3.10, there exists an injective homomorphism $l : \mathbb{V}_m^\omega \to H$. It remains to note that if $n = \mathrm{eo}(H)$, then nH is finite, so $n\mathbb{V}_m^\omega$ must be finite as well being isomorphic to a subgroup of the finite group nH, via the homomorphism l. This is possible only if $m|n$. QED

Lemma 3.37 provides a direct proof of Theorem 3.9 from the introduction of this chapter:

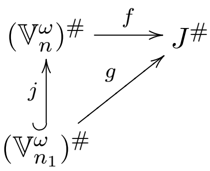
Proof of Theorem 3.9 We have to prove that for countably infinite bounded abelian groups G and H, there exists an embedding $G^\# \longrightarrow H^\#$ iff $\mathrm{eo}(G)|\mathrm{eo}(H)$. Let $m = \mathrm{eo}(G)$ and $n = \mathrm{eo}(H)$. By Lemma 3.11, G and H are weakly isomorphic (hence, weakly Bohr-homeomorphic by Lemma 3.8) to, respectively, $\mathbb{V}_m^\omega$ and $\mathbb{V}_n^\omega$. Therefore, $G^\# \hookrightarrow H^\#$ iff $(\mathbb{V}_m^\omega)^\# \hookrightarrow (\mathbb{V}_n^\omega)^\#$. Since $\mathrm{eo}(\mathbb{V}_m^\omega) = m$ and $\mathrm{eo}(\mathbb{V}_n^\omega) = n$, Lemma 3.37 implies that $(\mathbb{V}_m^\omega)^\# \hookrightarrow (\mathbb{V}_n^\omega)^\#$ if and only if $m|n$, i.e., $\mathrm{eo}(G)|\mathrm{eo}(H)$. QED

Our next aim is to relax the embeddings with respect to the Bohr topology to injective (or finitely many-to-one) continuous maps Bohr topology. The next lemma is a corollary of Theorem 3.10:

Lemma 3.38 *Let* $f : (\mathbb{V}_n^\omega)^\# \longrightarrow J^\#$ *be a continuous function, where* J *is an abelian group and* $m = \mathrm{eo}(J)$*. If* $n \nmid m^l$ *for every* l *(i.e.,* n *has a divisor coprime to* m*), then there exists an infinite subset* $B \subseteq \mathbb{V}_n^\omega$ *such that* $f\restriction_B$ *is constant.*

Proof. We can assume wlog that J is countable. Indeed, let J_1 be the subgroup of J generated by the image of f. Then J_1 is countable and $m_1 = \mathrm{eo}(J_1)$ divides m, so we can replace J by J_1 and m by m_1.

Our hypothesis implies that $n = n_1 n_2$, where $n_1 > 1$ and n_1 is coprime with m. The composition g of f with the embedding $j : (\mathbb{V}_{n_1}^{\omega})^{\#} \longrightarrow (\mathbb{V}_n^{\omega})^{\#}$, defined by $j(x) = n_2 \cdot x$ for all $x \in \mathbb{V}_{n_1}^{\omega}$, gives a continuous function $g : (\mathbb{V}_{n_1}^{\omega})^{\#} \longrightarrow J^{\#}$.



Since the only linear maps $\mathbb{V}_{n_1}^{\omega} \to J$ are the constant ones, by Theorem 3.10 there exists an infinite $A \subseteq \omega$ such that the restriction of g to $[A]^{n_1}$ is constant. Now just note that j is injective, therefore f is constant on the set $B = j([A]^{n_1}) \subseteq \mathbb{V}_n^{\omega}$. QED

Remark 3.39 It may happen that $f \restriction_{\mathcal{D}_{A,m}^{(m)}}$ is injective for every $A \subseteq \kappa$. Note that the set B in the above corollary has the form $B = n_2 \cdot \mathcal{D}_{A,n_1}^{(n_1)}$.

One can state Lemma 3.38 also in a contrapositive form:

Corollary 3.40 *Let J be an abelian group with* $\mathrm{eo}(J) = m$. *If there exists a continuous finitely many-to-one (in particular, injective) map $f : (\mathbb{V}_n^{\omega})^{\#} \longrightarrow J^{\#}$, then n divides some power of m.*

Now we can prove Corollary 3.13:

Proof of Corollary 3.13. Let p be a prime. We have to prove that $r_p(H) \geq r_p(G)$, if there exists a continuous injective function $f : G^{\#} \longrightarrow H^{\#}$, where G, H are abelian groups, H is bounded and $r_p(G)$ is infinite. Let $\kappa = r_p(G) \geq \omega$. Then G contains a direct sum $\mathbb{V}_p^{\kappa} \cong \bigoplus_{i<\kappa} V_i$, where $V_i \cong \mathbb{V}_p^{\omega}$ for every $i < \kappa$. Apply Theorem 3.10 to each V_i to get for every $i < \kappa$ an infinite set A_i of V_i such that $f(A_i) \subseteq H[p]$. Since f is injective, the sets $f(A_i)$ are pairwise disjoint, hence we conclude $|H[p]| \geq |\bigcup_{i<\kappa} A_i| \geq \kappa$. QED

As a consequence of Corollary 3.13 we want to emphasize that if there exists a continuous injective map $(\mathbb{V}_p^{\kappa})^{\#} \to H^{\#}$, then $r_p(H) \geq \kappa$ and, consequently, H contains a copy of $\mathbb{V}_p^{\kappa}$ (this entails the existence of an injective homomorphism $l : \mathbb{V}_p^{\kappa} \hookrightarrow H$). For uncountable κ this argument cannot be extended to arbitrary m in place of the prime p even if $f : G^{\#} \longrightarrow H^{\#}$ is an embedding. However, we know that the existence of an embedding $(\mathbb{V}_m^{\omega})^{\#} \to H^{\#}$ yields $m|\mathrm{eo}(H)$ by Lemma 3.37.

Let us prove Theorem 3.16 now.

Proof of Theorem 3.16. We have to prove that if $G^\# \to H^\#$ is an embedding, then $\mathrm{eo}(G)|\mathrm{eo}(H)$ and $r_p(G) \leq r_p(H)$ if $r_p(G)$ is infinite. Let $m = \mathrm{eo}(G)$. By item (3) of Remark 3.34, G contains a subgroup isomorphic to $\mathbb{V}_m^\omega$. So there exists an embedding $(\mathbb{V}_m^\omega)^\# \to H^\#$. Therefore, $m|\mathrm{eo}(H)$ by Lemma 3.37. For the inequality $r_p(G) \leq r_p(H)$ apply Corollary 3.13. QED

It was shown in [47] that the property of being bounded is preserved by Bohr-homeomorphism. Corollary 3.13 states that the property of possessing infinitely many p-torsion elements is preserved by weak Bohr-homeomorphism. The following corollary settles the weaker property of having non-trivial p-torsion elements at all.

Corollary 3.41 *Let G and H be infinite abelian groups such that one of them is bounded. If G^κ and H^κ are weakly Bohr-homeomorphic for some infinite cardinal κ, then $r_p(G)^\kappa = r_p(H)^\kappa$ for every prime p. In particular, $r_p(G) > 0 \iff r_p(H) > 0$ (i.e., G is p-torsion free if and only if H is p-torsion free).*

Proof. According to [47, Theorem 5.1], both groups are bounded. Now fix any prime p and apply Theorem 3.16 to G^κ and H^κ to get $r_p(G)^\kappa = r_p(G^\kappa) = r_p(H^\kappa) = r_p(H)^\kappa$. QED

It follows immediately from Remark 3.34 (a) and Lemma3.36 *(c)* that for groups of square-free essential order, (B) *implies almost isomorphism, hence both weak isomorphism and Bohr-homeomorphism.* Hence all five properties ((B), weak isomorphism, almost isomorphism and (weak) Bohr-homeomorphism), coincide for bounded abelian groups of square-free essential order. Let us see that this remains true for the larger class of all almost homogeneous bounded abelian groups.

Corollary 3.42 *For almost homogeneous bounded abelian groups G and H, TFAE:*

(a) G and H are almost isomorphic;

(b) G and H are weakly isomorphic;

(c) G and H are Bohr-homeomorphic.

(d) G and H are weakly Bohr-homeomorphic;

(e) G and H satisfy (B)*.*

Proof. Statements *(a)*, *(b)* and *(e)* are equivalent by *(c)* of Lemma 3.36. The implication *(a)* $\to$ *(c)* is Corollary 3.3, *(c)* $\to$ *(d)* is trivial and *(d)* $\to$ *(e)* follows from Theorem 3.16. QED

In Corollary 3.17 it is shown that Bohr-homeomorphism and weak Bohr-homeomorphism coincide for groups of the form $\mathbb{V}_m^\kappa$. It is not clear if this is true in general, as already pointed out in [47, § 6] (see also § 3.5). Corollary 3.42 gives a partial answer to this question.

We do not know whether *(a)* in the last corollary can be weakened to: *there exist continuous injective maps* $G^\# \longrightarrow H^\#$ *and* $H^\# \longrightarrow G^\#$ (see § 3.5 for a more specific question).

3.4.3 Retracts and ccs-subgroups

We conclude this chapter by discussing how Theorem 3.10 can be employed in the study of retracts and continuous cross sections in the Bohr topology (see Definition 3.48 below).

Recall the following questions of van Douwen ([77]):

Question 3.43 (Question 81, [79]) *Is it true that every countable subgroup* H *of an abelian group* $G^\#$ *is a retract of* $G^\#$*?*

This question is still open. As a matter of fact, the following more general question was answered negatively by Gladdines in 1995 ([48]):

Question 3.44 (Question 82, [79]) *Is it true that every countable closed subset of* $G^\#$ *is a retract of* $G^\#$*?*

Actually, Gladdines' counterexample shows that $\mathcal{D}_{\omega,2}^{(2)}$ is not a retract of $(\mathbb{V}_2^\omega)^\#$.

The space $\mathcal{D}_{\omega,2}^{(2)}$ has very interesting properties. In fact, consider the following lemma which states that $\mathcal{D}_{\omega,2}^{(2)}$ can be Bohr-embedded in every abelian group G (see [46] and [47, § 3] for more results in this line).

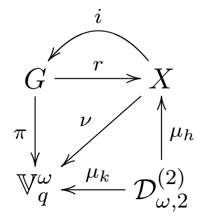
Lemma 3.45 ([38], Proposition 3.3) *Let* G *be a bounded abelian group and let* $f : \omega \longrightarrow G$ *be a function. Consider the map* $\mu_f : \mathcal{D}_{\omega,2}^{(2)} \longrightarrow G$ *defined by* $\mu_f(u,v) = f(u) - f(v)$ *for every* $(u,v) \in [\omega]^2$ *and* $\mu_f(0) = 0$*. Then:*

1. $\mu_f : \mathcal{D}_{\omega,2}^{(2)} \longrightarrow G^\#$ *is continuous;*
2. *if* f *is independent,* μ_f *is an embedding.*

Givens proved in [46] that every abelian group G possesses a closed countable subset F which is not a Bohr-retract of G. One can prove that F is always homeomorphic to $\mathcal{D}_{\omega,2}^{(2)}$, so Givens' result largely extends Gladdines' one. Here we employ Theorem 3.10 and the previous lemma to get a brief proof of the fact that $\mathcal{D}_{\omega,2}^{(2)}$ is not a Bohr-retract of G, for every bounded abelian group G:

Corollary 3.46 *Let G be any bounded abelian group. Then there exists an embedding $\mathcal{D}^{(2)}_{\omega,2} \hookrightarrow G^{\#}$ that makes $\mathcal{D}^{(2)}_{\omega,2}$ a closed subspace of $G^{\#}$ which is not a retract.*

Proof. Let m be the essential order of G and let p be a prime number that divides m. Then, as a corollary of Remark 3.34 (d), $(\mathbb{V}^{\omega}_{p})^{\#}$ embeds into $G^{\#}$ as a closed subgroup. Hence it suffices to work with $\mathbb{V}^{\omega}_{p}$ instead of G. For simplicity, we write $G = \mathbb{V}^{\omega}_{p}$ from now on. Let $h := \iota^{(1)}_{\omega,p} : \omega \longrightarrow \mathbb{V}^{\omega}_{p}$ be the canonical map. Then $\mu_h : \mathcal{D}^{(2)}_{\omega,2} \longrightarrow (\mathbb{V}^{\omega}_{p})^{\#}$ is an embedding by Lemma 3.45. Consider the subset $X = \mu_h(\mathcal{D}^{(2)}_{\omega,2})$ of G. Arguing as in [38, Lemma 2.5] one can easily prove that X is closed in G. Assume that there exists a Bohr-continuous retraction $r : G \longrightarrow X$. Take a prime $q \neq p$ and an independent function $k : \omega \longrightarrow (\mathbb{V}^{\omega}_{q})^{\#}$. By Lemma 3.45, $\mu_k : \mathcal{D}^{(2)}_{\omega,2} \longrightarrow (\mathbb{V}^{\omega}_{q})^{\#}$ is an embedding and the composition $\nu = \mu_k \circ \mu_h^{-1} : X \longrightarrow (\mathbb{V}^{\omega}_{q})^{\#}$ is an injective continuous map. Moreover, $\pi := \nu \circ r : G^{\#} \longrightarrow (\mathbb{V}^{\omega}_{q})^{\#}$ is a continuous map with $\pi(0) = 0$.



By Lemma 3.38 applied to $\pi \circ i \circ \mu_h$ (i denotes the inclusion of X in G), there exists an infinite subset $A \subseteq \omega$ such that $(\pi \circ i \circ \mu_h) \restriction_{([A]^2)} = 0$, thus $\pi \restriction_{\mu_h([A]^2)} = 0$. The contradiction follows from the fact that r coincides with the identity on $\mu_h([A]^2)$, so $\nu \restriction_{\mu_h([A]^2)} = \pi \restriction_{\mu_h([A]^2)} = 0$ while ν is injective. QED

Remark 3.47 Note that if $G = \mathbb{V}^{\omega}_{m}$, where $m > 2$, then for no infinite subset A of ω the restriction on $[A]^2$ of the function $\mu_f : \mathcal{D}^{(2)}_{\omega,2} \to G^{\#}$ defined in Lemma 3.45 can be of the form λ_h (as μ_h is continuous, whereas λ_h can be continuous only if $m = 2$, see *(1)* of Lemma 3.20). This fact does not contradict Theorem 3.31 since the theorem applies in the case of functions defined over $\mathcal{D}^{(2m)}_{\omega,m}$ with $m > 1$.

We recall the following:

Definition 3.48 *[25] If H is a subgroup of an abelian group G, then H is said to be a* ccs-subgroup *of $G^{\#}$ if there exists a continuous map $\Phi : (G/H)^{\#} \longrightarrow G^{\#}$ such that $\Phi(0) = 0$ and $\pi \circ \Phi = id_{G/H}$, where π denotes the canonical projection $\pi : G \longrightarrow G/H$ (i.e., Φ is a* continuous cross section *of π with $\Phi(0) = 0$).*

Observe that if $\Phi : (G/H)^{\#} \longrightarrow G^{\#}$ is an arbitrary continuous map such that $\pi \circ \Phi = id_{G/H}$, one can always suppose that $\Phi(0) = 0$. Indeed, if $\Phi(0) \neq 0$ then one compose Φ with the translation by $-\Phi(0)$ in G to define a new continuous map $\Phi' : (G/H)^{\#} \longrightarrow G^{\#}$ by $xH \mapsto \Phi'(x) := \Phi(xH) - \Phi(0)$ which is now a continuous cross section of π.

It is easy to see that the property of being a ccs-subgroup is stronger than being a retract in the following sense: if H is a ccs-subgroup of G, then using a continuous cross section $\Phi : (G/H)^{\#} \longrightarrow G^{\#}$ one defines a retraction $r : G \to H$ by letting $r(x) = x - \Phi(\pi(x))$, where $\pi : G \to G/H$ denotes the canonical projection. Note that this retraction is *partially linear* in the sense $r(x + h) = r(x) + h = r(x) + r(h)$ for every $x \in G$ and $h \in H$.

Here we recall some examples and basic properties from [25] and [39].

Lemma 3.49 *Let G be an abelian group and let $H \leq G$.*

(a) If H has finite index in G, then H is a ccs-subgroup of G.

(b) If H is a direct summand of G, then H is a ccs-subgroup of G.

(c) Let K be a subgroup of H. If K is ccs in H and H is ccs in G, then K is css in G (i.e., the property of being a ccs-subgroup is transitive).

(d) Let K be a subgroup of H. If K is ccs in G and H/K is ccs in G/K, then H is css in G.

Example 3.50 *For every abelian group G, the finitely generated subgroups of G are ccs-subgroups. More generally, (a), (b) and (c) of the above lemma imply that every subgroup H of G that is a sum of a finitely generated subgroup and a divisible subgroup is always a ccs-subgroup.*

Example 3.51 *For every prime p, the subgroup $p\mathbb{V}_{p^2}^{\omega} \cong \mathbb{V}_p^{\omega}$ of $\mathbb{V}_{p^2}^{\omega}$ is not a ccs-subgroup of $\mathbb{V}_{p^2}^{\omega}$.*

Both examples originally appeared in [25], the second one with a rather involved proof (consisting of the entire [25, §5]) developing in detail Kunen's approach of normal forms in the case of $\mathbb{V}_{p^2}^{\omega}$. We will deduce Example 3.51 directly from Corollary 3.54.

The next lemma shows that to resolve van Douwen's problem one can restrict only to essential subgroups.

Lemma 3.52 *If there exists an abelian group G and a subgroup H that is not a retract of $G^{\#}$, then this pair can be chosen also with the property H essential in G.*

Proof. Assume G is an abelian group and H is a subgroup of G that is not a retract of $G^{\#}$. Let D be the divisible hull of G. Then $H^{\#}$ is a topological subgroup of $D^{\#}$ and certainly H cannot be a retract of $D^{\#}$. Now let D_1 be the divisible hull of H in D. Then H is an essential subgroup of D_1. Since D_1 is divisible, there exists a subgroup L of D such that $D = D_1 \oplus L$. The desired pair is D_1 and H. Indeed, assume $r : D_1^{\#} \to H$ is a retraction. Composing with the continuous projection $p : D^{\#} \to D_1^{\#}$ we get a retraction $D^{\#} \to H$, a contradiction. QED

Our goal now is to characterize the essential ccs-subgroups of bounded abelian groups. We start with a technical fact.

Claim 3.53 *Let p be a prime number and let G be a bounded abelian p-group. If H is an essential ccs-subgroup of G, then $[G : H]$ is finite.*

Proof. Our hypothesis means that $H \geq G[p]$. Assume that $[G : H]$ is infinite. Since G is a bounded p-group, G/H is a bounded p-group as well, so G/H contains a copy of $\mathbb{V}_p^{\omega}$ by (d) of Remark 3.34. Now, let $\pi : G \longrightarrow G/H$ be the canonical projection and $\Phi : (G/H)^{\#} \longrightarrow G^{\#}$ a continuous cross section. Applying Theorem 3.10 to the restriction of Φ to $\mathbb{V}_p^{\omega}$ we find an infinite set $A \subseteq \omega$ such that $\Phi([A]^p) \subseteq G[p] \leq H$. So Φ takes the infinite set $B = [A]^p$ to $G[p]$ while π vanishes on $G[p] \leq H$. This proves that the composition $\pi \circ \Phi$ vanishes on B. On the other hand, $\pi \circ \Phi$ coincides with the identity since Φ is a section of π, and this yields a contradiction. QED

As a straightforward corollary, we obtain now the following

Corollary 3.54 *If G is a bounded p-group such that pG is infinite, then $G[p]$ is not a ccs-subgroup of G.*

Proof. Take $H = G[p]$. Then H obviously contains $G[p]$ and $[G : H]$ is infinite since $pG \cong G/H$ is infinite. Now Claim 3.53 applies. QED

Since $p\mathbb{V}_{p^2}^{\kappa} \cong \mathbb{V}_p^{\kappa}$ is infinite, this corollary immediately implies that $\mathbb{V}_p^{\kappa}$ is not a ccs-subgroup of $\mathbb{V}_{p^2}^{\kappa}$ (see Example 3.51 above). More precisely, for $\kappa = \omega$ this is [25, Theorem 35], for arbitrary κ this is [25, Remark 36].

Combining Claim 3.53 and Lemma 3.49, we can prove in the next theorem that an essential subgroup of a bounded abelian group is "almost never" a ccs-subgroup.

Theorem 3.55 *An essential subgroup H of a bounded abelian group G is a ccs-subgroup of G if and only if $[G : H]$ is finite.*

Proof. If $[G : H] < \infty$ then H is a ccs-subgroup of G by Lemma 3.49 *(a)*. Conversely, suppose that H is a ccs-subgroup of G. Write G as the direct sum of its p-components: $G = G_{p_1} \oplus \ldots \oplus G_{p_k}$, for some prime numbers $p_1, \ldots, p_k$. Then $H = H_{p_1} \oplus \ldots \oplus H_{p_k}$. By hypothesis, H is essential in G, i.e., H contains $Soc(G) \cong \bigoplus_{p \in \mathbb{P}} G[p] \cong G[p_1] \oplus \ldots \oplus G[p_k]$, so $H_{p_i} \geq G[p_i]$

for every $i = 1, \dots, k$. Since H is ccs in G, every H_{p_i} is a ccs-subgroup of G_{p_i} ([39]), so we can apply Claim 3.53 to $H_{p_i} \leq G_{p_i}$. We deduce that H_{p_i} has finite index in G_{p_i} for every i and, hence, $[G : H] < \infty$. QED

Note that if G in the above theorem is not bounded then the result can fail to be true.

Example 3.56 According to Example 3.50 the essential subgroup $\mathbb{Z}$ of $\mathbb{Q}$ is a ccs-subgroup of $\mathbb{Q}$ although $\mathbb{Z}$ has infinite index in $\mathbb{Q}$.

3.5 Questions

We saw that the invariant $\mathrm{eo}(G)$, along with $|G|$, provides for a complete classification, up to weak Bohr-homeomorphism, of all countable bounded abelian groups. The situation is similar for almost homogeneous bounded groups (but here one needs to take also the p-ranks). For non-almost-homogeneous bounded groups the situation changes completely even for the simplest *uncountable* bounded abelian group of essential order 4. Indeed, $G = \mathbb{V}_4^{\omega_1}$ and $H = \mathbb{V}_2^{\omega_1} \times \mathbb{V}_4^{\omega}$ are not weakly isomorphic, because $\omega_1 = |2G| > |2H| = \omega$. However, we do not know whether these groups are weakly Bohr-homeomorphic:

Question 3.57 *Can $(\mathbb{V}_4^{\omega_1})^{\#}$ be homeomorphically embedded into $(\mathbb{V}_2^{\omega_1} \times \mathbb{V}_4^{\omega})^{\#}$?*

Here is the question in the most general form:

Question 3.58 *Given a cardinal $\kappa \geq \omega$, an integer $s > 1$ and a prime number p, are $\mathbb{V}_{p^s}^{\kappa}$ and $\mathbb{V}_p^{\kappa} \times \mathbb{V}_{p^s}^{\omega}$ weakly Bohr-homeomorphic? Can this depend on p?*

Let us see that a positive answer to Question 3.58 for all prime p implies that bounded abelian groups G and H are weakly Bohr-homeomorphic if and only if (B) holds. Indeed, for infinite p-groups G, H of size κ with $\mathrm{eo}(G) = \mathrm{eo}(H) = p^s$, (B) yields an algebraic embedding of $\mathbb{V}_p^{\kappa} \times \mathbb{V}_{p^s}^{\omega}$ into G and H, where $\kappa = r_p(G) = r_p(H)$. Hence a positive answer to Question 3.58 entails an embedding of $(\mathbb{V}_{p^s}^{\kappa})^{\#}$ in $G^{\#}$ and $H^{\#}$. Since G and H are obviously isomorphic to subgroups of $\mathbb{V}_{p^s}^{\kappa}$, this proves that both G and H are separately weakly Bohr-homeomorphic to $\mathbb{V}_{p^s}^{\kappa}$.

A positive answer to the next question is equivalent to the strongest negative answer to Question 3.58.

Question 3.59 *Assume that p is a prime number, $s > 1$ is an integer, κ and λ are infinite cardinals such that $(\mathbb{V}_{p^s}^{\kappa})^{\#}$ can be homeomorphically embedded into $(\mathbb{V}_{p^{s-1}}^{\kappa} \times \mathbb{V}_{p^s}^{\lambda})^{\#}$. Must the inequality $\lambda \geq \kappa$ hold?*

A positive answer to Question 3.59 obviously gives a negative answer to Question 3.58 (hence to Question 3.57 as well, just take $\kappa = \omega_1 > \omega = \lambda$, $p = 2$ and $s = 2$).

A positive answer to the following stronger form of Question 3.59 is equivalent to the fact that weak Bohr-homeomorphism coincides with weak isomorphism for bounded abelian groups.

Question 3.60 *Assume that p is a prime number, $n \geq s > 1$ are integers, κ, κ' and λ are infinite cardinals such that $(\mathbb{V}_{p^s}^{\kappa})^{\#}$ can be homeomorphically embedded into $(\mathbb{V}_{p^{s-1}}^{\kappa'} \times \mathbb{V}_{p^n}^{\lambda})^{\#}$. Must the inequality $\lambda \geq \kappa$ hold?*

Note that in the above question only the case $\kappa \leq \kappa'$ is relevant (since the conclusion easily follows from Corollary 3.13 when $\kappa > \kappa'$). Moreover, n can be replaced by s in Question 3.60 if the answer to the following question is positive:

Question 3.61 *Assume that p is a prime number, $s > 1$ is an integer and κ is an infinite cardinal such that $(\mathbb{V}_{p^s}^{\kappa})^{\#}$ can be homeomorphically embedded into $H^{\#}$. Can $(\mathbb{V}_{p^s}^{\kappa})^{\#}$ be homeomorphically embedded also into $H[p^s]^{\#}$?*

For $s = 1$ the answer is positive.

The countable groups $\mathbb{V}_4^{\omega}$ and $\mathbb{V}_2^{\omega} \times \mathbb{V}_4^{\omega}$ are obviously weakly isomorphic, hence weakly Bohr-homeomorphic (see the discussion above).

Question 3.62 *(a)* ([59]) *Are $\mathbb{V}_4^{\omega}$ and $\mathbb{V}_2^{\omega} \times \mathbb{V}_4^{\omega}$ Bohr-homeomorphic?*

(b) Are weakly Bohr-homeomorphic bounded groups always *Bohr-homeomorphic?*

Question 3.63 *Suppose that G and H are bounded abelian groups such that $G^{\#}$ homeomorphically embeds into $H^{\#}$. Does G contain a subgroup of finite index that is isomorphic to a subgroup of H?*

Note that a positive answer to this question would obviously imply that weak Bohr-homeomorphism coincides with weak isomorphism. Hence a positive answer to this question would imply a positive answer to Question 3.59.

The following question involving the stronger version of Bohr-homeomorphism was already considered:

Question 3.64 ([59]) *Are Bohr-homeomorphic bounded abelian groups almost isomorphic?*

We do not know whether the weaker assumption of Corollary 3.40 already implies that $n|m$ even in the simplest situations like the following one:

Question 3.65 *Does there exist a continuous injective (finitely many-to-one) map* $(\mathbb{V}_4^\omega)^\# \longrightarrow (\mathbb{V}_2^\omega)^\#$*?*

The conclusion of Corollary 3.15 is precisely the second part of the condition (B). We are not aware if the first part of (B) holds true as well:

Question 3.66 *Does the existence of continuous injective maps* $G^\# \longrightarrow H^\#$ *and* $H^\# \longrightarrow G^\#$ *yield* $\mathrm{eo}(G) = \mathrm{eo}(H)$*?*

We do not know whether the embeddings in the "weakly Bohr-homeomorphic" version of Corollary 3.17 can be replaced by "continuous injections", see Question 3.65 or the general

Question 3.67 *If* $f : (\mathbb{V}_n^\kappa)^\# \longrightarrow (\mathbb{V}_m^\kappa)^\#$ *is continuous and injective, is it true that* $n|m$*?*

Question 3.68 *For distinct primes* p *and* q*, does there exist a continuous map* $(\mathbb{V}_p^\omega)^\# \longrightarrow (\mathbb{V}_q^\omega)^\#$ *with infinite image? What about* $p = 2$ *and* $q = 3$*?*

For subgroups H, K of an abelian group G let $H =^* K$ if $H \cap K$ has finite index in $H + K$. When G is torsion, $H =^* K$ iff there exists a finite subgroup F of G such that $H + F = K + F$. In particular, $H =^* K$ whenever H and K are finite.

Let us see first that if $H =^* K$ and H is a ccs subgroup of G, then also K is a ccs subgroup of G. Since $H \cap K$ is a ccs subgroup of H (having finite index), it is a ccs subgroup of G by *(c)* of Lemma 3.49. Now K contains a finite-index subgroup (namely, $H \cap K$) that is ccs in G. Then also K is ccs in G by Lemma 3.49 *(d)*. So we obtain: *Let* G *be an abelian group and let* H *be a subgroup. If* $H =^* L$ *for some direct summand* L *of* G*, then* H *is a ccs-subgroup of* G*.*

We expect that the following conjecture holds true (note that it holds for essential subgroups, according to Theorem 3.55, as a finite index subgroup H of G obviously satisfies $H =^* G$).

Conjecture 3.69 *For every bounded abelian group* G *and a subgroup* H *of* G*, the following properties are equivalent:*

1. H *is ccs-subgroup of* G*;*
2. $H =^* B$ *for some direct summand* B *of* G*.*

Note that this problem leads somehow far form the original van Douwen's problem, where the subgroup H in question can be assumed to be essential (so that Theorem 3.55 works). The next question can easily be reduced to the case of essential subgroup H.

Question 3.70 *Can "bounded abelian" be replaced by "divisible torsion" in Conjecture 3.69?*

Chapter 4

The weak $\mathbb{U}$-topology on abelian groups

A topological group G is said to be *ω-narrow* if for every open neighborhood V of 0_G there exists a countable subset A of G such that $AV = G$. In the literature, ω-narrow groups are also known as *ω-bounded* groups, but we prefer the former term in order to avoid ambiguity since "ω-bounded" has several different meanings in general topology and the theory of groups (see Definition 8.41).

The class of ω-narrow groups is pretty wide. For instance, it includes the classes of Lindelöf topological groups, of separable topological groups and of precompact topological groups. The reason for this wideness is the stability of ω-narrow groups with respect to all basic operations (see, for example, [50] or [71]). Indeed,

Proposition 4.1 *The class of ω-narrow topological groups has the following properties:*

a) *A continuous homomorphic image of an ω-narrow topological group is ω-narrow.*

b) *The topological product of an arbitrary family of ω-narrow topological groups is ω-narrow.*

c) *A subgroup of an ω-narrow topological group is ω-narrow.*

d) *If G is a topological group and H is a dense subgroup of G such that H is ω-narrow, then G is ω-narrow.*

e) *Let N be a closed invariant subgroup of a topological group G. If both N and the quotient group G/N are ω-narrow, so is G.*

As a matter of fact, ω-narrow groups are characterized precisely as the subgroups of topological products of second-countable topological groups (see [50] or [71, Theorem 3.4]):

Proposition 4.2 *A topological group G is ω-narrow if and only if there exists a family $\{H_i \mid i \in I\}$ of second-countable topological groups such that G is topologically isomorphic to a subgroup of $\prod_{i\in I} H_i$.*

Proposition 4.2 implies that the original topology of G is determined by a certain family of continuous homomorphisms of G to second-countable topological groups.

It is proved in [69] that there exists a *universal second-countable abelian group*, that is, a second-countable topological abelian group $\mathbb{U}$ such that every second-countable topological abelian group H is topologically isomorphic to a subgroup of $\mathbb{U}$. Moreover, we can suppose that $\mathbb{U}$ is divisible by means of the following proposition:

Proposition 4.3 ([9], Corollary 3) *Let K be a second-countable topological abelian group. Then there exists a second-countable divisible abelian group D containing K as a subgroup.*

Given a topological abelian group (G, τ), we consider on G the initial topology $T_{\mathbb{U}}(G)$ with respect to $\mathbb{U}$. We write $(G, \tau_{\ddagger})$ or simply $G^{\ddagger}$ to denote $(G, T_{\mathbb{U}}(G))$. Then, a topological group G is ω-narrow if and only if $G = G^{\ddagger}$. If the starting group G is discrete, we denote $(G, T_{\mathbb{U}}(G))$ by $(G, \tau_{\Box}(G))$ or simply by $G^{\Box}$. Then, $\tau_{\Box}(G)$ is the initial topology with respect to the family of *all* homomorphisms $g\colon G \to \mathbb{U}$, hence it is the *maximal ω-narrow topology* on G.
It is clear that the topological group $G^{\ddagger}$ is ω-narrow and also that $\tau_{\ddagger}$ is finer than the Bohr topology. In particular, since $G^{\#}$ is Hausdorff, then $G^{\Box}$ is Hausdorff as well (see Proposition 4.9).

4.1 Elementary properties of $T_{\mathbb{U}}(G)$

Remark 4.4 Let G be a topological abelian group, and let H be an ω-narrow group. If $f : G \to H$ is a continuous homomorphism, then $f : G^{\ddagger} \to H$ is continuous.

Proof. It follows from Proposition 1.7 2) since H coincides with $H^{\ddagger}$. QED

Remark 4.5 The functor $T_{\mathbb{U}}$ is ideal.

Proof. Let us fix an epimorphism $f\colon G \to H$, for some topological abelian groups G, H. We need to show that $f\colon G^{\ddagger} \to H^{\ddagger}$ is open. Indeed, let $K = ker f \subseteq G$. Consider the quotient homomorphism $q\colon G^{\ddagger} \to G^{\ddagger}/K^{\ddagger}$ and the canonical (algebraic) isomorphism $i\colon G^{\ddagger}/K^{\ddagger} \to H^{\ddagger}$. Then $f = i \circ q$. Since $f : G^{\ddagger} \to H^{\ddagger}$ is continuous by Remark 4.4 and q is open, i is continuous.

Further, since $G^\ddagger/K^\ddagger$ is ω-narrow as a continuous image of the ω-narrow group $G^\ddagger$, the homomorphism $i^{-1}\colon H^\ddagger \to G^\ddagger/K^\ddagger$ is continuous by Remark 4.4. Therefore, i is a topological isomorphism and f is open. QED

Proposition 4.6 *Let H be a subgroup of a topological abelian group G. Then:*

1) *$H^\ddagger$ is a topological subgroup of $G^\ddagger$;*

2) *$G^\ddagger/H^\ddagger$ is topologically isomorphic to $(G/H)^\ddagger$.*

Proof. 1) It follows from Proposition 1.9 *(a)* jointly with the fact that $\mathbb{U}$ can be assumed to be divisible.
2) This is a consequence of Remark 4.5 and Proposition 1.6 *(d)*. QED

Proposition 4.7 *Let G be an abstract abelian group.*

a) *if $G^\ddagger$ is discrete, then $|G| \leq \omega$.*

b) *Let $H \leq G$. Then $Int(H) \neq \emptyset \iff H$ is open in $G^\ddagger \implies [G : H]$ is countable.*

Proof. a) It is obvious that a discrete ω-narrow group is countable.
b) The first equivalence is well-known. Now, suppose that H is open in $G^\ddagger$. This is equivalent to say that $(G/H)^\ddagger$ is discrete (using Proposition 4.6), and this implies $[G : H] \leq \omega$ by a). QED

Proposition 4.8 *Let G be an abelian group. Then $G^\square$ is discrete if and only if $|G| \leq \omega$.*

Proof. If $G^\square$ is discrete, then G is countable by a) of Proposition 4.7. The other implication is clear by the definition of $\tau_\square$.
QED

Proposition 4.9 *For every abelian group G, $G^\square$ is Hausdorff. In particular, every subgroup of $G^\square$ is closed.*

Proof. Since $G^\#$ is Hausdorff and the identity map of $G^\square$ onto $G^\#$ is a continuous isomorphism, $G^\square$ is Hausdorff as well. The other statement is a consequence of Proposition 1.9 *(b)*. QED

Clearly, the maximal ω-narrow topologies shares all the general properties described in § 1.3.1 with the Bohr topology. On the other hand, in the next three results we present some basic differences between $G^\square$ and $G^\#$.

Corollary 4.10 *Let G be an arbitrary abelian group. Then every countable subgroup of $G^\square$ is closed and discrete in $G^\square$.*

Proof. Let H be a countable subgroup of G. By Proposition 4.6, $G^\square$ induces on H the maximal ω-narrow topology. According to Proposition 4.8, the maximal ω-narrow group topology on a countable group is discrete, and H is closed according to Lemma 1.4. QED

Corollary 4.11 *Every countable subset of $G^\square$ is closed and discrete in $G^\square$.*

Proof. Take a countable subset E of G. Then the subgroup $\langle E\rangle$ of $G^\square$ generated by E is closed and discrete in $G^\square$, by Corollary 4.10. In particular, $E \subseteq \langle E\rangle$ is closed and discrete. QED

Corollary 4.10 and Corollary 4.11 fail to hold for the Bohr topology. Indeed, by a theorem of Protasov in [66], every infinite precompact topological group contains a countable non-closed subset. One cannot extend the proposition below to the groups of the form $G^\#$ either.

Proposition 4.12 *Every countable subset E of a topological abelian group $G^\square$ is C-embedded in $G^\square$.*

Proof. Let f be a real-valued function on E. By Corollary 4.11, E is discrete and f is automatically continuous. Consider the identity map $i_H\colon H \to H \leq D$, where H denotes the subgroup of G generated by E and D is the discrete divisible hull of H. There exists a homomorphic extension $j\colon G^\square \to D$ of i_H, and j is continuous since D is countable and discrete (hence, second-countable). For every $x \in D$, set $G_x = j^{-1}(x)$. Then $\{G_x \mid x \in D\}$ is a partition of $G^\square$ into clopen (i.e., closed and open) subsets such that every G_x contains at most one element of E. Define a function $\widetilde{f}\colon G^\square \to \mathbb{R}$ by setting $\widetilde{f}(y) = f(e)$ if $y \in G_x$ and $e \in E \cap G_x$, and $\widetilde{f}(y) = 0$ if $y \in G_x$ and $G_x \cap E = \emptyset$. Then $\widetilde{f}$ is a continuous extension of f, and we are done. QED

We finish this section with a remark about product of groups. From Proposition 1.6 *(a)* we know that:

Proposition 4.13 *Let $G_1, \ldots, G_n$ be abelian groups. Then $G_1^\ddagger \times \ldots \times G_n^\ddagger$ is topologically isomorphic to $(G_1 \times \ldots \times G_n)^\ddagger$.*

It is natural to ask whether Proposition 4.13 extends to infinite product. The answer is negative, as the following example shows:

Example 4.14 Take $G = \mathbb{Z}^\mathbb{N}$. Then $G^\square$ contains $\mathbb{Z}^{(\mathbb{N})} = \bigoplus_\mathbb{N} \mathbb{Z}$ as a closed discrete subgroup, by Corollary 4.10, while $\mathbb{Z}^{(\mathbb{N})}$ is dense in $(\mathbb{Z}^\square)^\mathbb{N}$. This implies that $(\mathbb{Z}^\mathbb{N})^\square$ is not topologically isomorphic to $(\mathbb{Z}^\square)^\mathbb{N}$. In fact, we have shown that the topology of $(\mathbb{Z}^\mathbb{N})^\square$ is strictly finer than the topology of $(\mathbb{Z}^\square)^\mathbb{N}$, since the identity isomorphism $i\colon (\mathbb{Z}^\mathbb{N})^\square \to (\mathbb{Z}^\square)^\mathbb{N}$ is evidently continuous and not open.

4.2 Specific properties of the maximal ω-narrow topology

We show here that, for an uncountable abelian group G, the corresponding topological group $G^{\square}$ is neither a P-space, nor a Baire space, nor $\mathbb{R}$-factorizable (see Definition 4.18).

We start with the following fact:

Proposition 4.15 *If G is an abelian group, then all compact subsets of $G^{\square}$ are finite. Hence, $G^{\square}$ is locally compact if and only if it is countable.*

Proof. Take a compact subset K of $G^{\square}$. To prove that K is finite, take any countable subset A of K. By Corollary 4.11, A is closed so, as a closed subset of K, it must be also compact. Therefore A is finite. This proves that K is finite as well.

If G is countable, then the group $G^{\square}$ is discrete by Proposition 4.8 and, hence, locally compact. Conversely, if G is locally compact, we can take an open neighborhood U of the neutral element in $G^{\square}$ with compact closure, say K. Since the group $G^{\square}$ is ω-narrow, we can cover it with a countable family of translations of U and of K. Since K is finite, the group G is also countable. QED

Lemma 4.16 *The inequality $\psi(G^{\square}) \leq \omega$ holds for every abelian group G with $|G| \leq 2^{\omega}$.*

Proof. It is well known (see the equivalence of (b) and (c) of [26, Theorem 4.6]) that $\psi(G^{\#}) \leq \omega$ whenever $|G| \leq 2^{\omega}$. Since $i\colon G^{\square} \to G^{\#}$ is continuous, $\psi(G^{\square}) \leq \psi(G^{\#}) \leq \omega$. QED

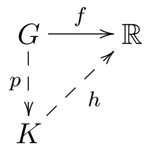
Theorem 4.17 *Let G be an uncountable abelian group. Then $G^{\square}$ is not a P-group.*

Proof. First suppose that $|G| \leq 2^{\omega}$. By Lemma 4.16, we can find a countable family γ of open subsets of $G^{\square}$ such that $\bigcap \gamma = \{0_G\}$. Since $G^{\square}$ is not discrete by Proposition 4.8, this immediately implies that $G^{\square}$ is not a P-group.

If $|G| \geq 2^{\omega}$, take a subgroup H of G such that $|H| = 2^{\omega}$. The claim just proved implies that $H^{\square}$ is not a P-group. By Proposition 4.6, $H^{\square}$ is a topological subgroup of $G^{\square}$, and the property of being a P-group is preserved under taking subgroups. So $G^{\square}$ is not a P-group. QED

Recall the following definition.

Definition 4.18 *A topological group G is called* $\mathbb{R}$-factorizable *([72, 73]) if for every continuous real-valued function f on G, one can find a second-countable group K, a continuous homomorphism $p\colon G \to K$, and a continuous real-valued function h on K such that $f = h \circ p$.*



The class of $\mathbb{R}$-factorizable groups contains — among others — all precompact groups, all Lindelöf groups, arbitrary subgroups of σ-compact groups ([73]), and it is properly contained in the class of ω-narrow groups. During quite a long period of time, only "sporadic" examples of ω-narrow non $\mathbb{R}$-factorizable groups were known (see [71, Example 5.14]). The next theorem implies that groups with this combination of properties exist in abundance.

Theorem 4.19 *The topological group $G^{\square}$ is not $\mathbb{R}$-factorizable, for every uncountable abelian group G.*

Proof. First we construct an abelian group H of cardinality $\aleph_1$ which is a homomorphic image of G. To this aim, identify G with its image $i(G) \subseteq D$, where D is the divisible hull of G and $i\colon G \hookrightarrow D$ is the natural monomorphism. Then G is an essential subgroup of the direct sum $D \cong \mathbb{Q}^{(r(D))} \times \bigoplus_{p\in\mathbb{P}} \mathbb{Z}(p^\infty)^{(r_p(D))}$ (see § 1.1). Choose ω_1 summands $\{L_\alpha \mid \alpha < \omega_1\}$ of the direct sum D and define L to be the direct sum $\bigoplus_{\alpha<\omega_1} L_\alpha$. It is clear that $|L| = \aleph_1$. Consider the natural projection $p\colon D \to L$. The image $H = p(G)$ is an essential subgroup of L, therefore it contains a maximal independent set M of independent elements of L. Since $r(L) = |L| = \aleph_1$, we have that $|M| = \aleph_1$ and hence $|H| = \aleph_1$.

For every $x \in L$ distinct from the neutral element 0_L, denote by $supp(x)$ the minimal set $A \subseteq \omega_1$ such that $x \in \bigoplus\{L_\alpha \mid \alpha \in A\}$. We also put $supp(0_L) = \emptyset$. Let us equip L with the ω-box topology. A base of such a topology is generated by the family

$$\mathcal{B} = \{x + U_\alpha \mid x \in L,\ \alpha < \omega_1\},$$

where $U_\alpha = \{x \in L \mid supp(x) \cap \alpha = \emptyset\}$ is an open subgroup of L. With this topology, L becomes a Lindelöf P-group (see [24, Corollary 2.5]). Hence H, being a subgroup of L, is an ω-narrow P-group. In addition, we have that $w(H) \le w(L) = \aleph_1$. Therefore, if H is not closed in L, it is not Lindelöf, and it follows from [74, Lemma 3.3] that it is not $\mathbb{R}$-factorizable. In its turn, this implies that G is not $\mathbb{R}$-factorizable either, since H is a continuous homomorphic image of G ([74, Theorem 3.1]).

If H is closed in L, we can construct a topology τ'_H on H which is finer than the ω-box topology τ_H induced from L and such that (H, τ'_H) is still an ω-narrow P-group but no more Lindelöf. To this purpose, we need the following claim.

Claim. *Let F be the discrete group $\mathbb{Q} \oplus \bigoplus_{p\in\mathbb{P}} \mathbb{Z}(p^\infty)$. Then there exists a discontinuous homomorphism $h\colon H \to F$.*

Proof of the claim. Let $\{C_\alpha \mid \alpha < \omega_1\}$ be a local base at the neutral element of H. It suffices to construct a homomorphism $h\colon H \to F$ and a set $\{x_\alpha \mid \alpha < \omega_1\}$ such that $x_\alpha \in C_\alpha$ and $h(x_\alpha) \neq 0_F$, for each $\alpha < \omega_1$. Indeed, h will not be continuous since for every open neighborhood U of 0_H, $h(U)$ is not contained in the open set $\{0_F\}$.
Pick an element $x_0 \in C_0$ distinct from 0_L. Suppose that for some $\alpha < \omega_1$, we have constructed an independent set $X_\alpha = \{x_\beta \mid \beta < \alpha\} \subseteq H$ such that $x_\beta \in C_\beta$, for every $\beta < \alpha$. Then it is possible to pick an element $x_\alpha \in C_{\alpha+1} \setminus \langle X_\alpha\rangle$, and the set $\{x_\beta \mid \beta \leq \alpha\}$ is independent.
Let $X = \{x_\alpha \mid \alpha \in \omega_1\}$. Define a function $f\colon X \to F$ by setting $f(x_\alpha) = y_\alpha$, where y_α is any element of F distinct from 0_F. Since X is independent and F is divisible, there exists a homomorphism $h\colon H \to F$ that extends f. The claim is proved.

To finish the proof of the theorem, consider a finer group topology τ'_H on H obtained by declaring $K = ker(h)$ to be open in H. In other words, τ'_H is the upper bound of the topologies τ_H and τ^*, where τ^* is the coarsest topology on H that makes the homomorphism $h\colon H \to F$ to be continuous. Then (H, τ'_H) is a topological group of weight $\aleph_1$ which is also a P-group. Note that $[H : K] \leq |F| \leq \omega$ implies that τ'_H is an ω-narrow group topology.

We claim that the group $H' = (H, \tau'_H)$ is not Lindelöf. Indeed, if H' were Lindelöf, then the identity homomorphism $i\colon H' \to H$ would be open, since both groups are P-spaces (see [75, Lemma 2.4]). Clearly, this is impossible since the topology τ'_H is strictly finer than τ_H.

It follows from [74, Theorem 3.1] that if a P-group T is a continuous homomorphic image of an $\mathbb{R}$-factorizable group, then T is $\mathbb{R}$-factorizable as well. Since H' is an ω-narrow P-group and the restriction of p to G is a continuous homomorphism of $G^\square$ to H' (see Remark 4.4), we conclude that the group $G^\square$ cannot be $\mathbb{R}$-factorizable. QED

Theorem 4.20 *If G is an uncountable abelian group, then $G^\square$ is a first category space.*

Proof. Let us consider G as a subgroup of its divisible hull $D = D(G) \cong \mathbb{Q}^{(r(D))} \times \bigoplus_{p\in\mathbb{P}} \mathbb{Z}(p^\infty)^{(r_p(D))}$. We can rewrite D as the direct sum of groups L_α, with $\alpha < \kappa = |D|$, where each L_α is isomorphic either to $\mathbb{Q}$ or $\mathbb{Z}(p^\infty)$, for some prime p. Take a countable partition $\{A_i \mid i \in \omega\}$ of the set κ, where $|A_i| = \kappa$ for each $i \in \omega$, and put $B_n = \bigcup_{i=1}^n A_i$. Then $|B_n| = \kappa$ and $B_n \subseteq B_{n+1}$, for every $n \in \omega$. Consider the subgroups $G_n = G \cap \bigoplus_{\alpha\in B_n} L_\alpha$ of G, $n \in \omega$. From Proposition 4.9 it follows that every G_n is closed in $G^\square$.

In addition, the index of every G_n in G is uncountable. Indeed, using the fact that G is essential in D, one can easily see that the family $\{x+G_n \mid x \in L_\alpha,\ \alpha \in \kappa \setminus B_n\}$ of cosets of G_n in G has cardinality κ, i.e., $[G : G_n] = \kappa$. According to Proposition 4.7 b), this implies that, for every n, the closed subgroups G_n have empty interior in $G^\square$. To finish the proof it suffices to note that $G = \bigcup_{n=0}^{\omega} G_n$. QED

4.3 Problems

If G is MAP, then G^+ and, consequently, $G^\ddagger$ are Hausdorff. Nevertheless, a precise characterization of those topological group G such that $G^\ddagger$ is Hausdorff is missing.

Problem 4.21 *Give a characterization of those topological Hausdorff abelian groups G such that $G^\ddagger$ is Hausdorff.*

Observe that we can suppose that G is metrizable in the previous problem. Indeed, fix $x \neq 0_G$ in G. Since G is Hausdorff, there exists $V_0 \in \mathcal{N}_G(0)$ such that $x \notin V_0$. By induction, construct a sequence $\{V_n\}_{n=1}^{\infty}$ of open neighborhood of 0_G such that V_n is symmetric and $V_n + V_n \subseteq V_{n-1}$ for every $n \geq 1$. Then $H := \bigcap_{n=1}^{\infty} V_n \leq G$. Consider the quotient $q : G \to G/H$. Then $\{q(V_n)\}_n$ is the base of a metrizable topology σ on G/H such that σ is weaker then the quotient topology of G/H. Now, observe that $q(x)$ and $q(0) = H$ are separated in G/H.

In what follows G and H denote abelian groups.
Since the group $G^\square$ is ω-narrow, it follows from [71, Theorem 4.29] that $c(G^\square) \leq 2^\omega$, where $c(G^\square)$ is the cellularity of $G^\square$. This suggests the following:

Problem 4.22 *Is it true that $c(G^\square) = \min\{|G|, 2^\omega\}$?*

It is known that the tightness of $G^\#$ is countable, for an arbitrary abelian group G (see [3, Chapter 9]). On the other hand, from Corollary 4.10 it follows that the tightness of $G^\square$ is uncountable, for each uncountable group G.

Problem 4.23 *Is there any upper bound for the tightness of $G^\square$?*

It is easy to verify that the weight and character of an ω-narrow topological group coincide (see [71, Proposition 4.1]). We do not know whether these cardinal invariants of $G^\square$ can be calculated only in terms of the cardinality of G:

Problem 4.24 *Suppose that G and H are groups and $|G| = |H|$. Does the equality $w(G^{\square}) = w(H^{\square})$ hold? If so, what is the exact expression for $w(G^{\square})$ in terms of $|G|$?*

It was proved in [76] that $G^{\#}$ is normal if and only if G is uncountable. So we ask:

Problem 4.25 *Characterize the abelian groups G such that $G^{\square}$ is normal.*

Problem 4.26 *Characterize the abelian groups G such that:*

a) *$G^{\square}$ is paracompact;*

b) *$G^{\square}$ is zero-dimensional;*

c) *$G^{\square}$ carries a linear topology.*

Problem 4.27 *Find out which algebraic properties of groups G and H ensure the existence of a topological isomorphism (or homeomorphism) between two (uncountable) groups $G^{\square}$ and $H^{\square}$.*

More concretely, on the same line of van Douwen's Question 3.1,

Question 4.28 *If G and H are uncountable and of the same cardinality, are $G^{\square}$ and $H^{\square}$ topologically isomorphic?*

Concerning the relation with Bohr-retractions:

Problem 4.29 *Let H be an uncountable subset of G. If $r\colon G \to H$ is a Bohr-retraction, is $r\colon G^{\square} \to H^{\square}$ continuous? Conversely, if $r\colon G^{\square} \to H^{\square}$ is a continuous retraction, is $r\colon G^{\#} \to H^{\#}$ continuous?*

The same question can be formulated considering a subgroup $H \leq G$ and a cross section $\varphi\colon G/H \to G$.

Problem 4.30 *Let G be an arbitrary abelian group.*

a) *Is $G^{\square}$ complete?*

b) *Is $G^{\square}$ realcompact?*

c) *Is $G^{\square}$ Dieudonné complete?*

Chapter 5

Quasi-convexity

5.1 Basic facts on quasi-convexity

Let us recall some known properties of quasi-convex sets. See also [5] and [19].

It follows from the definition that every quasi-convex set $E \subseteq G$ is necessarily symmetric (i.e., $-e \in E$, for every $e \in E$) and closed in G. In particular, note that:

Remark 5.1 If G is compact and $E \subseteq G$ is quasi-convex, then E is compact.

From the definition of quasi-convexity, it is also clear that:

Fact 5.2 *Given G, the intersection of an arbitrary family of quasi-convex subsets of G is still a quasi-convex subset of G.*

Fact 5.3 *Consider a group G and a subset E of G. If τ and τ' are two topologies on G such that $(G,\tau)^\wedge = (G,\tau')^\wedge$, then $Q_{(G,\tau)}(E) = Q_{(G,\tau')}(E)$.*

By the definition of the Bohr topology, $(G,\tau)^\wedge = (G,\tau^+)^\wedge$ for every topological group (G,τ). Hence the previous corollary applies for the pair of topologies τ, τ^+, for every MAP topology τ. In particular, *one can reduce the computation of the quasi-convex hull in MAP groups to the case of precompact topologies.* Consider also the following more precise remark:

Remark 5.4 Let (G,τ) be a non-necessarily precompact nor MAP group. Consider the subgroup $N := \overline{\{0\}}$ (known as the *von Neumann kernel*), where the closure is taken with respect to τ^+, and the quotient $q : G \to G/N$. Then G/N, equipped withe the Bohr modification of the quotient topology, is MAP and precompact, and every character $\chi \in G^\wedge$ can be factorized as follows:

$$\begin{array}{ccc} G & \xrightarrow{q} & G/N \\ & \overset{\circlearrowleft}{\underset{\chi}{\searrow}} & \downarrow \widetilde{\chi} \\ & & \mathbb{T} \end{array}$$

In particular, $Q_G(E) = q^{-1}(Q_{G/N}(q(E)))$ for every $E \subseteq G$. Therefore, *the computation of the quasi-convex hull can be reduced to the case of precompact groups*. We will often use this fact in what follows.

Now fix G and let us consider two group topologies τ, τ' on G such that $\tau \geq \tau'$. Then $Q_{(G,\tau)}(E) \subseteq Q_{(G,\tau')}(E)$ for every $E \subseteq G$ (see Corollary 5.6). The proof of this fact is based on the following:

Lemma 5.5 *If $f : G \to H$ is a continuous homomorphism of abelian topological groups and $E \subseteq G$, then $f(Q_G(E)) \subseteq Q_H(f(E))$.*

Proof. Let $x \in G$ and assume that $f(x) \notin Q_H(f(E))$. There exists a continuous character $\chi : H \to \mathbb{T}$ such that $\chi(f(E)) \subseteq \mathbb{T}_+$ while $\chi(f(x)) \notin \mathbb{T}_+$. Now, the character $\xi = \chi \circ f : G \to \mathbb{T}$ obviously witnesses $x \notin Q_G(E)$. This proves $f(Q_G(E)) \subseteq Q_H(f(E))$. QED

Corollary 5.6 *If $\tau \geq \tau'$ are two topologies on a group G, then for every $E \subseteq G$ we have*

$$Q_{(G,\tau)}(E) \subseteq Q_{(G,\tau')}(E).$$

Proof. Just observe that the identity map $id : (G, \tau) \to (G, \tau')$ is continuous and apply Lemma 5.5. QED

More consequences of Lemma 5.5 are the following ones.

Corollary 5.7 *Consider a family $\{G_i \mid i \in I\}$ of topological abelian groups and let $E_i \subseteq G_i$ for every $i \in I$. Equip $G := \prod_i G_i$ with the product topology or a finer one, and put $E := \prod_i E_i$. Then we have that $Q_G(E) \subseteq \prod_i Q_{G_i}(E_i)$.*

Proof. Apply Lemma 5.5 to the projections. QED

In particular this proves that:

Corollary 5.8 *The product of quasi-convex sets is still quasi-convex.*

Proof. If E_i are quasi-convex, then $\prod_i E_i \subseteq Q_G(\prod_i E_i) \subseteq \prod_i Q_{G_i}(E_i) = \prod_i E_i$, and this implies $\prod_i E_i = Q_G(\prod_i E_i)$. QED

Let us see that in general we do not have equality in Corollary 5.7 (see also [19, § 4.2]). With Theorem 0.5 in mind, it is easy to find an example: take any MAP group G and $E_1 = E_2 = \{x\}$, for $0 \neq x \in G$. Then $Q_G(E_1 \times E_2) = Q_G(\{(x,x)\}) = \{(0,0), \pm(x,x)\} \subsetneq \{0, \pm x\} \times \{0, \pm x\} = Q_G(E_1) \times Q_G(E_2)$.

Another corollary of Lemma 5.5:

Corollary 5.9 *If $f : G \to H$ is a continuous homomorphism and $E \subseteq H$ is quasi-convex, then $f^{-1}(E)$ is quasi-convex in G.*

Proof. By Lemma 5.5, $f(Q_G(f^{-1}(E))) \subseteq Q_H(E) = E$, then $Q_G(f^{-1}(E)) \subseteq f^{-1}(E)$. QED

5.2 Quasi-convexity of subgroups

If H is a subgroup of G, then clearly $\chi(H) \subseteq \mathbb{T}_+$ if and only if $\chi(H) = \{0\}$, for every $\chi : G \to \mathbb{T}$. In other words, the polar of H coincides with the *annihilator* of H (which is denoted by $H^\perp$). This easy observation permits to give the following characterization of quasi-convex subgroups.

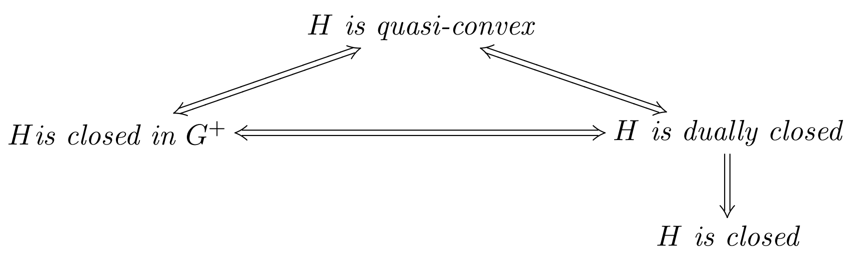
Proposition 5.10 *A subgroup H of a topological group G is quasi-convex in G if and only if G/H is MAP.*

A subgroup $H \leq G$ such that G/H is MAP is called *dually closed*. So, *H is quasi-convex in G if and only if H is dually closed in G.*
Moreover, it is well-known that dually closed subgroups are exactly the closed subgroups in the Bohr topology:

Proposition 5.11 *For $H \leq G$, H is dually closed iff H is closed in G^+.*

As a consequence, every dually closed subgroup is closed. Let us summarize what we have seen so far in the following lemma.

Lemma 5.12 *If $H \leq G$, then:*

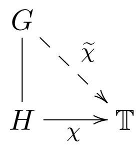


In general, the implication "H is dually closed" $\implies$ "H is closed" is not an equivalence, even in MAP abelian groups. For examples of discrete (hence, closed by Lemma 1.4) subgroups of MAP abelian groups that are dense in the Bohr topology (hence, not dually closed, according to Proposition 5.11) see Banaszczyk [11, Remark 5.5], Hooper [55] and Higasikawa [54]. Nevertheless, in some cases the equivalence holds:

Remark 5.13 For finite subgroups, for subgroups of precompact groups and of locally compact groups, it is true that "closed $\iff$ dually closed".

Another class of subgroups of topological groups that is crucial when dealing with quasi-convexity is the class of *dually embedded subgroups*, that is defined as follows:

Definition 5.14 *A subgroup H of a topological group G is* dually embedded *if each continuous character χ of H can be extended to a continuous character $\widetilde{\chi}$ of G.*



Example 5.15 *In locally compact groups and in precompact abelian groups, every subgroup is dually embedded.*

Lemma 5.16 *Let H be a dually embedded subgroup of a topological abelian group G. Then $Q_H(E) = H \cap Q_G(E)$ for every subset E of H.*

Proof. By definition, $Q_H(E) \subseteq H$. To check that $Q_H(E) \subseteq Q_G(E)$, apply Lemma 5.5 with the inclusion $i : H \to G$. Conversely, suppose $x \in (H \cap Q_G(E)) \setminus Q_H(E)$. Then there exists $\chi \in H^\wedge$ such that $\chi(E) \subseteq \mathbb{T}_+$ and $\chi(x) \notin \mathbb{T}_+$. Now just extend (by the dual embeddeness of H) χ to a continuous character of G and we find that $x \notin Q_G(E)$, a contradiction. QED

Corollary 5.17 *Let H be a dually embedded subgroup of G, and let $E \subseteq H$. Then E is quasi-convex in H if and only if E is quasi-convex in G.*

Another consequence of Lemma 5.16 is that when H contains $Q_G(E)$ we get $Q_G(E) = Q_H(E)$, as formulated in the following corollary.

Corollary 5.18 *Let $E \subseteq H \leq G$ be such that $Q_G(E) \subseteq H$. Then $Q_H(E) = Q_G(E)$.*

Proof. Since H^+ is dually embedded in G^+ by Example 5.15, we apply Lemma 5.16 and we get $Q_{H^+}(E) = H \cap Q_{G^+}(E)$. Now use Fact 5.3 and the hypothesis $Q_G(E) \subseteq H$ to conclude that $Q_H(E) = Q_G(E)$. QED

In particular, this is a useful tool in the case in which E is finite since, by Theorem 0.4, $Q_G(E) \subseteq \langle E \rangle$. So:

Corollary 5.19 *Let $E \subseteq G$ with E finite. Then $Q_{\langle E \rangle}(E) = Q_G(E)$.*

5.3 Elementary quasi-convex subsets

Given G, recall that an elementary quasi-convex subset of G is of the form $\chi^{-1}(\mathbb{T}_+)$, for some $\chi \in G^\wedge$.
For example, the elementary quasi-convex sets of $\mathbb{Z}$ are the sets $\mathcal{W}_\alpha$ introduced and studied in § 2.2. Observe that every set $\mathcal{W}_\alpha$ is the polar of the singleton $\{\alpha\}$:

$$\mathcal{W}_\alpha = \{\alpha\}^\triangleright,$$

which is equivalent to say $\mathcal{W}_\alpha = \{\alpha\}^\triangleleft$, according to Example 1.14.
Although we have already analyzed several properties of the $\mathcal{W}_\alpha$'s in § 2.2, now we want to stress their more significant aspect with respect to quasi-convexity.

It is clear that for every pair of elements $\alpha, \beta \in \mathbb{T}$, $\mathcal{W}_\alpha \subseteq \mathcal{W}_\beta$ if and only if $\beta \in Q_\mathbb{T}(\alpha)$. More generally,

Lemma 5.20 *For every $\alpha_1, \dots, \alpha_t, \beta \in \mathbb{T}$, $\bigcap_{i=1}^t \mathcal{W}_{\alpha_i} \subseteq \mathcal{W}_\beta$ if and only if $\beta \in Q_\mathbb{T}(\{\alpha_1, \dots, \alpha_t\})$.*

We already shown in Lemma 2.3 that this result can be strengthen whenever $\alpha_1, \dots, \alpha_t$ are independent.

By means of Lemma 5.20, we can offer a more precise version of Corollary 2.7 that involves the calculation of the quasi-convex hull of singletons.

Corollary 5.21 *For $\alpha, \beta \in \mathbb{T} \setminus \{0\}$ the following properties are equivalent:*

(a) $\beta = \pm\alpha$;

(b) $\alpha \in Q_\mathbb{T}(\beta)$;

(c) $\beta \in Q_\mathbb{T}(\alpha)$;

(d) $\mathcal{W}_\alpha \subseteq \mathcal{W}_\beta$;

(e) $\mathcal{W}_\beta \subseteq \mathcal{W}_\alpha$;

(f) $\mathcal{W}_\alpha = \mathcal{W}_\beta$.

The elementary quasi-convex subsets of $\mathbb{Z}$ will play a prominent role in the study of the (especially, finite) quasi-convex subsets of $\mathbb{T}$, as we will show in the following sections. This should not surprise the reader because it simply expresses the fact that $\mathbb{T}^\wedge \cong \mathbb{Z}$ and $\mathbb{Z}^\wedge \cong \mathbb{T}$. Actually, this gives also a motivation to study the elementary quasi-convex sets of $\mathbb{T}$, which already appeared implicitly in § 2.2. Their relevance will be clear in some of the next sections, namely § 5.4 and §6.2 (more concretely, see the proofs of Theorem 6.12 and Theorem 6.13 in §6.2.1).
For every integer $m \geq 1$, put $\mathbb{T}_m := [-\frac{1}{4m}, \frac{1}{4m}] \subseteq \mathbb{T}$. In particular,

$$\mathbb{T}_+ = \mathbb{T}_1 \supseteq \mathbb{T}_2 \supseteq \ldots \supseteq \mathbb{T}_n \supseteq \mathbb{T}_{n+1} \supseteq \ldots .$$

The proof of the following lemma can be deduced from Fact 2.2 (see also [5, 19]).

Lemma 5.22 *For every $m \geq 1$, $(\mathbb{T}_m)^\triangleright = \{0, \pm 1, \ldots, \pm m\}$.*

Every set $\mathbb{T}_n$ is a connected quasi-convex neighborhood of $0 \in \mathbb{T}$. Moreover, it turns out that *every (non-trivial) connected quasi-convex neighborhood of $0_\mathbb{T}$ is precisely of this form* ([5]). For reader's convenience, we include here a brief proof of this fact. Let N be a non-trivial connected quasi-convex neighborhood of 0. For being homeomorphic to a real interval (and symmetric and closed), it must have the form $\{x \in \mathbb{T} : ||x|| \leq r\}$, for a certain $r \in (0, 1/2)$. One easily verifies that r has to be smaller or equal to $1/4$. Let k be the greatest integer such that $x \leq \frac{1}{4k}$. Just note that $N^\triangleright = \{n \in \mathbb{Z} : |n| \leq k\} = \mathbb{T}_k^\triangleright$, therefore $N = N^{\triangleright\triangleleft} = \mathbb{T}_k^{\triangleright\triangleleft} = \mathbb{T}_k$.

The elementary quasi-convex sets of $\mathbb{T}$ have the form $V_n = \{n\}^\triangleright = \{x \in \mathbb{T}_+ \mid nx \in \mathbb{T}_+\}$. Then $V_n = \bigcup_{i=1}^n \frac{i}{n} + \mathbb{T}_n$.

5.4 qc-dense subsets

Given a non quasi-convex subset $E \subseteq G$, it can happen that the quasi-convex hull $Q_G(E)$ of E is the biggest possible, i.e., $Q_G(E) = G$. In this case we say that *E is qc-dense in G*. Here we give a more precise formulation.

Definition 5.23 *Given a topological group G and a quasi-convex subset K, a subset $H \subseteq K$ is said to be* qc-dense *in K if $Q_G(H) = K$.*

Obviously, dense sets are qc-dense. Now we shall see an example of a tiny compact (far from being dense) subset of $\mathbb{T}$ that is qc-dense.

Example 5.24 *The set $\{\pm 2^{-n} \mid n \geq 0\} = \{0\} \cup \{\pm 2^{-(n+1)} \mid n \geq 0\} \subseteq \mathbb{T}$ is compact and qc-dense in $\mathbb{T}$.*

The compactness is clear, while the qc-density is a direct consequence of the following claim.

Claim 5.25 *Put* $K := \{0\} \cup \{\pm 2^{-(n+1)} \mid n \geq 0\}$. *Then* $K^{\triangleright} = \{0\}$

Proof. Fix $0 \neq \chi \in \mathbb{T}^{\wedge} \cong \mathbb{Z}$. We need to show that there exists $x \in K$ such that $\chi(x) \notin \mathbb{T}_+$.
Write $\chi = 2^k \cdot m$, where $k \geq 0$ and m is odd, and take $x = 2^{-(k+1)} \in K$. Then $\chi(x) = \frac{m}{2} \equiv_1 1/2 + \mathbb{Z}$, hence $\chi(x) \notin \mathbb{T}_+$. So $K^{\triangleright} = \{0\}$. QED

Now let us consider $K_{(n)_{n \geq n_0}} := \{0\} \cup \{\pm 2^{-(n+1)} \mid n \geq n_0\} \subseteq \mathbb{T}_{2^{n_0-1}}$ for some fixed $n_0 \geq 1$. Then we have:

Proposition 5.26 $K_{(n)_{n \geq n_0}}$ *is compact and qc-dense in* $\mathbb{T}_{2^{n_0-1}}$.

Proof. Observe first that $K_{(n)}$ is compact since $2^{-(n+1)} \to 0$.
For the qc-dense property, it suffices to show that

$$K^{\triangleright}_{(n)_{n \geq n_0}} = (\mathbb{T}_{2^{n_0-1}})^{\triangleright}.$$

Since $K_{(n)_{n \geq n_0}} \subseteq \mathbb{T}_{2^{n_0-1}}$, it is clear that $K^{\triangleright}_{(n)_{n \geq n_0}} \supseteq (\mathbb{T}_{2^{n_0-1}})^{\triangleright}$. So, using Lemma 5.22, we only need to show that $K^{\triangleright}_{(n)_{n \geq n_0}} \subseteq \{0, \pm 1, \ldots, \pm 2^{n_0-1}\}$. Indeed, fix $z > 2^{n_0-1}$ and let us show that there exists $x \in K_{(n)_{n \geq n_0}}$ such that $zx \notin \mathbb{T}_+$. Let ℓ be (uniquely determined) such that $2^{\ell} < z \leq 2^{\ell+1}$ (observe that $\ell \geq n_0 - 1$ since $z > 2^{n_0-1}$). Put $x := \frac{1}{2^{\ell+2}}$; then $x \in K_{(n)_{n \geq n_0}}$ since $\ell + 2 \geq n_0 + 1$. Now, $\frac{1}{4} < zx < \frac{1}{2}$, so $zx \notin \mathbb{T}_+$ and we are done.
QED

Example 5.27 *The set* $K := \{0\} \cup \{\pm 3^{-(n+1)} \mid n \geq 0\}$ *is compact and qc-dense in* $\mathbb{T}$.

Proof. The compactness is clear, so it suffices to show that $K^{\triangleright} = \{0\}$. To this end, fix $0 \neq \chi \in \mathbb{T}^{\wedge} \cong \mathbb{Z}$ and write $\chi = 3^k \cdot m$, where $k \geq 0$ and $3 \nmid m$. Now, for $x = 3^{-(k+1)} \in K$ we have $\chi(x) = \frac{m}{3} \equiv_1 \pm 1/3 + \mathbb{Z}$, hence $\chi(x) \notin \mathbb{T}_+$. Therefore $K^{\triangleright} = \{0\}$. QED

The next result shows that the image of a qc-dense set by means of a continuous epimorphism is still qc-dense:

Corollary 5.28 *Let* $f : G \to H$ *be continuous epimorphism of abelian topological groups, and let* E *be qc-dense in* G. *Then* $f(E)$ *is qc-dense in* H.

Proof. Just apply Lemma 5.5. QED

5.5 Unconditional quasi-convexity and potential quasi-convexity

In [64] Markov introduced the following notion:

Definition 5.29 *For a group G, a subset H of G is said to be* unconditionally closed in G *if it is closed in every Hausdorff group topology on G.*

Example 5.30 *For every group G and for every $n > 1$, $G[n] = \{x \in G \mid nx = 0\} \leq G$ is unconditionally closed.*

More precisely, every infinite unconditionally closed subgroup of G is of this form, as established by Perel'man (according to Markov [64]). Here we present a stronger result from [42].

Theorem 5.31 *Let G be an infinite abelian group. Then, for a subgroup H of G, the following properties are equivalent:*

(1) there exists $n \geq 0$ such that $H = G[n]$;

(2) H is unconditionally closed in G;

(3) H is closed in every precompact group topology on G.

We introduced here a new notion of quasi-convexity inspired by Definition 5.29 and Theorem 5.31.

Definition 5.32 *Let G be an abelian group. A subset E of G is* unconditionally quasi-convex in G *if E is quasi-convex in every MAP topology on G.*

Observe that this is equivalent to ask it only for precompact group topologies (see Remark 5.4).

Example 5.33 *According to Theorem 0.5, A_x is unconditionally quasi-convex for every group G and every $x \in G$.*

We include here a result that gives a necessary condition for unconditional quasi-convexity in $\mathbb{Z}$. We will apply this lemma in the following sections.

Lemma 5.34 *Let $E \subseteq \mathbb{Z}$. If E is unconditionally quasi-convex in $\mathbb{Z}$, then for every $z \in \mathbb{Z} \setminus E$ there exists $\chi : \mathbb{Z} \to \mathbb{T}$ such that $\chi(E) \subseteq \mathbb{T}_+$, $\chi(z) \notin \mathbb{T}_+$ and $\chi(1) \in \mathbb{T}$ is irrational.*

Proof. Consider the MAP topology τ_α induce on $\mathbb{Z}$ by the embedding $\chi_\alpha : 1 \to \alpha \in \mathbb{T} \setminus \mathbb{Q}/\mathbb{Z}$. Fix $z \in \mathbb{Z} \setminus E$. Since E is unconditionally quasi-convex in $\mathbb{Z}$, there exists $\xi \in (\mathbb{Z}, \tau_\alpha)^\wedge$ such that χ separates z and E. We want to show that $\xi(1)$ is irrational. By the definition of the topology, $(\mathbb{Z}, \tau_\alpha)^\wedge \cong \mathbb{T}^\wedge \cong \mathbb{Z}$, so that ξ has the form $\xi = k\chi_\alpha$ for some integer k, i.e., $\xi = \chi_{k\alpha}$. Therefore, $\xi(1) = k\alpha$ is irrational. QED

More properties about unconditionally quasi-convex subsets of $\mathbb{Z}$ will be studied in § 6.3. In the following section we show that unconditional closedness and unconditional quasi-convexity coincide for subgroups.

5.5.1 Unconditionally quasi-convex subgroups

In light of what we considered in § 5.2, we are in position to prove the equivalence between unconditional closedness and unconditional quasi-convexity for subgroups.

Theorem 5.35 *Let G be an infinite abelian group. Then, for a subgroup H of G, the following properties are equivalent:*

(1) H is unconditionally closed in G;

(2) H is unconditionally quasi-convex in G.

Proof. Let H be an unconditionally closed subgroup of G. In particular, it is closed in every precompact topology. So, by Lemma 5.11, it is also dually closed, i.e., quasi-convex, in every precompact topology. The converse follows from the fact that if H is quasi-convex (in some precompact topology), then H is (dually closed, hence) closed by Lemma 5.12. This shows that if H is unconditionally quasi-convex in G, then it is closed in every precompact group topology on G. Now we apply Theorem 5.31 and we conclude. QED

Observe that the previous result, jointly with Theorem 5.31, gives a complete algebraic description of the unconditionally quasi-convex subgroups.

5.5.2 Potential quasi-convexity

Markov also defined in [64] the *potentially dense subsets* of a group G as those subsets of G that are dense for some Hausdorff topology on G. We re-interpret this notion of potentiality in our framework as follows:

Definition 5.36 *A subset E of an abelian group G is said to be* potentially quasi-convex in G *if there exists a MAP topology τ on G such that E is quasi-convex in (G, τ).*

Remark 5.37 It is clear that "Unconditional quasi-convexity" implies "potential quasi-convexity". Actually these two notions coincide in the case of *finite subgroups*. This follows from the fact that, by Theorem 0.4 *(1)*, every finite subgroup H of a MAP group G is quasi-convex.
This equivalence does not hold, in general, for finite *subsets*, as we will show in Example 5.40.

Note also that the notion of potential quasi-convexity is equivalent to ask quasi-convexity with respect to the discrete topology, namely:

Remark 5.38 Let $E \subseteq G$. Then E is potentially quasi-convex in G if and only if E is quasi-convex in (G, τ_d).

Proof. One implication is clear since the discrete topology is MAP. Conversely, if E is potentially quasi-convex then it is also quasi-convex in (G, τ_d) by Corollary 5.6. QED

In particular, we deduce that the potential quasi-convexity admits a description in algebraic terms. We will see in Theorem 6.33 that also the unconditional quasi-convexity of *finite* sets admits an algebraic characterization.

Before formulating Example 5.40, let us consider the following claim.

Claim 5.39 *Let α be in $\mathbb{T}$. Then, the following properties are equivalent:*

- $\alpha, 3\alpha \in \mathbb{T}_+$*, and* $2\alpha \notin \mathbb{T}_+$*;*
- $||\alpha|| = \frac{1}{4}$.

Indeed, if $||\alpha|| = \frac{1}{4}$ *then it is clear that* $\alpha, 3\alpha \in \mathbb{T}_+$*, and* $2\alpha = \frac{1}{2} + \mathbb{Z} \notin \mathbb{T}_+$*. Conversely, suppose that* $\alpha \in \mathbb{T}_+$ *and* $2\alpha \notin \mathbb{T}_+$*. Then* $\alpha \in (\frac{1}{8} + \mathbb{Z}, \frac{1}{4} + \mathbb{Z}] \cup [-\frac{1}{4} + \mathbb{Z}, -\frac{1}{8} + \mathbb{Z})$*. Wlog, suppose that* $\alpha \in (\frac{1}{8} + \mathbb{Z}, \frac{1}{4} + \mathbb{Z}]$*. Therefore* $3\alpha \in (\frac{3}{8} + \mathbb{Z}, \frac{3}{4} + \mathbb{Z}]$*. In particular,* $3\alpha \in \mathbb{T}_+$ *implies* $\alpha = \frac{1}{4} + \mathbb{Z}$ *and the claim is proved.*

Example 5.40 *Take* $E = \{0, \pm 1, \pm 3\} \subseteq \mathbb{Z}$*. It is potentially quasi-convex in* $\mathbb{Z}$ *but not unconditionally quasi-convex.*

Proof. Observe that by Claim 5.39 there exists a (unique, up to sign) homomorphism $\chi : \mathbb{Z} \to \mathbb{T}$ such that $\chi(1), \chi(3) \in \mathbb{T}_+$ and $\chi(2) \notin \mathbb{T}_+$, namely $\chi : 1 \to \alpha$ with $||\alpha|| = \frac{1}{4}$. This shows that E is potentially quasi-convex in $\mathbb{Z}$. On the other hand, it also proves that E is not unconditional quasi-convex in $\mathbb{Z}$, according to Lemma 5.34. QED

The previous example is crucial to understand the essence of finite unconditionally quasi-convex sets. Indeed, the proof of the already mentioned

Theorem 6.33 will be based on the idea of Example 5.40.

More examples of potentially quasi-convex subsets of $\mathbb{Z}$ will be considered in the next chapter. Indeed, we will discuss several quasi-convex subsets of $\mathbb{T}$ that are contained in a cyclic group (see Corollary 6.1, Corollary 6.3, Theorem 6.12, Theorem 6.13, among others). Then the following remark applies:

Remark 5.41 Let $E = \{0, \pm e_1, \ldots, \pm e_j, \ldots\} \subseteq \mathbb{Z}$. If

$$\alpha E = \{0, \pm e_1\alpha, \ldots, \pm e_j\alpha, \ldots\}$$

is quasi-convex in $\langle\alpha\rangle \leq \mathbb{T}$, for some $\alpha \in \mathbb{T}$, then E is potentially quasi-convex in $\mathbb{Z}$.

Proof. Just observe that $E = \chi_\alpha^{-1}(\alpha E)$, where $\chi_\alpha : \mathbb{Z} \to \mathbb{T}$ is defined by $1 \to \alpha$, so E is quasi-convex in $\mathbb{Z}$ equipped with the initial topology with relation to χ_α. QED

5.6 Int-quasi-convex sets

Consider the following definition ([14]):

Definition 5.42 *Let G be a topological abelian group. We say that $E \subseteq G$ is* Int-quasi-convex in G *if for every $e \in G \setminus E$ there exists $\chi \in G^\wedge$ such that $\chi(E) \subseteq Int(\mathbb{T}_+) = (-\frac{1}{4}, \frac{1}{4})$ and $\chi(e) \notin \mathbb{T}_+$.*

Clearly, every Int-quasi-convex set is also quasi-convex.

Consider also the following:

Definition 5.43 *A subset E of an abelian group G is* potentially Int-quasi-convex in G *if there exists a MAP topology τ on G such that E is Int-quasi-convex in (G, τ).*

Similarly to what happens with potential quasi-convexity (see Remark 5.38), a subset $E \subseteq G$ is potentially Int-quasi-convex in G if and only if E is Int-quasi-convex in (G, τ_d).

Let us consider $E = \{0, \pm 1, \pm 3\}$. Then E is potentially quasi-convex in $\mathbb{Z}$ by Example 5.40. On the other hand, for every topology τ on $\mathbb{Z}$, E is not Int-quasi-convex in $(\mathbb{Z}, \tau)$, according to the following example.

Example 5.44 *The set $E = \{0, \pm 1, \pm 3\}$ is not Int-quasi-convex in $(\mathbb{Z}, \tau)$, for every topology τ on $\mathbb{Z}$. Indeed, according to Claim 5.39, there no exists a homomorphism $\chi : \mathbb{Z} \to \mathbb{T}$ such that $\chi(1), \chi(3) \in Int(\mathbb{T}_+)$ and $\chi(2) \notin \mathbb{T}_+$.*

In particular, this is an example of quasi-convex set that is not Int-quasi-convex. Nevertheless, there are many subsets of $\mathbb{T}$ that are quasi-convex and Int-quasi-convex in $\mathbb{T}$. Before giving an example (namely, Example 5.46), let us consider an additional definition and some remarks.

Let $E \subseteq G$. Analogously to the quasi-convex hull, we can also define the *Int-quasi-convex hull* $Q_G^\circ(E)$ *of* E *in* G as the intersection of all the Int-quasi-convex subsets of G that contain E.
The following proposition is the counterpart of Theorem 0.4 *(1)* for the Int-quasi-convexity.

Proposition 5.45 *Let E be a finite subset of a MAP group G. Then $Q_G^\circ(E) \subseteq \langle E \rangle$.*

Proof. Fix $y \in G \setminus \langle E \rangle$. There exists a (possibly discontinuous) homomorphism $\chi : G \to \mathbb{T}$ such that $\chi(E) = \{0\}$ and $||\chi(y)|| > \frac{1}{4}$. By Corollary 1.19, we can find a continuous ξ that approximates χ in the following sense: $||\xi(e)|| < \frac{1}{4}$ for every $e \in E$ and $||\xi(y)|| > \frac{1}{4}$. Hence we are done. QED

Example 5.46 *Let E be a quasi-convex set in $\mathbb{T}$ contained in $\mathbb{Z}_m \leq \mathbb{T}$, for some natural number m such that $4 \nmid m$. Then E is Int-quasi-convex in $\mathbb{Z}_m$.*

Proof. According to Proposition 5.45, it suffices to show that for every $z \in \langle E \rangle \setminus E$ there exists a continuous character χ such that $\chi(E) \subseteq Int(\mathbb{T}_+)$ and $\chi(z) \notin \mathbb{T}_+$. So, fix such a z. If $E = \{0\}$ then it is trivial, so suppose that $E \neq \{0\}$. Since E is quasi-convex, we can find a continuous character χ that separates E and z. Now just observe that the image of $\langle E \rangle$ in $\mathbb{T}$ by means of any homomorphism $\mathbb{Z}_m \to \mathbb{T}$ does not contain $\frac{1}{4} + \mathbb{Z}$ (and, consequently, $-\frac{1}{4} + \mathbb{Z}$), in particular $\chi(E) \subseteq Int(\mathbb{T}_+)$ and we are done. QED

Clearly, this can be generalized as follows:

Lemma 5.47 *If $E \subseteq \mathbb{T}$ is such that $E \cap \mathbb{Z}(2^\infty) \subseteq \{0, \frac{1}{2} + \mathbb{Z}\}$, then E is quasi-convex in $\mathbb{T}$ if and only if E is Int-quasi-convex.*

Theorem 5.48 *Let E be a finite subset of $\mathbb{Z}$. Then the following properties are equivalent:*

(a) E is unconditionally quasi-convex in $\mathbb{Z}$;

(b) E is potentially Int-quasi-convex in $\mathbb{Z}$.

Proof. $(a) \implies (b)$ Since E is unconditionally quasi-convex, then, in particular, E is quasi-convex in $\mathbb{Z}$ equipped with the 5-adic topology τ_5. Therefore, for every $z \in \mathbb{Z} \setminus E$, every character of $(\mathbb{Z}, \tau_5)$ that separates E and z is such that $\chi(E) \subseteq Int(\mathbb{T}_+)$, i.e., E is Int-quasi-convex in $(\mathbb{Z}, \tau_5)$. This yields (b).

$(b) \implies (a)$ Fix a MAP topology μ on $\mathbb{Z}$, and let $z \in \mathbb{Z} \setminus E$. We need to show that we can find a μ-continuous character that separates E and z. Since E is potentially Int-quasi-convex in $\mathbb{Z}$, there exists an algebraic homomorphism $\chi : \mathbb{Z} \to \mathbb{T}$ such that $\chi(E) \subseteq \mathrm{Int}(\mathbb{T}_+)$ and $\chi(z) \notin \mathbb{T}_+$. Now apply Corollary 1.19 (possible since E is finite) to find a μ-continuous approximation of χ, i.e., there exists $\xi \in (\mathbb{Z}, \mu)^\wedge$ such that $\xi(E) \subseteq Int(\mathbb{T}_+)$ and $\xi(z) \notin \mathbb{T}_+$. QED

In § 6.3 we will prove a stronger result, namely Theorem 6.33.

5.7 Problems

We already commented in § 5.1 that, in general, there is no equality between $Q_G(E_1 \times E_2)$ and $Q_G(E_1) \times Q_G(E_2)$. We propose the following:

Problem 5.49 *Give sufficient conditions on a pair of sets $E_1, E_2 \subseteq G$ in order to guarantee the equality $Q_G(E_1 \times E_2) = Q(E_1) \times Q_G(E_2)$.*

Lemma 5.47 gives a sufficient condition on a subset E of $\mathbb{T}$ that assures the equivalence between "quasi-convexity" and "Int-quasi-convexity" for E. Nevertheless, the following general problem is open:

Problem 5.50 *Characterize those (finite) quasi-convex sets that are also Int-quasi-convex.*

Chapter 6

Finite quasi-convex sets

In this chapter we switch to the case of finite quasi-convex sets. We start in § 6.1 with the subsets of the torus group $\mathbb{T}$; we offer many relevant examples that underline how the properties of the sets $\mathcal{W}_\alpha$ proved in § 2.2 and §5.3 can be used to obtain quasi-convex subsets of $\mathbb{T}$.
In § 6.2 we study in detail those quasi-convex subsets of $\mathbb{T}$ that are contained in a cyclic subgroup $\langle\alpha\rangle \leq \mathbb{T}$; Lemma 6.8 offers a characterization of such sets. Theorem 6.9, which is based on Theorem 2.9, describes the structure of the quasi-convex subsets of $\langle\alpha\rangle \leq \mathbb{T}$, with $\langle\alpha\rangle \cong \mathbb{Z}$, in terms of "blocks" and "gaps" of multiples of α. Several consequences of this new description are commented. We deal with the case $o(\alpha) < \infty$ in §6.2.2.
In § 6.3 we give a characterization of the finite unconditional quasi-convex subsets of $\mathbb{Z}$ (Theorem 6.33) involving the notion of Int-quasi-convexity introduced in § 5.6 and the new notion of S-potential quasi-convexity (see Definition 6.28).

6.1 Finite quasi-convex sets of $\mathbb{T}$

Throughout this section, we will deal with several examples of finite subsets E of $\mathbb{T}$ that are contained in some cyclic subgroup $\langle\alpha\rangle \leq \mathbb{T}$, so it is extremely important to recall that the computation of their quasi-convex hulls can be done equivalently with respect to $\langle E\rangle \leq \langle\alpha\rangle$ or $\mathbb{T}$ (see Corollary 5.19).

Corollary 6.1 *For every $\alpha \in \mathbb{T}$, the set $A_\alpha = \{0, \pm\alpha\}$ is quasi-convex in $\mathbb{T}$.*

Proof. Apply Corollary 5.21. QED

Corollary 6.2 *If $\alpha_1 \dots \alpha_t \in \mathbb{T}$ are independent, then the set $E = A_{\alpha_1} \cup \dots \cup A_{\alpha_t} = \{0, \pm\alpha_1, \dots, \pm\alpha_t\}$ is quasi-convex in $\mathbb{T}$.*

Proof. Assume $\beta \in Q_{\mathbb{T}}(E)$. This is equivalent to $E^{\triangleright} \subseteq \{\beta\}^{\triangleright} = \mathcal{W}_\beta$. As $E^{\triangleright} = \bigcap_{i=1}^m \mathcal{W}_{\alpha_i}$, the latter property is equivalent to $\beta \in E$, according to Corollary 2.4. QED

For $x \in G$ and $k \in \mathbb{N}$, let

$$E_{x,n} := \{0, \pm x, \pm 2x, \dots, \pm nx\}.$$

Clearly, $E_{x,1} = A_x$ for every x.

Corollary 6.3 *For every irrational $\alpha \in \mathbb{T}$ and $n \in \mathbb{N}$, the set $E_{\alpha,n}$ is quasi-convex.*

Proof. It is clear that $Q_{\mathbb{T}}(E_{\alpha,n}) = Q_{\mathbb{T}}(\{\alpha, 2\alpha, \dots, n\alpha\})$. Assume that $\beta \in Q_{\mathbb{T}}(\{\alpha, 2\alpha, \dots, n\alpha\})$. Since $\{\alpha, 2\alpha, \dots, n\alpha\}^{\triangleright} = \bigcap_i \{i\alpha\}^{\triangleright} = \bigcap_i \mathcal{W}_{i\alpha}$ and $\mathcal{W}_\beta = \{\beta\}^{\triangleright}$, we conclude that $\bigcap_i \mathcal{W}_{i\alpha} \subseteq \mathcal{W}_\beta$. According to Corollary 2.5, for irrational α this yields $\beta = \pm i\alpha$ for some $i = 1, 2, \dots, n$, i.e., $\beta \in E_{\alpha,n}$. QED

The next result shows that, in some cases, the sum of quasi-convex sets is still quasi-convex. Hence we can use finite quasi-convex sets contained in cyclic groups (as Corollary 6.1 and Corollary 6.3) to get quasi-convex subsets of $\mathbb{T}$. In particular, it is a tool to easily construct examples of quasi-convex sets of $\mathbb{T}$ that are not contained in a cyclic group.

Theorem 6.4 *Let $\alpha_1, \alpha_2, \dots, \alpha_t \in \mathbb{T}$ be independent. If $Q_i \subseteq \langle \alpha_i \rangle$ is finite and quasi-convex in $\langle \alpha_i \rangle$ (or, equivalently, in $\mathbb{T}$), then $E = Q_1 + Q_2 + \dots + Q_t$ is quasi-convex in $\mathbb{T}$.*

Proof. Since E is finite, by Corollary 5.19 we have to show that $E = Q_{\langle E \rangle}(E)$. Therefore it is sufficient to consider $\gamma = r_1\alpha_1 + \dots + r_t\alpha_t \in \langle E \rangle \setminus E$ and show that there exists a character $\chi \in E^{\triangleright}$ such that $\chi(\gamma) \notin \mathbb{T}_+$.
By the choice of γ, there exists an index $j \in \{1, \dots, t\}$ such that $r_j\alpha_j \notin Q_j$. Wlog, we can suppose that $j = 1$. Let $0 < q_1^{(i)} < \dots < q_{u_i}^{(i)}$ be integer numbers, for $i = 1, \dots, t$, such that $Q_i = \{0, \pm q_1^{(i)}\alpha_i, \dots, \pm q_{u_i}^{(i)}\alpha_i\}$, and set $M := \max_{h=2,\dots,t} \{q_{u_h}^{(h)}, |r_h|\}$. Since Q_1 is quasi-convex, there exists an integer $= l_0$ such that $l_0 q_i^{(1)}\alpha_1 \in \mathbb{T}_+$ for every $i = 1, \dots, u_1$ and $l_0 r_1 \alpha_1 \notin \mathbb{T}_+$. Choose $0 < \delta < 1/4$ such that (possible since α_1 is irrational)

$$J_i = (l_0 q_i^{(1)}\alpha_1 - \delta, l_0 q_i^{(1)}\alpha_1 + \delta) \subseteq \mathbb{T}_+, \quad i = 1, \dots, u_1,$$

$$J_r = (l_0 r_1 \alpha_1 - \delta, l_0 r_1 \alpha_1 + \delta) \not\subseteq \mathbb{T}_+$$

and define $I_1 = \left(l_0\alpha_1 - \frac{\delta}{tq_{u_1}}, l_0\alpha_1 + \frac{\delta}{tq_{u_1}}\right)$ and $I_h = \left(0, \frac{\delta}{tM}\right)$ for $h = 2, \dots, t$. By Corollary 1.20, there exists an integer k_0 such that $k_0\alpha_i \in I_i$ for $i =$

$1,\dots,t$ (thus $k_0 E_{\alpha_h,M} \subseteq \left(-\frac{\delta}{t},\frac{\delta}{t}\right)$ for every $h = 2,\dots,t$). Then, for every $i = 1,\dots,u_1$, we get $k_0(q_i^{(1)}\alpha_1 + E_{\alpha_2,M} + \ldots + E_{\alpha_t,M}) \subseteq J_i$. This proves that $k_0 \in E^{\triangleright}$. On the other hand, $k_0(r_1\alpha_1 + E_{\alpha_2,M} + \ldots + E_{\alpha_t,M}) \subseteq J_r$, and this implies $k_0\gamma \in J_r \nsubseteq \mathbb{T}_+$. QED

6.1.1 Some examples

As an application of Theorem 6.4, we get:

Example 6.5 *Let $\alpha_1,\dots,\alpha_t \in \mathbb{T}$ be independent. Then:*

- $A_{\alpha_1} + \ldots + A_{\alpha_t}$ *is quasi-convex in $\mathbb{T}$;*
- *for every $k_1,\dots,k_t \in \mathbb{N}$, the set $E_{\alpha_1,k_1} + \ldots + E_{\alpha_t,k_t}$ is quasi-convex in $\mathbb{T}$.*

For $x \in G$ and $k \in \mathbb{N}$, let

$$U_{x,n} := \{0, \pm x, \pm nx\}.$$

Clearly, $U_{x,1} = A_x$ and $U_{x,2} = E_{x,2}$.
For further use, put also

$$F_{x,n} := \{0, \pm 2x, \pm nx\},$$

for every $n \geq 2$. Then $F_{x,n} = U_{2x,n/2}$ whenever n is even.

The hypothesis of linear independence in Theorem 6.4 seems to be necessary:

Example 6.6 *Take $\alpha \in \mathbb{T}$ be irrational.*

- $Q_1 = A_\alpha = \{0, \pm\alpha\}$ *and* $Q_2 = U_{\alpha,5} = \{0, \pm\alpha, \pm 5\alpha\}$ *are quasi-convex in $\mathbb{T}$ but $Q_1 + Q_2$ is not quasi-convex;*
- $Q_1 = E_{\alpha,2} = \{0, \pm\alpha, \pm 2\alpha\}$ *and* $Q_2 = A_{6\alpha} = \{0, \pm 6\alpha\}$ *are quasi-convex in $\mathbb{T}$, but $Q_1 + Q_2$ is not quasi-convex;*
- $Q = U_{\alpha,4} = \{0, \pm\alpha, \pm 4\alpha\}$ *is quasi-convex in $\mathbb{T}$ but $Q + Q$ is not quasi-convex.*

A direct verification of these facts is possible. In any case, we will consider again this example in § 6.2 (see Example 6.14).

6.2 Quasi-convex sets in cyclic subgroups of $\mathbb{T}$

It follows from the definition of quasi-convexity that for every $\alpha \in \mathbb{T}$, $\mathcal{W}_\alpha \cdot \alpha = \langle \alpha \rangle \cap \mathbb{T}_+$ is quasi-convex in $\langle \alpha \rangle$. Similarly, we also have that

$$\text{for every } \alpha \in \mathbb{T} \text{ and for every subset } \mathcal{A} \text{ of } \mathbb{Z},$$

$$\left(\bigcap_{n \in \mathcal{A}} \mathcal{W}_{n\alpha} \right) \cdot \alpha \text{ is quasi-convex in } \langle \alpha \rangle. \tag{6.1}$$

This observation can be pushed forward to easily generate finite quasi-convex sets in a cyclic group. Indeed,

$$\text{if } \alpha \in \mathbb{T} \text{ is irrational and } k \text{ is a positive integer, then}$$

$$\left(\bigcap_{n \in \mathcal{A}} \mathcal{W}_{n\alpha} \right) \cdot \alpha \cap E_{\alpha,k} \text{ is finite and quasi-convex in } \langle \alpha \rangle, \tag{6.2}$$

$$\text{for every subset } \mathcal{A} \text{ of } \mathbb{Z}.$$

With these simple tools we are able to show some interesting examples.

Example 6.7 *Let $\alpha \in \mathbb{T}$ be irrational.*

(1) The sets $U_4 = \{0, \pm\alpha, \pm 4\alpha\}$ and $U_8 = \{0, \pm\alpha, \pm 8\alpha\}$ are quasi-convex in $\mathbb{T}$. Indeed, by Corollary 1.20 we can find a positive integer n such that $||n\alpha|| \in (\frac{1}{8} + \frac{1}{16}, \frac{1}{4})$. Then $U_4 = E_{\alpha,4} \cap (\mathcal{W}_{n\alpha}) \cdot \alpha$. On the other hand, $U_8 = E_{\alpha,8} \cap (\mathcal{W}_{n_1\alpha} \cap \mathcal{W}_{n_2\alpha}) \cdot \alpha$, where n_1 and n_2 are such that $||n_1\alpha|| \in (\frac{1}{8}, \frac{1}{8} + \frac{1}{40})$ and $||n_2\alpha|| \in (\frac{1}{8} - \frac{1}{32}, \frac{1}{8} - \frac{1}{56})$.

(2) The set $F_3 = \{0, \pm 2\alpha, \pm 3\alpha\}$ is quasi-convex. To see it, take n such that $||n\alpha|| \in (\frac{3}{8}, \frac{3}{8} + \frac{1}{24})$. Then $F_3 = E_{\alpha,3} \cap (\mathcal{W}_{n\alpha}) \cdot \alpha$.

Both these examples will be generalized in Theorem 6.12 and Theorem 6.13.

The following easy lemma states that every quasi-convex subset Q of $\langle \alpha \rangle$ admits one of the two representations (6.1) or (6.2), depending on whether Q is finite or not.

Lemma 6.8 *Let $\alpha \in \mathbb{T} \setminus \{0\}$, and let $Q \subseteq \langle \alpha \rangle$ be quasi-convex in $\langle \alpha \rangle$. Then there exists a set $\mathcal{A} \subseteq \mathbb{Z}$ such that $Q = (\bigcap_{n \in \mathcal{A}} \mathcal{W}_{n\alpha}) \cdot \alpha$. Moreover,*

1) if α is rational, then $\mathcal{A}$ can be taken finite;

2) if α is irrational and Q is finite, then there exists a finite $\mathcal{A}' \subseteq \mathcal{A}$ and an integer $k \geq 0$ such that $Q = (\bigcap_{n \in \mathcal{A}'} \mathcal{W}_{n\alpha}) \cdot \alpha \cap E_{\alpha,k}$.

Proof. The main assertion is trivial: just take $\mathcal{A} = Q^{\triangleright}$.
Suppose that α is rational. Wlog, we can suppose $\alpha = \frac{1}{m} + \mathbb{Z}$, for some $m > 1$. Then just take $\mathcal{A} = Q^{\triangleright} \cap \{0, 1, \dots, m-1\}$. So *1)* is clear. To check *2)*, note that $\bigcap_{n\in\mathcal{A}} \mathcal{W}_{n\alpha}$ is not finite, then we need to intersect it with $E_{\alpha,k}$ (which is quasi-convex by Corollary 6.3), where k is the greater integer such that $\pm k\alpha \in Q$. QED

It is clear now that the elementary quasi-convex sets of $\mathbb{Z}$ determine the structure of any quasi-convex set contained in a cyclic group. The following theorem gives an explicit description of this phenomenon.
Given $\alpha \in \mathbb{T}$ irrational, we will say that $Q \subsetneq \langle\alpha\rangle$ contains a *block of length* n (with $0 < n < o(\alpha)$) if there exists $k \in \mathbb{Z}$ such that

$$\{k\alpha, (k+1)\alpha, \dots, (k+n-1)\alpha\} \subseteq Q$$

and that Q has a *gap of length* l (with $l > 0$) if there exists $k \in \mathbb{Z}$ such that

$$\{k\alpha, (k+1)\alpha, \dots, (k+l-1)\alpha\} \cap Q = \emptyset.$$

Theorem 6.9 *Let $\alpha \in \mathbb{T}$ be irrational, and let $Q \subsetneq \langle\alpha\rangle$ be quasi-convex in $\langle\alpha\rangle$. If there exists an integer $m \geq 2$ such that Q contains a block of length $m+1$, then:*

(1) the minimum length of every gap of Q is $m-1$;

(2) $E_{\alpha,r} \subseteq Q$, where $r = \frac{m}{2}$ if m is even and $r = \frac{m-1}{2}$ if m is odd.

Proof. For our hypothesis, the set $\mathcal{W}_{n\alpha}$ contains $\{k, k+1, \dots, k+m\}$ for every $n \in \mathcal{A} = Q^{\triangleright}$ (see Lemma 6.8). Therefore $a_{n\alpha} = \left[\frac{1}{2||n\alpha||}\right] \geq m$, and from Theorem 2.9 we deduce that the length of the gaps of every $\mathcal{W}_{n\alpha}$ is at least $m-1$. This proves *(1)*.
To check *(2)*, observe that $||n\alpha|| \leq \frac{1}{2m}$ since $a_{n\alpha} \geq m$ for every $n \in \mathcal{A}$. If $m = 2r$, it easily implies that $\{0, \pm1, \dots, \pm r\}$ is contained in $\mathcal{W}_{n\alpha}$ for every $n \in \mathcal{A}$ and, therefore, $E_{\alpha,r} \subseteq Q$. The case when m is odd is similar. QED

It should be observed that this theorem implies that the quasi-convex sets that do not contain α are *slim* in the sense that they contain only small blocks of length 1 or 2. Indeed,

Corollary 6.10 *Let $\alpha \in \mathbb{T}$ be irrational. If Q is quasi-convex in $\langle\alpha\rangle$ and $(k-1)\alpha, k\alpha, (k+1)\alpha \in Q$ for some integer k, then $\alpha \in Q$.*

Example 6.11 *If $\alpha \in \mathbb{T}$ is irrational, then $\alpha \in Q_{\mathbb{T}}(\{2\alpha, 3\alpha, 4\alpha\})$.*

The following theorems generalize Example 6.7:

Theorem 6.12 *The set $U_{\alpha,k} = \{0, \pm\alpha, \pm k\alpha\}$ is quasi-convex in $\mathbb{T}$, for every irrational $\alpha \in \mathbb{T}$ and every $k \neq 3$;*

Theorem 6.13 *The set* $F_{\alpha,k} = \{0, \pm 2\alpha, \pm k\alpha\}$ *is quasi-convex in* $\mathbb{T}$*, for every irrational* $\alpha \in \mathbb{T}$ *and every* $k \neq 6$.

We deal with the proof of both these theorems in §6.2.1.

Now compare the following example with Theorem 6.4 and Example 6.6.

Example 6.14 *Let* $\alpha \in \mathbb{T}$ *be irrational.*

- *From Corollary 6.1 and Theorem 6.12 we deduce that the sets* $Q_1 = \{0, \pm\alpha\}$ *and* $Q_2 = \{0, \pm\alpha, \pm 5\alpha\}$ *are quasi-convex in* $\mathbb{T}$*. On the other hand, the sum* $Q_1 + Q_2 = \{0, \pm\alpha, \pm 2\alpha, \pm 4\alpha, \pm 5\alpha, \pm 6\alpha\}$ *contains the block* $E_{\alpha,2} = \{-2\alpha, -\alpha, 0, \alpha, 2\alpha\}$ *of length* 5 *and it has a gap of length* 1*, therefore it cannot be quasi-convex by Theorem 6.9.*
- *The sets,* $Q_1 = \{0, \pm\alpha, \pm 2\alpha\}$ *and* $Q_2 = \{0, \pm 6\alpha\}$ *are quasi-convex in* $\mathbb{T}$ *by, respectively, Corollary 6.3 and Corollary 6.1, but* $Q_1 + Q_2 = \{0, \pm\alpha, \pm 2\alpha, \pm 4\alpha, \pm 5\alpha, \pm 6\alpha, \pm 7\alpha, \pm 8\alpha\}$ *is not quasi-convex since it contains the block* $E_{\alpha,2} = \{-2\alpha, -\alpha, 0, \alpha, 2\alpha\}$ *and it has a gap of length* 1.
- *The set* $Q = \{0, \pm\alpha, \pm 4\alpha\}$ *is quasi-convex in* $\mathbb{T}$ *by Theorem 6.12. However, the sum*

$$Q + Q = \{0, \pm\alpha, \pm 2\alpha, \pm 3\alpha, \pm 4\alpha, \pm 5\alpha, \pm 8\alpha\}$$

  is not quasi-convex since it contains a block $E_{\alpha,5}$ *of length* 11 *and it has a gap of length* 2.

Actually, $L_{\alpha,k} = \{0, \pm\alpha, \pm 2\alpha, \pm 3\alpha, \pm 4\alpha, \pm 5\alpha, \pm k\alpha\}$ is not quasi-convex for every $7 \leq k < 15$ by Theorem 6.9 *(1)*. This fact and its natural generalization will motivate the definition of the sets $R_{\alpha,k}$ in § 6.2.2. We anticipate here that from Corollary 6.26 we will deduce, for example, that also $L_{\alpha,15}$ is not quasi-convex since α is irrational.

On the other hand, $\{0, \pm\alpha, \pm 2\alpha, \pm 3\alpha, \pm 4\alpha, \pm 5\alpha, \pm 16\alpha\}$ is quasi-convex, and more generally

$$\{0, \pm\alpha, \pm 2\alpha, \pm 3\alpha, \pm 4\alpha, \pm 5\alpha, \pm 16\alpha, \dots, \pm(16+k)\alpha\}$$

is quasi-convex for every $0 \leq k \leq 11$. Indeed, every character χ such that $\chi(\alpha) \in (\frac{1}{20} - \frac{1}{20 \cdot 16}, \frac{1}{20})$ makes the job for every k.

6.2.1 Proof of Theorem 6.12 and Theorem 6.13

As a consequence of Claim 5.39, we deduce:

Example 6.15 *The set* $U_{\alpha,3} = \{0, \pm\alpha, \pm 3\alpha\}$ *is quasi-convex in* $\mathbb{T}$ *if and only if* $||m\alpha|| = \frac{1}{4}$*, for a certain integer* m.

In particular, $U_{\alpha,3} = \{0, \pm\alpha, \pm 3\alpha\}$ is not quasi-convex in any infinite subgroup $\langle\alpha\rangle \leq \mathbb{T}$.

If $k \neq 3$ then Theorem 6.12 states that $U_{\alpha,k}$ is quasi-convex in $\mathbb{T}$ for every irrational $\alpha \in \mathbb{T}$. Let us prove this result.

Proof of Theorem 6.12. First note that for $k = 1, 2$ it has been proved in Corollary 6.1 and Corollary 6.3, so we can suppose $k \geq 4$. Since $U_{\alpha,k} \subseteq E_{\alpha,k}$ and $E_{\alpha,k}$ is quasi-convex by Corollary 6.3, it suffices to see that for every $1 < m < k$ there exists $n \in E^{\triangleright}$ such that $nm\alpha \notin \mathbb{T}_+$. To this end, define $Z_m := ((\mathbb{T} \setminus V_m) \cap \mathbb{T}_+) \times \mathbb{T}_+) \subseteq \mathbb{T}^2$, where $V_n = \{x \in \mathbb{T} \mid nx \in \mathbb{T}_+\}$ for every positive integer n are the elementary quasi-convex sets of $\mathbb{T}$ (see §5.3). Fix $2 < m < k$. It is easy to see that the interval $\Delta := (\frac{1}{4m}, \frac{3}{4m})$ is contained in $(\mathbb{T} \setminus V_m) \cap \mathbb{T}_+$, therefore we have $W_m := \Delta \times \mathbb{T}_+ \subseteq Z_m$.
Now, let N be the closed subgroup of $\mathbb{T}^2$ consisting of all pairs of the form (z, kz), where $z \in \mathbb{T}$. By Corollary 1.20, the cyclic subgroup $\langle\alpha, k\alpha\rangle$ is dense in N. Therefore, it suffices to show that N intersects the interior of W_m (indeed, if $(z, kz) \in \mathrm{Int}(W_m)$, then $z, kz \in \mathbb{T}_+$ while $mz \notin \mathbb{T}_+$; by Corollary 1.20 take n such that z can be approximated by $n\alpha$ and we are done).
To prove it we use the representation of $\mathbb{T}^2$ as a quotient of the unit square $S = [0, 1] \times [-1/4, 3/4]$ in the plane with the usual identifications of the opposite sides. Let $s := \left[\frac{k}{4m}\right]$, so that $s \leq k/4m$ and

$$s > \frac{k}{4m} - 1. \tag{6.3}$$

Consider the line L in $\mathbb{R}^2$ determined by the graph of the linear function $f(x) = kx - s$. Its image in $\mathbb{T}^2$ is precisely the subgroup N. Let $a_1 = (s/k, 0), a_2 = ((s + 1)/k, 0) \in S$. Consider the translates L_i of L passing through the points a_i $(i = 1, 2)$, i.e., corresponding to the linear functions $f_1(x) = kx - s$ and $f_2(x) = kx - s - 1$, respectively. Let Γ_i be the segment of L_i restricted to S. It suffices to prove that $\Gamma_1 \cup \Gamma_2$ meets $\mathrm{Int}(W_m)$. To this end we consider two cases.

(i) $f_1(1/4m) = k/4m - s < 1/4$. As $0 \leq f_1(1/4m) < 1/4$, Γ_1 intersects the left side $\{1/4m\} \times [-1/4, 1/4]$ of W_m in an internal point. Therefore, Γ_1 intersects $\mathrm{Int}(W_m)$.

(ii) $f_1(1/4m) \geq 1/4$ (i.e., $s \leq \frac{k-m}{4m}$). Then $f_1(3/4m) > 3/4$, or equivalently $s < \frac{3k-3m}{4m}$ (as $k > m$ entails $\frac{3k-3m}{4m} > \frac{k-m}{4m} \geq s$). This proves $f_2(3/4m) = f_1(3/4m) - 1 > -1/4$. On the other hand, $f_2(1/4m) < 0$ by (6.3). Therefore the points $(1/4m, 0)$ and $(3/4m, -1/4)$ remain in different half planes w.r.t. the segment Γ_2, thus Γ_2 intersects $\mathrm{Int}(W_m)$.

For the case $m = 2$, just take $W_2' := \Delta' \times \mathbb{T}_+ \subseteq Z_2$, where $\Delta' := (1/8, 1/4)$. Define $s, a_i, L_i, \Gamma_i, f_i$ as above. If $f_1(1/8) < 1/4$ then we conclude as in (i). Suppose therefore that $f_1(1/8) \geq 1/4$, i.e. $s \leq \frac{k-2}{8}$. If $k = 4$ then $s = 0$ and $(1/4, 0) \in \Gamma_1$, so Γ_1 meets W_2' in an internal point. If $k > 4$, then $\frac{k-3}{4} > \frac{k-2}{8} \geq s$. This implies that $f_2(1/4) > -1/4$, while $f_2(1/8) < 0$

by (6.3). Thus the points $(1/8, 0)$ and $(1/4, -1/4)$ remain in different half planes w.r.t. the segment Γ_2, and Γ_2 intersects $\mathrm{Int}(W_2')$. The theorem is proved.

Proof of Theorem 6.13. If $k = 1, 2, 3, 4$ it follows, respectively, from Corollary 6.3, Corollary 6.1, Example 6.7 *(1)* and Corollary 6.3. So suppose that $k \geq 5$ (and $k \neq 6$). Since $F_{\alpha,k} \subseteq E_{\alpha,k}$ and $E_{\alpha,k}$ is quasi-convex by Corollary 6.3, it suffices to see that we can separate $F_{\alpha,k}$ from m, for every $1 \leq m < k$, $m \neq 2$.

Let N be the closed subgroup of $\mathbb{T}^2$ consisting of all pairs of the form (z, kz), where $z \in \mathbb{T}$. It follows by Corollary 1.20 that $\langle \alpha, k\alpha \rangle$ is dense in N. Therefore it is sufficient to show that N meets the interior of $Z_{2,m} := ((\mathbb{T} \setminus V_m) \cap V_2) \times \mathbb{T}_+)$ for every $m \in \{1, \dots, k-1\} \setminus \{2\}$ (the sets V_m are the elementary quasi-convex sets of $\mathbb{T}$; see § 5.3).

Fix $m \in \{3, \dots, k-1\} \setminus \{4\}$. There exists an integer $\ell \geq 1$ such that the interval $\Delta^\ell := (\frac{\ell}{4m}, \frac{\ell+2}{4m})$ is contained in $(\mathbb{T} \setminus T_m) \cap T_2$. Indeed, if $m \geq 6$ we can take $\ell = 1$ (because $\frac{3}{4m} \leq \frac{1}{8}$, thus $(\frac{1}{4m}, \frac{3}{4m}) \subseteq [0, \frac{1}{8}] \subseteq T_2$); if $m = 3$ and $m = 5$ take $\ell = 5$ and $\ell = 9$ respectively. Choose such an ℓ and let $W_{\ell,m} = W_m := \Delta^\ell \times \mathbb{T}_+ \subseteq Z_{2,m}$.

We use the representation of $\mathbb{T}^2$ as a quotient of the unit square $S = [0, 1] \times [-1/4, 3/4]$ in the plane with the usual identifications of the opposite sides. Define $s := \left[\frac{\ell k}{4m}\right]$, so that $s \leq \ell k/4m$ and

$$s > \frac{\ell k}{4m} - 1. \tag{6.4}$$

Consider the line L in $\mathbb{R}^2$ determined by the graph of the linear function $f(x) = kx - s$. Its image in $\mathbb{T}^2$ is precisely the subgroup N. Let $a_1 = (s/k, 0), a_2 = ((s+1)/k, 0) \in S$. Consider the translates L_i of L passing through the points a_i $(i = 1, 2)$, i.e., corresponding to the linear functions $f_1(x) = kx - s$ and $f_2(x) = kx - s - 1$, respectively. Let Γ_i be the segment of L_i restricted to S. It suffices to prove that $\Gamma_1 \cup \Gamma_2$ meets $\mathrm{Int}(W_m)$.

(i) If $f_1(\ell/4m) < 1/4$ then Γ_1 intersects internally the segment $\{\ell/4m\} \times [-1/4, 1/4]$. Thus Γ_1 intersects $\mathrm{Int}(W_m)$.

(ii) If $f_1(\ell/4m) \geq 1/4$ (i.e., $s \leq \frac{\ell k - m}{4m}$), then Γ_2 meets W_m in an internal point. To check it, note that $f_2(\ell/4m) < 0$ by (6.4) and that $f_2((\ell+2)/4m) > -1/4 (\Leftrightarrow s < \frac{(\ell+2)k - 3m}{4m})$ according to the following calculation: $\frac{(\ell+2)k-3m}{4m} > \frac{\ell k - m}{4m} \geq s$. Therefore the points $(\ell/4m, 0)$ and $((\ell+2)/4m, -1/4)$ remain in different half planes w.r.t. the segment Γ_2, thus Γ_2 intersects $\mathrm{Int}(W_m)$.

For the case $m = 1$, take $\Delta' := (1/4, 3/8)$ and $W_4' := \Delta' \times \mathbb{T}_+ \subseteq Z_{2,1}$. Now, $s = \left[\frac{k}{4}\right]$, a_i, L_i, Γ_i, f_i are as before.

If $f_1(1/4) < 1/4$ it is clear that we have finished. Suppose that $f_1(1/4) \geq 1/4$ (i.e., $s \leq \frac{k-1}{4}$). Then (since $k \geq 5$) $f_2(3/8) > -1/4$, or equivalently $s < \frac{3k-6}{8}$ (because $\frac{3k-6}{8} > \frac{k-1}{4} \geq s$), and this, together with the fact that $f_2(1/4) < 0$ by definition of s, assures that Γ_2 meets the interior of W_4'.

If $m = 4$, take $\Delta'' := (1/16, 1/8) \cup (3/8, 7/16)$ and $W''_4 := \Delta'' \times \mathbb{T}_+ \subseteq Z_{2,4}$. Now define $s := \left[\frac{k}{16}\right]$, $a_i = ((s+i-1)/k, 0)$, $f_i = kx - s - (i-1)$ for $i = 1, 2, 3$ and L_i, Γ_i similarly to what we did before.
If $f_1(1/16) < 1/4$ we are done. Suppose $f_1(1/16) \geq 1/4 (\Leftrightarrow s \leq \frac{k-4}{16})$. If $k > 8$, then we can do the same as in (ii): $f_2(1/8) > -1/4$, i.e. $s < \frac{k-6}{8}$ (because $\frac{k-6}{8} > \frac{k-4}{16} \geq s$) and $f_2(1/16) < 0$ by (6.4). The same idea still holds when $k = 7$ (because $f_2(1/16) = -9/16 < 0$ and $f_2(1/8) = -1/8 > -1/4$) and when $k = 8$ ($f_2(1/16) = -1/2 < 0$ and $f_2(1/8) = 0 > -1/4$). If $k = 5$ just note that $(2/5, 0) \in \Gamma_3 \cap \mathrm{Int}(W''_4)$.

6.2.2 Quasi-convex sets in finite cyclic groups

In Corollary 6.3 we have shown that $E_{x,k} \subseteq \langle x \rangle$ is quasi-convex in $\mathbb{T}$ and, equivalently (by Corollary 5.19), in $\langle x \rangle$, for every irrational $x \in \mathbb{T}$. On the other hand, the equality $E_{x,k} = Q_{\langle x \rangle}(E_{x,k})$ is not guaranteed in the case in which x is rational. Here we deal with the quasi-convexity of the sets of the form $E_{\frac{1}{4m},k} \subseteq \langle \frac{1}{4m} + \mathbb{Z} \rangle \cong \mathbb{Z}_{4m} \leq \mathbb{T}$, with $m, k \geq 1$. Again, recall that $Q_{\mathbb{T}}(E_{\frac{1}{4m},k}) = Q_{\langle \frac{1}{4m} + \mathbb{Z} \rangle}(E_{\frac{1}{4m},k})$ by Corollary 5.19.
For every $m \geq 1$, put $\alpha_m := \frac{1}{4m} + \mathbb{Z}$. Observe first that

Fact 6.16 *$E_{\alpha_m,2m} = \langle \alpha_m \rangle$; in particular, $E_{\alpha_m,2m}$ is quasi-convex in $\langle \alpha_m \rangle$.*

So, we will assume $k < 2m$. Now let us see that $E_{\alpha_m,k}$ is qc-dense in $\langle \alpha_m \rangle$ whenever $m < k < 2m$.

Lemma 6.17 *If $m < k < 2m$, then $Q_{\langle \alpha_m \rangle}(E_{\alpha_m,k}) = \langle \alpha_m \rangle$.*

Proof. Fix $m < k < 2m$ and put $E := E_{\alpha_m,k}$. It suffices to show that $E^{\triangleright} = \{0\}$. Indeed, suppose that $\chi \in E^{\triangleright}$. Then, by Fact 2.2, $\chi(\alpha_m) \in \mathbb{T}_k$. Now just note that $\mathbb{T}_k \cap \langle \alpha_m \rangle = \{0\}$ by our choice of k; this implies that $\chi = 0$. QED

Now fix $k \leq m$ and let us characterize the quasi-convex hull of $E_{\alpha_m,k}$. Since $k \leq m$, there exist $q \geq 1$ and $0 \leq r < k$ such that $m = qk + r$.

Theorem 6.18 *Fix $k \leq m$ and let $q \geq 1$, $0 \leq r < k$ be such that $m = qk+r$. Then $Q_{\langle \alpha_m \rangle}(E_{\alpha_m,k}) = E_{\alpha_m,[m/q]}$.*

Proof. We want to calculate $(E_{\alpha_m,k})^{\triangleright\triangleleft}$. Observe that $\langle \alpha_m \rangle^{\wedge} = \langle \alpha_m \rangle$, so every $\chi \in \langle \alpha_m \rangle$ is of the form $\chi_s = s \cdot id_{\langle \alpha_m \rangle} : x \mapsto sx$. Now, if $\chi_s \in (E_{\alpha_m,k})^{\triangleright}$, then $\frac{si}{4m} \in \mathbb{T}_+$ for every $i \in \{1, \dots, k\}$, therefore — by Lemma 5.22 — $\frac{s}{4m} \leq \frac{1}{4k}$, which is equivalent to $s \leq \frac{m}{k} = q + \frac{r}{k}$. Now, $\frac{r}{k} < 1$, so we have $\chi_s \in (E_{\alpha_m,k})^{\triangleright} \implies s \leq q$, i.e., $(E_{\alpha_m,k})^{\triangleright} = \{\chi_s \mid 0 \leq s \leq q\}$. This yields $(E_{\alpha_m,k})^{\triangleright\triangleleft} = \langle \alpha_m \rangle \cap \bigcap_{s=1}^{q} V_s$, where the V_s's are the elementary quasi-convex subsets of $\mathbb{T}$ (see §5.3), hence $(E_{\alpha_m,k})^{\triangleright\triangleleft} = \{\frac{\ell}{4m} : ||\frac{\ell}{4m}|| \leq \frac{1}{4q}\} = \{\frac{\ell}{4m} : ||\ell|| \leq \frac{m}{q}\} = \{\frac{\ell}{4m} : ||\ell|| \leq [\frac{m}{q}]\} = E_{\alpha_m,[m/q]}$. QED

Corollary 6.19 *If m, k are such that $m = qk+r$ with $q \geq 1$ and $0 \leq r < k$, then $E_{\alpha_m,k}$ is quasi-convex in $\langle \alpha_m \rangle$ if and only if $q > r$.*

Proof. This follows from Theorem 6.18 since $k = [m/q]$ if and only if $q > r$. QED

This immediately yields the following

Corollary 6.20 *If $k|m$, then $E_{\alpha_m,k}$ is quasi-convex in $\langle \alpha_m \rangle$.*

Example 6.21 *The set $E_{\frac{1}{4m},2}$ is quasi-convex in $\langle \frac{1}{4m} + \mathbb{Z} \rangle \leq \mathbb{T}$ if and only if $m \neq 3$.*
Indeed, if $m = 1$ then $E_{\frac{1}{4},2} = \langle \frac{1}{4} + \mathbb{Z} \rangle$ is quasi-convex (see also Fact 6.16). Suppose now that $m \geq 2$. Write $m = 2q + r$ with $q \geq 1$ and $r \in \{0, 1\}$. By Corollary 6.19, $E_{\frac{1}{4},2}$ is quasi-convex if and only if $q \neq 1 \neq r$, i.e., $m \neq 3$.

Example 6.22 *The set $E_{\frac{1}{4m},3}$ is quasi-convex in $\langle \frac{1}{4m} + \mathbb{Z} \rangle \leq \mathbb{T}$ if and only if $m \notin \{2, 4, 5, 8\}$.*
Indeed, if $m = 1$ then $E_{\frac{1}{4m},3} = \langle \frac{1}{4m} + \mathbb{Z} \rangle$ is quasi-convex (see also Fact 6.16). If $m = 2$, then $E_{\frac{1}{4m},3} \subsetneq Q_{\langle \frac{1}{4m}+\mathbb{Z} \rangle}(E_{\frac{1}{4m},3}) = \langle \frac{1}{4m} + \mathbb{Z} \rangle$ by Lemma 6.17. Now suppose $m \geq 3$ and write $m = 3q + r$ with $q \geq 1$ and $r \in \{0, 1, 2\}$. Then, by Corollary 6.19, $E_{\frac{1}{4m},3}$ is not quasi-convex if $r = 1$ and $q = 1$ (i.e., $m = 4$) and if $r = 2$ and either $q = 1$ (i.e., $m = 5$) or $q = 2$ (i.e., $m = 8$).

Example 6.23 *Let $m > 1$. Then $E_{\frac{1}{4m},m-1}$ is quasi-convex in $\langle \frac{1}{4m} + \mathbb{Z} \rangle \leq \mathbb{T}$ if and only if $m = 2$.*
Indeed, if $m = 2$ then $E_{\frac{1}{8},1} = A_{\frac{1}{8}}$ is quasi-convex by Theorem 0.5 (or, equivalently, by Corollary 6.20). Now suppose $m \geq 3$ and write $m = q(m-1) + r$. Then $q = r = 1$, hence $E_{\frac{1}{4m},m-1}$ is not quasi-convex by Corollary 6.19.

Example 6.24 *Let $m > 2$. Then $E_{\frac{1}{4m},m-2}$ is quasi-convex in $\langle \frac{1}{4m} + \mathbb{Z} \rangle \leq \mathbb{T}$ if and only if $m \in \{3, 4\}$.*
Indeed, if $m = 3$ then $E_{\frac{1}{12},1} = A_{\frac{1}{12}}$ is quasi-convex by Theorem 0.5. If $m = 4$, then $m - 2 = 2|4$ and $E_{\frac{1}{16},2}$ is quasi-convex by Corollary 6.20. Now suppose $m \geq 5$ and write $m = q(m-2) + r$. Then necessarily $q = 1$ and $r = 2$, hence $E_{\frac{1}{4m},m-1}$ is not quasi-convex by Corollary 6.19.

In order to discuss more examples of quasi-convex subsets of $\mathbb{T}$, let us consider a set of the form $E_{\alpha,s} \cup \{\pm t\alpha\}$, for some $\alpha \in \mathbb{T}$ and $t > s$. If $s < t < 3s$ then $Q_{\mathbb{T}}(E_{\alpha,k} \cup \{\pm t\alpha\}) = Q_{\mathbb{T}}(E_{\alpha,3s})$, according to Theorem 6.9 *(1)*. This motivates the following notation:

$$R_{\alpha,s} := E_{\alpha,s} \cup \{\pm 3s/\alpha\}.$$

Lemma 6.25 *Let $\alpha \in \mathbb{T}$ and $s \geq 1$. If $R_{\alpha,s}$ is quasi-convex in $\mathbb{T}$, then $||n\alpha|| = \frac{1}{4s}$ for some $n \geq 1$.*

Proof. Suppose that $R_{\alpha,s}$ is quasi-convex in $\mathbb{T}$. In particular, there exists n such that $\{0, 1, \ldots, s, 3x\} \subseteq \mathcal{W}_{n\alpha}$. Since $R_{\alpha,s} \supseteq E_{\alpha,s}$, from Theorem 2.9 we deduce that $a_{n\alpha} = \left[\frac{1}{2||n\alpha||}\right] \in \{2s, 2s+1\}$. If $a_{n\alpha} = 2s+1$, then $2s+1 \leq \frac{1}{2||n\alpha||} \lneq 2s+2 \iff \frac{1}{2(2s+2)} \lneq ||n\alpha|| \leq \frac{1}{2(2s+1)}$, otherwise $2s \leq \frac{1}{2||n\alpha||} \lneq 2s+1 \iff \frac{1}{2(2s+1)} \lneq ||n\alpha|| \leq \frac{1}{4s}$. Now just observe that if $||n\alpha|| < \frac{1}{4s}$ then $3s \notin \mathcal{W}_{n\alpha}$ and we are done. QED

Corollary 6.26 *If $\alpha \in \mathbb{T}$ is irrational, then $R_{\alpha,s}$ is not quasi-convex in $\mathbb{T}$, for every $s \geq 1$.*

Lemma 6.27 *Let $m \geq 3$ and $s \geq 1$. If $3s|m$, then $R_{\frac{1}{4ms},s}$ is quasi-convex in $\mathbb{T}$.*

Proof. Put $\alpha = \frac{1}{4ms}$. Since $R_{\alpha,s} \subseteq E_{\alpha,3s}$ and $E_{\alpha,3s}$ is quasi-convex by Corollary 6.20, it suffices to separate those elements of $\mathbb{T}$ of the form $\frac{\ell}{4ms} + \mathbb{Z}$ with $s < \ell < 3s$. To this aim, let us consider the character $\chi_m : x \mapsto mx$ of $\langle \alpha + \mathbb{Z} \rangle$. Then $\chi_m(R_{\alpha,s}) \subseteq \mathbb{T}_+$ since $\chi_m(E_{\alpha,s}) = \{0, \pm\frac{1}{4s}, \ldots, \pm\frac{1}{4}\} \subseteq \mathbb{T}_+$ and $\chi_m(\frac{3}{4m} + \mathbb{Z}) = \frac{3}{4} + \mathbb{Z} \in \mathbb{T}_+$, and $\frac{1}{4} < ||\chi(\frac{\ell}{4ms} + \mathbb{Z})|| < \frac{3}{4}$ for every $s < \ell < 3s$. QED

6.3 Unconditional quasi-convexity for finite subsets of $\mathbb{Z}$

Theorem 0.5 offers a nice example of an unconditionally quasi-convex finite set. On the other hand, there are simple examples of finite sets, some of them contained in $\mathbb{Z}$, that are not unconditionally quasi-convex (see Example 5.40).

In this section we characterize the finite unconditionally quasi-convex subsets of $\mathbb{Z}$. To this aim, let us introduce a new notion of quasi-convexity for subsets of $\mathbb{Z}$ which is weaker then potential Int-quasi-convexity.

Definition 6.28 *For a subset $E \subseteq \mathbb{Z}$, we say that E is* S-potentially quasi-convex in $\mathbb{Z}$ *if for every $z \in \mathbb{Z} \setminus E$, there exists $\chi : \mathbb{Z} \to \mathbb{T}$ such that the following conditions hold:*

$$\chi(E) \subseteq \mathbb{T}_+ \text{ and } \chi(z) \notin \mathbb{T}_+; \tag{S-pot a}$$

$$\text{if } \chi(e_1) = \frac{1}{4} + \mathbb{Z} \text{ for some } 0 < e_1 \in E,$$

$$\text{then } \chi(e) \neq \frac{3}{4} + \mathbb{Z} \text{ for every } 0 < e \in E. \tag{S-pot b}$$

Lemma 6.29 *If E is S-potentially quasi-convex in $\mathbb{Z}$, then it is potentially quasi-convex (in particular, E is symmetric).*

Proof. This is a consequence of (S-pot a), according to Remark 5.38. QED

Observe that every potentially Int-quasi-convex in $E \subseteq \mathbb{Z}$ is S-potentially quasi-convex, as announced in the following lemma.

Lemma 6.30 *Let E be a potentially Int-quasi-convex subset of $\mathbb{Z}$. Then E is S-potentially quasi-convex.*

Proof. By our hypothesis, for every $z \in \mathbb{Z} \notin E$ there exists a homomorphism $\chi : \mathbb{Z} \to \mathbb{T}$ such that $\chi(E) \subseteq Int(\mathbb{T}_+)$ and $\chi(z) \notin \mathbb{T}_+$. Therefore (S-pot a) holds and (S-pot b) plays no role in this case. QED

Here we discuss some examples.

Example 6.31 *For every $k \geq 1$, $E_{1,k} = \{0, \pm1, \dots, \pm k\}$ is S-potentially quasi-convex in $\mathbb{Z}$.*
Indeed, fix $z \in \mathbb{Z} \setminus E$. Then (S-pot a) *holds according to Corollary 6.3 combined with Remark 5.41. Now,* (S-pot b) *holds as a consequence of the following more general fact:*

Claim. *For every $\chi : \mathbb{Z} \to \mathbb{T}$ such that $\chi(E_{1,k}) \subseteq \mathbb{T}_+$, if $\chi(e) = \frac{1}{4} + \mathbb{Z}$ for some $e \in E_{1,k}$ then $e = \pm k$.*
To see it, suppose that $\chi(e) = \frac{1}{4} + \mathbb{Z}$ for some $e \neq \pm k$ (in particular, $e + 1 \in E_{1,k}$); wlog, assume $e > 0$. Since $\chi(1), \dots, \chi(k) \in \mathbb{T}_+$, $||\chi(1)|| \leq \frac{1}{4k} \leq \frac{1}{4}$ according to Lemma 5.22. Therefore $\chi(e+1) = \chi(e) + \chi(1) \notin \mathbb{T}_+$ and this contradicts the fact that $\chi(E_{1,k}) \subseteq \mathbb{T}_+$.

Now, the claim implies that if $\chi(e_1) = \frac{1}{4} + \mathbb{Z}$ for some $e_1 \in E_{1,k}$, then there no exists $e \in E_{1,k}$ such that $e \neq -e_1$ and $\chi(e) = \frac{3}{4} + \mathbb{Z}$. This finishes the example.

Example 6.32 *If $k \notin 3+4\mathbb{Z}$, then $U_{1,k} = \{0, \pm1, \pm k\}$ is S-potentially quasi-convex in $\mathbb{Z}$.*
Indeed, fix $z \in \mathbb{Z} \setminus E$. Then (S-pot a) *holds according to Theorem 6.12 combined with Remark 5.41. Now,* (S-pot b) *holds according to the following:*

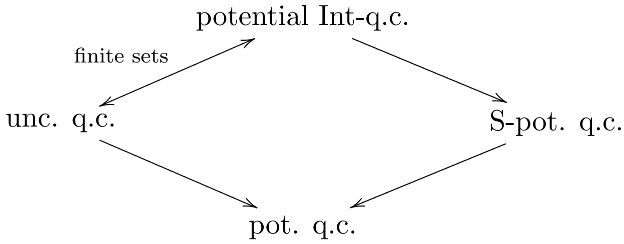
Claim. *Fix $\chi : \mathbb{Z} \to \mathbb{T}$. Suppose that one of the following two properties holds:*

- $\chi(1) = \frac{1}{4} + \mathbb{Z}$ *and* $\chi(k) = \frac{3}{4} + \mathbb{Z}$;
- $\chi(1) = \frac{3}{4} + \mathbb{Z}$ *and* $\chi(k) = \frac{1}{4} + \mathbb{Z}$.

Then $k \in 3 + 4\mathbb{Z}$.

We observe that Example 6.32 does not characterize those sets of the form $U_{1,k}$ that are S-potentially quasi-convex (see also Conjecture 6.42). Indeed, let us consider $E = U_{1,7}$. Then Example 6.32 does not apply, but E is S-potentially quasi-convex since every $z \notin E$ can be separated from E by the characters $\chi_7, \chi_9 : \mathbb{Z} \to \mathbb{T}$, $\chi_7 : n \mapsto \frac{n}{7} + \mathbb{Z}$ and $\chi_9 : n \mapsto \frac{n}{9} + \mathbb{Z}$.

According to Remark 5.37, Theorem 5.48, Lemma 6.29 and Lemma 6.30, the following diagram holds:



The next theorem states that "unconditional q.c.", "S-potential q.c." and "potential Int-q.c." actually coincide for finite subsets.

Theorem 6.33 *Let* $E \subseteq \mathbb{Z}$ *be finite. Then the following properties are equivalent:*

(a) E *is unconditionally quasi-convex in* $\mathbb{Z}$*;*

(b) E *is potentially Int-quasi-convex in* $\mathbb{Z}$*;*

(c) E *is S-potentially quasi-convex in* $\mathbb{Z}$.

Recall that $\{x\}_f$ denotes the fractional part of x, for every $x \in \mathbb{R}$, and that $\widetilde{\chi} : \mathbb{Z} \to \mathbb{R}$ (with $\widetilde{\chi}(1) \in [0, 1)$) is the natural lifting of χ, for every $\chi : \mathbb{Z} \to \mathbb{T}$ (see Lemma 1.2).

Proof. The equivalence *(a)* $\iff$ *(b)* is Theorem 5.48, while *(b)* $\implies$ *(c)* is Lemma 6.30. Let us show that *(c)* $\implies$ *(b)*.
Suppose that E is S-potentially quasi-convex (in particular, E is symmetric). Fix $z \in \mathbb{Z} \setminus E$. We can suppose wlog that $z > 0$. According to (*S-pot a*), there exists a (not necessarily continuous) $\chi : \mathbb{Z} \to \mathbb{T}$ such that $\chi(E) \subseteq \mathbb{T}_+$ and $\chi(z) \notin \mathbb{T}_+$.

[case 1] Suppose that $\chi(e) \neq \frac{1}{4} + \mathbb{Z}$ for every $e \in E$, i.e., $\chi(E) \subseteq \mathrm{Int}(\mathbb{T}_+)$. Then E is Int-quasi-convex in $(\mathbb{Z}, \tau_d)$ and we are done.

[case 2] Suppose now that there exists $e_1 \in E$ such that $\chi(e_1) = \frac{1}{4} + \mathbb{Z}$, i.e., $\{\widetilde{\chi}(e_1)\}_f = 1/4$. We suppose wlog that $e_1 > 0$.

Put $E_\uparrow := \{e \in E \mid e > 0, 0 < \{\widetilde{\chi}(e)\}_f \leq 1/4\}$, $E_\downarrow := \{e \in E \mid e > 0, \{\widetilde{\chi}(e)\}_f \geq 3/4\}$ and $E_0 := \{e \in E \mid e > 0, \{\widetilde{\chi}(e)\}_f = 0\}$. Since $\chi(E) \subseteq \mathbb{T}_+$, it is clear that if $0 < e \in E$ then either $e \in E_\uparrow$, $e \in E_\downarrow$ or $e \in E_0$ (i.e., $E_\uparrow, E_\downarrow, E_0$ is a partition of the subset of all the strictly positive elements of E). Moreover, according to (S-pot b), $\{\widetilde{\chi}(e)\}_f \neq 3/4$ for every $e \in E$ such that $e > 0$, therefore $E_\downarrow = \{e \in E \mid e > 0, \{\widetilde{\chi}(e)\}_f > 3/4\}$.

Put

$$\delta_\uparrow := \min\{\{\widetilde{\chi}(e)\}_f \mid e \in E_\uparrow\}$$

and

$$\delta_\downarrow := \begin{cases} \min\{\{\widetilde{\chi}(e)\}_f - \frac{3}{4} \mid e \in E_\downarrow\} & \text{if } E_\downarrow \neq \emptyset; \\ 1/4 & \text{if } E_\downarrow = \emptyset; \end{cases}$$

Observe that:

- $\delta_\uparrow > 0$, because $\{\widetilde{\chi}(e)\}_f > 0$ for every $e \in E_\uparrow$ and $E_\uparrow$ is finite;
- $\delta_\downarrow > 0$, because $\{\widetilde{\chi}(e)\}_f - 3/4 > 0$ for every $e \in E_\downarrow$ and $E_\downarrow$ is finite (here we use (S-pot b)).

Also recall that $1/4 < \{\widetilde{\chi}(z)\}_f < 3/4$ since $\chi(z) \notin \mathbb{T}_+$. So

$$\delta := \min\{\delta_\uparrow, \delta_\downarrow, \{\widetilde{\chi}(z)\}_f - 1/4, 3/4 - \{\widetilde{\chi}(z)\}_f\} > 0$$

and we can choose ε' such that

$$0 < \varepsilon' < \delta.$$

In particular, observe that $\varepsilon' < \delta_\uparrow$, and this suffices to state that

$$\varepsilon' < \{\widetilde{\chi}(y)\}_f \text{ for every } y \in \{z\} \cup E_\uparrow \cup E_\downarrow. \tag{6.5}$$

Define $\xi : \mathbb{Z} \to \mathbb{T}$ by $\xi(1) = \chi(1) - \varepsilon/N$, where $N := \max\{|y| : y \in \{z\} \cup E\}$ and ε is such that $0 < \varepsilon < \varepsilon'$ and $\{\widetilde{\chi}(1)\}_f - \varepsilon/N > 0$. We claim that

$$\xi(E) \subseteq \mathrm{Int}(\mathbb{T}_+) \tag{6.6}$$

and

$$\xi(z) \notin \mathbb{T}_+. \tag{6.7}$$

In order to prove (6.6) and (6.7), we need some additional considerations. Observe fist that to prove (6.6) it suffices to show that $\xi(e) \in \mathrm{Int}(\mathbb{T}_+)$ for every $0 < e \in E$. Now, for every $n \in \mathbb{N}$, we have:

$$\widetilde{\xi}(n) = \widetilde{\chi}(n) - n\varepsilon/N. \tag{6.8}$$

Fix $y \in \{z\} \cup E_\uparrow \cup E_\downarrow$, then $y\varepsilon/N \leq \varepsilon$, so (6.8) implies

$$\widetilde{\chi}(y) - \varepsilon \leq \widetilde{\xi}(y) < \widetilde{\chi}(y). \tag{6.9}$$

Let us show that this implies

$$\{\widetilde{\chi}(y)\}_f - \varepsilon \leq \{\widetilde{\xi}(y)\}_f < \{\widetilde{\chi}(y)\}_f. \tag{6.10}$$

Indeed, according to (6.5), we have that $y\varepsilon/N \leq \varepsilon < \{\widetilde{\chi}(y)\}_f$, therefore

$$[\widetilde{\chi}(y)] = [\widetilde{\chi}(y) - \varepsilon] = [\widetilde{\chi}(y)] = [\widetilde{\chi}(y) - y\varepsilon/N],$$

and this coincides with $[\widetilde{\xi}(y)]$ by (6.8). This permits to rewrite (6.9) in terms of fractional parts as follows:

$$\{\widetilde{\chi}(y) - \varepsilon\}_f \leq \{\widetilde{\xi}(y)\}_f < \{\widetilde{\chi}(y)\}_f.$$

Moreover, $\varepsilon < \{\widetilde{\chi}(y)\}_f$ also implies $\{\widetilde{\chi}(y) - \varepsilon\}_f = \{\widetilde{\chi}(y)\}_f - \varepsilon$, and this proves (6.10).
Now we are in position to prove (6.6). Fix $e > 0$. Then — as already noted — $e \in E_\uparrow \cup E_\downarrow \cup E_0$. We consider three cases.

- Suppose that $e \in E_\uparrow$. Then $\{\widetilde{\chi}(e)\}_f \leq 1/4$, and from (6.10) we deduce $\{\widetilde{\xi}(e)\}_f < \{\widetilde{\chi}(e)\}_f \leq 1/4$. Then $\widetilde{\xi}(e) \in \mathrm{Int}(\mathbb{T}_+)$.
- Suppose that $e \in E_\downarrow$. Then, by our choice of ε, $\varepsilon < \varepsilon' < \delta \leq \delta_\downarrow \leq \{\widetilde{\chi}(e)\}_f - 3/4$, in particular $\{\widetilde{\chi}(e)\}_f - \varepsilon > 3/4$. Then from (6.10) we get $\{\widetilde{\xi}(e)\}_f \geq \{\widetilde{\chi}(e)\}_f - \varepsilon > 3/4$. Therefore $\xi(e) \in \mathbb{T}_+$.
- Suppose that $e \in E_0$. Then $\chi(e) =: \ell \in \mathbb{Z}_+ = \{1, 2, \ldots\}$ by definition of E_0. By (6.8), $\widetilde{\xi}(e) = \ell - e\varepsilon/N$. Now, $e\varepsilon/N \leq \varepsilon < \delta \leq 1/4$, so $\{\widetilde{\xi}(e)\}_f = \{\ell - e\varepsilon/N\}_f = 1 - e\varepsilon/N > 3/4$. Hence, $\xi(e) \in \mathrm{Int}(\mathbb{T}_+)$.

Now we prove (6.7). We have $\varepsilon < \{\widetilde{\chi}(z)\}_f - 1/4$, so $\{\widetilde{\chi}(z)\}_f - \varepsilon > 1/4$. Then from (6.10) we deduce $1/4 < \{\widetilde{\chi}(z)\}_f - \varepsilon \leq \{\widetilde{\xi}(z)\}_f < \{\widetilde{\chi}(z)\}_f < 3/4$, in particular $\xi(z) \notin \mathbb{T}_+$.

So we have proved (6.6) and (6.7). Then E is Int-quasi-convex in $(\mathbb{Z}, \tau_d)$ and we are done. QED

Applying Theorem 6.33 to Example 6.31 and Example 6.32, we deduce:

Example 6.34 *For every $k \geq 1$, $E_{1,k} = \{0, \pm 1, \ldots, \pm k\}$ is unconditionally quasi-convex in $\mathbb{Z}$.*

Example 6.35 *If $k \notin 3 + 4\mathbb{Z}$, then $U_{1,k} = \{0, \pm 1, \pm k\}$ is unconditionally quasi-convex in $\mathbb{Z}$.*

Observe that as a consequence of Theorem 6.33 we get that S-potential quasi-convexity and potential quasi-convexity are not equivalent. Indeed, according to Example 5.40, $\{0, \pm 1, \pm 3\}$ is potentially quasi-convex but not S-potentially quasi-convex.

The next result extends Theorem 6.33 to any group G that possesses a non-torsion element.

Theorem 6.36 *Let E be a finite subset of $\mathbb{Z}$. Then E is S-potentially quasi-convex in $\mathbb{Z}$ if and only if $xE = \{xe \mid e \in E\}$ is unconditionally quasi-convex in every MAP group G with a non-torsion element x.*

Proof. Suppose that E is S-potentially quasi-convex in $\mathbb{Z}$. Hence, by Theorem 6.33, E is unconditionally quasi-convex in $\mathbb{Z}$. Now, fix G and $x \in G$ non-torsion. Then xE is unconditionally quasi-convex in $\langle x \rangle \cong \mathbb{Z}$, hence in G according to Corollary 5.19 (since E is finite). The converse is trivial. QED

In particular, from Example 6.34 we deduce that $E_{x,k}$ is unconditionally quasi-convex in every infinite cyclic group $\langle x \rangle$. If $\langle x \rangle$ is finite, then the only Hausdorff topology on $\langle x \rangle$ is the discrete one, so it is obsolete to speak about "unconditional quasi-convexity" in $\langle x \rangle$.
The following result describes for which *finite* cyclic groups all sets $E_{x,k}$ are quasi-convex.

Proposition 6.37 *Let G be a group and $x \in G$ a torsion element. Then $E_{x,k}$ is quasi-convex in G for every $k \in \mathbb{N}$ iff $o(x) \leq 5$.*

Proof. Assume that $d = o(x) > 5$. Pick k in such a way that $E_{x,k}$ contains — roughly speaking — more than the half of the elements of $\langle x \rangle$, but still $E_{x,k} \neq \langle x \rangle$ (e.g., $k = \frac{d-1}{2} - 1$ when d is odd, $k = \frac{d}{2} - 1$ otherwise). It is clear now that the polar of $E_{x,k}$ is trivial, hence $Q_G(E_{x,k}) = \langle x \rangle$.
Conversely, suppose that $o(x) \leq 5$. Then for every $k > 1$, the set $E_{x,k}$ coincides with the cyclic subgroup $\langle x \rangle$, hence $E_{x,k}$ is quasi-convex (see Remark 5.37). On the other hand, the quasi-convexity of $E_{x,1}$ was established in Theorem 0.5 for arbitrary elements x. QED

6.4 Problems

Concerning a possible generalization of Theorem 6.12 and Theorem 6.13:

Conjecture 6.38 *It has been conjectured in [14] that for every irrational $\alpha \in \mathbb{T}$ the set $\{0, \pm r\alpha, \pm s\alpha\}$ is quasi-convex if and only if $s \neq 3r$. This would give a complete characterization of the five element quasi-convex subsets of $\mathbb{T}$ that are contained in an infinite cyclic group.*

Lemma 6.27 gives a partial answer to the following

Problem 6.39 *Characterize for which $\alpha \in \mathbb{Q}/\mathbb{Z}$ and $s \geq 1$ the set $R_{\alpha,s}$ is quasi-convex in $\langle \alpha \rangle$.*

More problems on the same line are:

Problem 6.40 *Characterize for which $\alpha \in \mathbb{Q}/\mathbb{Z}$ and $s \geq 1$ the set $U_{\alpha,s}$ is quasi-convex in $\langle \alpha \rangle$.*

Problem 6.41 *Characterize for which $\alpha \in \mathbb{Q}/\mathbb{Z}$ and $s \geq 2$ the set $F_{\alpha,s}$ is quasi-convex in $\langle \alpha \rangle$.*

We also propose the following

Conjecture 6.42 *For every $k \neq 3$, $U_{1,k}$ is unconditionally quasi-convex in $\mathbb{Z}$.*

Recall that this is equivalent to say that $U_{1,k}$ is S-potentially quasi-convex or potentially Int-quasi-convex by Theorem 6.33. If Conjecture 6.42 is true, this would clearly generalize Example 6.35.

Chapter 7

Countably infinite quasi-convex sets

Here we compute the quasi-convex hull of some converging to 0 sequences in $\mathbb{T}$. More specifically, in Theorem 7.2 we establish that under mild conditions the range of a sequence of negative powers of 2 is quasi-convex in $\mathbb{T}$. It is convenient to fix the following notation before formulating the result.

Notation 7.1 *For a sequence* $\underline{a} = (a_n)_n$ *put*

$$K_{\underline{a}} := \{0\} \cup \{\pm 2^{-(a_n+1)} \mid n \in \mathbb{N}\} \subseteq \mathbb{T}.$$

Theorem 7.2 *Let* $\underline{a} = (a_n)_n$ *be a sequence of positive integers, and suppose that* $a_{n+1} - a_n > 1$ *for every* $n \in \mathbb{N}$. *Then* $K_{\underline{a}}$ *is quasi-convex in* $\mathbb{T}$.

Observe that in Theorem 7.2 we suppose $a_0 > 0$ (that is, $\frac{1}{2} \notin K_{\underline{a}}$). If we add the term $\frac{1}{2}$ to $K_{\underline{a}}$, then the quasi-convexity of $K_{\underline{a}}$ granted by Theorem 7.2 is lost in a substantial way. Indeed,

Theorem 7.3 *Let* $\underline{a} = (a_n)_n$ *be a sequence of integers such that* $a_0 = 0$ *and that* $a_{n+1} - a_n > 1$ *for every* $n \in \mathbb{N}$. *Then the quasi-convex hull* $Q_{\mathbb{T}}(K_{\underline{a}})$ *of* $K_{\underline{a}}$ *is given by:*

$$Q_{\mathbb{T}}(K_{\underline{a}}) = K_{\underline{a}} \cup (1/2 + K_{\underline{a}}).$$

From the subtle difference between Theorem 7.2 and Theorem 7.3, one realizes how delicate is the property of quasi-convexity. Along this chapter the reader may also observe how different is to work with quasi-convex sets — which requires hard calculations — from the relatively easy manipulations in convex sets.

Note that the lacunarity condition $a_{n+1} - a_n > 1$ for every $n \in \mathbb{N}$ cannot be omitted in Theorem 7.2. In fact, we have already observed in Example 5.24 that if $a_n = n$ for every $n \in \mathbb{N}$, then we obtain a sequence which behaves in the opposite way (see also the more general Proposition 5.26): $Q_{\mathbb{T}}(K_{\underline{a}}) = \mathbb{T}$ *whenever* $a_n := n$ *for every* $n \in \mathbb{N}$. In this example, every pair of members of $\underline{a}$ consists of adjacent integers (i.e., $a_{n+1} = a_n + 1$ for every n); what we get is a qc-dense set $K_{\underline{a}}$. We conjecture that if $\underline{a}$ contains even finitely many pairs of adjacent members, then $K_{\underline{a}}$ is not quasi-convex (see Conjecture 7.46).

As a matter of fact, the sets considered in Theorem 7.2 are in the 2-torsion part of $\mathbb{T}$. Thus it is not surprising that this result admits a counterpart in $\mathbb{Z}$ endowed with the 2-adic topology, namely:

Theorem 7.4 *Fix a sequence* $\underline{a} = (a_n)_n$ *of positive integers such that* $a_{n+1} - a_n > 1$ *for every* $n \in \mathbb{N}$. *Then* $\{0\} \cup \{\pm 2^{a_n - 1} \mid n \in \mathbb{N}\}$ *is quasi-convex in* $\mathbb{Z}$ *equipped with the* 2*-adic topology.*

We will deal with this in [29].
Finally, our motivation to study quasi-convex subsets was to close some open problems left in [23] (see also § 8). More details are given in § 7.5.

7.1 Representation via binary blocks

We introduce some additional notation and we fix the background in order to make the exposition of this chapter self-contained.

For every $x \in [0, 1)$ there exists a unique sequence $(b_i)_{i=1}^{\infty}$ such that $x = \sum_{i=1}^{\infty} \frac{b_i}{2^i} = \frac{b_1}{2} + \ldots + \frac{b_s}{2^s} + \ldots$, and $(b_i)_i$ verifies

- $b_i \in \{0, 1\}$ for every i;
- there exist infinitely many 0's in $(b_i)_i$.

Fact 7.5 *Let* $x \neq 0$ *be in* $\mathbb{T}$. *We take as binary coordinates of* x *the sequence* $(b_i)_i$ *corresponding to the unique representative* r *of* x *belonging to the fundamental domain* $[0, 1)$.

As a consequence, we have that every $0 \neq x \in \mathbb{T}$ is uniquely determined by intervals $[n_j, m_j]$ of non-negative integers $n_j < m_j$ such that $n_{j+1} > m_j$ for every j and:

- $b_i = 1$ for every $i \in (n_j, m_j]$, and
- $b_i = 0$ for every $i \in (m_j, n_{j+1}]$.

So one can describe any $x \in \mathbb{T} \setminus \{0\}$ in terms of "blocks": $x = x_1 + x_2 + \ldots + x_j + \ldots$, where each block x_j is of the form $\frac{1}{2^{n_j+1}} + \frac{1}{2^{n_j+2}} + \ldots + \frac{1}{2^{m_j}} = \frac{1}{2^{n_j}} - \frac{1}{2^{m_j}}$. We will say that a block $x = \frac{1}{2^{n+1}} + \ldots + \frac{1}{2^m}$ is:

- *degenerate* if $n+1=m$, i.e., $x=\frac{1}{2^{n+1}}$, otherwise we will call it *non-degenerate*;
- *initial* if $n=0$, i.e. $x=\frac{1}{2}+\ldots+\frac{1}{2^m}$.

For $b\in\{0,1\}$ let $b^*:=1-b$ (i.e., $b\mapsto b^*$ is the unique non-identical involution of $\{0,1\}$). Now we can extend this involution to an involution $x\mapsto x^*$ of $[0,1)$ putting $x^*:=(b_i^*)$, where (b_i) were the binary coordinates of $x\in[0,1)$. It induces a self-map of $\mathbb{T}$ that is precisely the involution $x\mapsto -x$, since $x+x^*=0$ in $\mathbb{T}$. This simple observation leads to the following:

Remark 7.6 Write $x=x_1+x_2+...+x_t+...$, and denote by g_i the "gap" between x_i and x_{i+1}, for every i. Then g_i^* are proper blocks, and $x^*=g_1^*+g_2^*+....$ is the decomposition of $x^*=-x$ in blocks.
In particular, every initial block is the opposite of a degenerate block: $\frac{1}{2}+\ldots+\frac{1}{2^m}=1-\frac{1}{2^m}\equiv_1-\frac{1}{2^m}$.

Observe that the number of blocks for an element x can be finite. Actually, this property characterizes the Prüfer group $\mathbb{Z}(2^\infty)$.

Example 7.7 *For any $x\in\mathbb{T}$, $x\in\mathbb{Z}(2^\infty)$ if and only if x is given by a finite number of binary coordinates, and this is equivalent to say that x is a finite sum of blocks.*

In our exposition the characters $\xi_k\in\mathbb{T}^\wedge$ — defined by $x\mapsto 2^k\cdot x$ for all $x\in\mathbb{T}$, for every non-negative integer k — will play a prominent role. In particular observe that ξ_0 is the identity character: $\xi_0(x)=x$ for every $x\in\mathbb{T}$. Since every integer $n\neq 0$ can be written uniquely as $m\cdot 2^{\nu_2(n)}$, where $\nu_2(n)$ is the biggest non-negative integer k such that $2^k|n$ and $m\geq 1$ is odd, the characters $m\xi_k$ describe all non-trivial characters of $\mathbb{T}$. Hence

$$\mathbb{T}^\wedge=\bigcup_{m\in 2\mathbb{Z}+1}\pm m\Lambda, \tag{7.1}$$

where $\Lambda:=\{2^k\mid k\in\mathbb{N}\}$.

7.1.1 Characters and blocks

Here we show some examples about how our convention about the block structure of the elements of $\mathbb{T}$ fits with our description of the characters. According to (7.1), we focus on the characters in Λ.

Example 7.8 *Let x be a degenerate block: $x=\frac{1}{2^{n+1}}$, for some $n\in\mathbb{N}$. Then, for $k\in\mathbb{N}$, $\xi_k(x)\in\mathbb{T}_+$ if and only if $k\neq n$.*
Indeed,

$$\xi_k(x)=\xi_k\left(\frac{1}{2^{n+1}}\right)=\frac{1}{2^{n+1-k}},$$

hence $\xi_k(x) \in \mathbb{T}_+ \iff n+1-k \neq 1 \iff k \neq n$.
In particular, from Remark 7.6 it follows that if x *is an initial block, namely* $x = \frac{1}{2} + \ldots + \frac{1}{2^m}$, *then* $\xi_k(x) \in \mathbb{T}_+$ *if and only if* $k \neq m-1$.

Let $x = \frac{b_1}{2} + \frac{b_2}{2^2} + \ldots + \frac{b_{k-1}}{2^{k-1}} + \frac{b_k}{2^k} + \frac{b_{k+1}}{2^{k+1}} + \frac{b_{k+2}}{2^{k+2}} + \ldots$. Then

$$\xi_k(x) = \underbrace{\frac{b_1}{2^{1-k}} + \frac{b_2}{2^{2-k}} + \ldots + \frac{b_{k-1}}{2^{k-1-k}} + \frac{b_k}{2^{k-k}}}_{\in \mathbb{Z}} + \frac{b_{k+1}}{2^{k+1-k}} + \frac{b_{k+2}}{2^{k+2-k}} + \ldots \equiv_1$$

$$\equiv_1 \frac{b_{k+1}}{2} + \frac{b_{k+2}}{2^2} + \ldots .$$

This trivial observation suggests the following notation: for $m \in \mathbb{N}$, put $t_m(x) := \sum_{n \geq m}^{\infty} \frac{b_n}{2^n}$. We say that $t_m(x)$ is the *m-tail* of x. Then

Remark 7.9 For every $m \geq 1$ and every $x \in \mathbb{T}$, $\xi_m(x) \equiv_1 \xi_m(t_{m+1}(x))$.

Corollary 7.10 *Let* x *be in* $\mathbb{Z}(2^\infty)$, *and write* $x = \ldots + \frac{1}{2^m}$, *for some* $m \geq 1$. *Then* $\xi_k(x) \equiv_1 0$ *for every* $k \geq m$.

Proof. Just note that $\xi_m(x) = \frac{1}{2^{m-m}} \equiv_1 0$ and that $t_k(x) = 0$ for every $k \geq m+1$. QED

Example 7.11 *Let* x *be a non-degenerate block:* $x = \frac{1}{2^{n+1}} + \ldots + \frac{1}{2^m}$, *for some* $0 < n < m-1$. *Then* $\xi_k(x) \in \mathbb{T}_+$ *if and only if* $k \neq n-1, m-1$. *Indeed, let us show the necessity. Since* x *is non-degenerate, i.e.* $n+2 \leq m$,

$$\xi_{n-1}(x) = \frac{1}{2^{n+1-(n-1)}} + \frac{1}{2^{n+2-(n-1)}} + \ldots + \frac{1}{2^{m-(n-1)}} = \frac{1}{2^2} + \frac{1}{2^3} + \varepsilon$$

with $\varepsilon \in [0, \frac{1}{8})$, *hence* $\xi_{n-1}(x) \notin \mathbb{T}_+$. *Moreover,* $\xi_{m-1}(x) = \xi_{m-1}(t_m(x)) = \frac{1}{2} \notin \mathbb{T}_+$. *Now we show the sufficiency. Fix* $k \neq n-1, m-1$ *and let us prove that* $\xi_k(x) \in \mathbb{T}_+$. *If* $k \geq m$ *then we are done by Corollary 7.10. If* $n \leq k \leq m-2$, *then* $\xi_k(x) = \xi_k(t_{k+1}(x)) = \frac{1}{2} + \frac{1}{2^2} + \varepsilon$ *with* $\varepsilon \in [0, \frac{1}{4})$. *If* $k \leq n-2$, *then* $n+1-k \geq 3$ *and* $\xi_k(x) = \frac{1}{2^i} + \frac{1}{2^{i+1}} + \ldots$ *with* $i \geq 3$, *i.e.,* $\xi_k(x) \in [\frac{1}{2^3} + \frac{1}{2^4}, \frac{1}{4})$.

7.2 Quasi-convex subsets of $\mathbb{T}$ generated by convergent sequences

In this section we give a proof of Theorem 7.2, namely we show that $K_{\underline{a}}$ is quasi-convex in $\mathbb{T}$ whenever $a_0 > 0$ and $a_{n+1} - a_n > 1$ for every $n \in \mathbb{N}$. We observe that a direct computation of $Q_{\mathbb{T}}(K_{\underline{a}})$ seems really involved. So, in order to give a neat and short proof of the theorem, we introduce some notation, preliminary results and even examples which will smooth out the way.

Fix a sequence $\underline{a} = (a_n)_n$ of non-negative integers. Recall from § 7.1 that $\Lambda = \{2^k \mid k \in \mathbb{N}\}$ is sufficient to describe the whole dual group $\mathbb{T}^\wedge$ (see (7.1)). In particular, we can describe $K_{\underline{a}}^{\triangleright}$ by means of $\Lambda \cap K_{\underline{a}}^{\triangleright}$. To this aim, let us consider the following notation.

Notation 7.12 *For every positive odd integer m, let J_m denote the set of all non-negative integers k such that $m\xi_k \in K_{\underline{a}}^{\triangleright}$:*

$$J_m := \{k \in \mathbb{N} \mid m\xi_k \in K_{\underline{a}}^{\triangleright}\}.$$

Although J_m clearly depends on $\underline{a}$, we prefer not to use heavy notation since no confusion will be possible.

Then $\Lambda \cap K_{\underline{a}}^{\triangleright} = \{2^k \mid k \in J_1\}$; more generally, $m\Lambda \cap K_{\underline{a}}^{\triangleright} = \{m2^k \mid k \in J_m\}$ for every odd m. This suggests that it is significant to describe the sets of the form J_m. Let us give here a characterization for $m = 1, 3$.

Claim 7.13 *The following properties hold:*

1. $J_1 = \{k \in \mathbb{N} \mid k \neq a_n, \ \forall\, n \in \mathbb{N}\} = \mathbb{N} \setminus \{a_n \mid n \in \mathbb{N}\}$;
2. $J_3 = \{k \in \mathbb{N} \mid k \neq a_n - i, \ i = 0, 2, \ \forall\, n \in \mathbb{N}\}$.

Proof. Let $x = 2^{-(a_n+1)}$ be in $K_{\underline{a}}$. Then, by Example 7.8, $\xi_k(x) \in \mathbb{T}_+$ if and only if $k \neq a_n$. This proves *1.*
For *2.* just observe that

$$3\xi_k(x) = \frac{3}{2^{a_n+1-k}} \in \mathbb{T}_+ \iff a_n + 1 - k \neq 1, 3 \iff k \neq \begin{cases} a_n \\ a_n - 2 \end{cases}$$

QED

Observe that if $m > 3$ then J_m can be empty, as the following example shows:

Example 7.14 *Consider the sequence $\underline{a}$ defined by $a_n := 2n$ for every $n \in \mathbb{N}$. Then $J_5 = \emptyset$.*

Proof. Note that:

$$5\xi_k(2^{-(2n+1)}) = \frac{5}{2^{2n+1-k}} \in \mathbb{T}_+ \iff 2n + 1 - k \neq \begin{cases} 1 \\ 3 \\ 4 \end{cases} \iff$$

$$\iff k \neq \begin{cases} 2n \\ 2n - 2 \\ 2n - 3 \end{cases}$$

Since $2^{-(2n+1)} \in K_{\underline{a}}$ for every n, in particular we get that $k \in J_5$ if and only if $k \neq 2n, 2n-3$ for every $n \in \mathbb{N}$, i.e., $J_5 = \emptyset$. QED

We denote by M the set of all (positive odd) integers m such that $J_m \neq \emptyset$.

Observe that Claim 7.13 implies that J_1 contains J_3 for every $\underline{a}$. The following more general property holds:

Lemma 7.15 *For every $m \in M$, $J_1 \supseteq J_m$.*

Proof. Fix $m \in M$, and fix $k \in \mathbb{N}$. According to Claim 7.13, we need to show that if $m\xi_k(x) \in \mathbb{T}_+$ for every $x = 2^{-(a_n+1)} \in K_{\underline{a}}$, then $k \neq a_n$ for every $n \in \mathbb{N}$. Indeed, we have that

$$m\xi_k(x) \in \mathbb{T}_+ \implies \frac{m}{2^{a_n+1-k}} \neq \frac{m}{2} \iff a_n + 1 - k \neq 1 \iff k \neq a_n.$$

QED

Now that we have managed to describe $K_{\underline{a}}^{\triangleright}$ by means of the J_m's, we want to push it further to get a factorization of $Q_{\mathbb{T}}(K_{\underline{a}})$ as an intersection of some "simpler" quasi-convex sets. For every $m \in M$, put

$$Q_m := \bigcap_{k \in J_m} (m\xi_k)^{-1}(\mathbb{T}_+) \subseteq \mathbb{T}.$$

It is clear that every Q_m is quasi-convex (by definition) and that

$$Q_{\mathbb{T}}(K_{\underline{a}}) = \bigcap_{m \in M} Q_m. \tag{7.2}$$

We anticipate here that the proof of Theorem 7.2 consists in proving the equality $Q_{\mathbb{T}}(K_{\underline{a}}) = Q_1 \cap Q_3$. Obviously, this simplifies a lot the calculation of $Q_{\mathbb{T}}(K_{\underline{a}})$. For this reason, it will be crucial to characterize those elements of $\mathbb{T} \setminus \{0\}$ that are in Q_1. We dedicate the entire § 7.2.1 to this.

7.2.1 The characterization of Q_1

Throughout this section, $\underline{a} = (a_n)_n$ is an increasing sequence of non-negative integers. Observe that in some cases we will suppose a stronger hypothesis on $\underline{a}$, namely the lacunarity condition $a_{n+1} - a_n > 1$ for every $n \in \mathbb{N}$.

We give some examples of elements of Q_1.

Example 7.16 *The following properties are equivalent:*

(1) $\frac{1}{2} \in Q_1$;

(2) $\frac{1}{2} \in K_{\underline{a}}$;

(3) $a_0 = 0$.

Indeed, (2) $\iff$ (3) *follows from the definition of* $K_{\underline{a}}$*. The implication* (2) $\implies$ (1) *is obvious since* $K_{\underline{a}} \subseteq Q_1$*. Now suppose that* (1) *holds and let us show* (3)*. Since* $\xi_0(\frac{1}{2}) = \frac{1}{2} \notin \mathbb{T}_+$*, we have that* $0 \notin J_1$*, i.e.,* $0 \in \{a_n \mid n \in \mathbb{N}\}$*. Clearly, this is equivalent to say that* $a_0 = 0$ *and we are done.*

Example 7.17 *Let* x *be a degenerate block:* $x = \frac{1}{2^{h+1}}$*. Then* $x \in Q_1$ *if and only if* $h = a_m$*, for some* $m \in \mathbb{N}$*.*
Indeed, by the definition of Q_1*,* $x \in Q_1 \iff \xi_k(x) \in \mathbb{T}_+$ *for every* $k \in J_1$*, hence, by Example 7.8,* $x \in Q_1 \iff h \neq k$ *for every* $k \in J_1$*, i.e.,* $h \in \{a_n \mid n \in \mathbb{N}\}$ *according to Claim 7.13.*
Since Q_1 *is symmetric, by Remark 7.6 we also have that if* x *is an initial block, namely* $x = \frac{1}{2} + \ldots + \frac{1}{2^m}$*, then* $x \in Q_1$ *if and only if* $m = a_h + 1$ *for some* $h \in \mathbb{N}$*.*

For every $m \in \mathbb{N}$, we will say that $d_m := \frac{1}{2^{a_m+1}}$ is a *degenerate admissible block* and that

$$i_m := \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{a_m}} + \frac{1}{2^{a_m+1}} \equiv_1 -d_m$$

is an *initial admissible block.* Therefore, $K_{\underline{a}} = \{d_m \mid m \in \mathbb{N}\} \cup \{i_m \mid m \in \mathbb{N}\} \cup \{0\}$.
Observe that if $a_0 = 0$, then $d_0 = i_0 = \frac{1}{2}$ and this is the only block that is simultaneously degenerate admissible and initial admissible.

Example 7.18 *Let* x *be a non-degenerate block:* $x = \frac{1}{2^{n+1}} + \ldots + \frac{1}{2^m}$*, for some* $0 < n < m-1$*. Then* $x \in Q_1$ *if and only if* $n = a_g + 1$ *and* $m = a_\ell + 1$ *for some* $g, \ell \in \mathbb{N}$*.*
Indeed, $x \in Q_1 \iff \xi_k(x) \in Q_1$ *for every* $k \in J_1$*, and this is equivalent to* $n-1, m-1 \in \{a_n \mid n \in \mathbb{N}\}$ *by Example 7.11, i.e., there exist* $g, \ell \in \mathbb{N}$ *such that* $n = a_g + 1$ *and* $m = a_\ell + 1$*.*

For every $n < m \in \mathbb{N}$, put

$$c_{n,m} := \frac{1}{2^{a_n+2}} + \frac{1}{2^{a_n+3}} + \ldots + \frac{1}{2^{a_m}} + \frac{1}{2^{a_m+1}} = d_n - d_m.$$

We will call $c_{n,m}$ an *admissible block.*

Remark 7.19 Observe that an admissible block $c_{n,m}$ is degenerate if and only if $a_m = a_{n+1} = a_n + 1$, so $m = n + 1$ and $c_{n,m} = d_m \in K_{\underline{a}}$. In particular, if $a_{n+1} - a_n > 1$ for every $n \in \mathbb{N}$ then every admissible block is non-degenerate.

In the next result we characterize those $x \in Q_1$ that consist precisely of one block. This is a consequence of Example 7.17 and Example 7.18.

Corollary 7.20 *Suppose that* $x = \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \ldots + \frac{1}{2^m}$. *Then* $x \in Q_1$ *if and only if one of these three possibilities holds:*

- x *is degenerate admissible;*
- x *is initial admissible;*
- x *is non-degenerate admissible.*

Proof. $\Rightarrow$: Suppose that $x \in Q_1$. If x is degenerate, i.e. $n+1 = m$, then x is degenerate admissible by Example 7.17. So suppose that x is non-degenerate, i.e., $n+1 < m$. If $n = 0$, then x is an initial admissible block by Example 7.17, otherwise x is a non-degenerate admissible block by Example 7.18.
$\Leftarrow$: This is Example 7.17 and Example 7.18, respectively. QED

Now we deal with the case when $x \in Q_1$ consists of more than one block. We first introduce the following claims that describe general properties of Q_1.

Claim 1. Suppose that $x \in Q_1$. Let h be any positive integer such that $b_h = 1$ and $b_{h+1} = 0$. Then $h = a_\ell + 1$ for some $\ell \in \mathbb{N}$. In particular, if $x = x_1 + \ldots \in Q_1$ and $x_1 = \frac{1}{2}$, then $a_0 = 0$.
Indeed, if $h \neq a_\ell + 1$ for all ℓ, then $h - 1 \in J_1$ by Claim 7.13. Now,

$$\xi_{h-1}(x) \equiv_1 \xi_{h-1}(t_h(x)) = \xi_{h-1}\left(\frac{1}{2^h} + 0 + \frac{b_{h+2}}{2^{h+2}} + \ldots\right) \equiv_1 \frac{1}{2} + \varepsilon,$$

with $\varepsilon \in [0, \frac{1}{4})$. Therefore $\xi_{h-1}(x) \notin \mathbb{T}_+$, but this contradicts the fact that $x \in Q_1$. Hence, h is of the desired form.
Note that if $x_1 = \frac{1}{2}$ (i.e., $h = 1$), then $1 = a_\ell + 1$ for some $\ell \in \mathbb{N}$ implies $a_\ell = 0$, and this is possible only if $\ell = 0$.

Roughly speaking, Claim 1 states that the last summand of every block of $x \in Q_1$ is of the form $\frac{1}{2^{a_\ell+1}}$. On the other hand, Claim 2 describes the first summand of every non-degenerate block of $x \in Q_1$.

Claim 2. Suppose that $x \in Q_1$. Let $h \geq 2$ be such that $b_{h-1} = 0, b_h = 1$ and $b_{h+1} = 1$. Then $h = a_g + 2$, for some $g \in \mathbb{N}$.
Indeed, suppose that $h \neq a_g + 2$ for every $g \in \mathbb{N}$. Then $h - 2 \in J_1$, and

$$\xi_{h-2}(x) \equiv_1 \xi_{h-2}(t_{h-1}(x)) = \xi_{h-2}\left(0 + \frac{1}{2^h} + \frac{1}{2^{h+1}} + \frac{b_{h+2}}{2^{h+2}} + \ldots\right) \equiv_1$$

$$\equiv_1 \frac{1}{2^2} + \frac{1}{2^3} + \varepsilon$$

with $\varepsilon \in [0, \frac{1}{8})$. Therefore $\xi_{h-2} \notin \mathbb{T}_+$ and this contradicts our assumption $x \in Q_1$.

For the third claim we impose on $\underline{a}$ the assumption $a_{n+1} - a_n > 1$ for every $n \in \mathbb{N}$. It states that if $x = x_1 + \ldots + x_j + x_{j+1} + \ldots \in Q_1$ is such that x_j is degenerate, for some $j \geq 1$, then x_j is admissible and $x_{j+i} = 0$ for every $i \geq 1$ (in particular, $x \in \mathbb{Z}(2^\infty)$). In other words, every degenerate block $x \in Q_1$ is admissible and terminal.

Claim 3. Suppose that $a_{n+1} - a_n > 1$ for every $n \in \mathbb{N}$. Let $x = x_1 + \ldots \in Q_1$ be such that x contains a degenerate block x_j, for some $j \geq 1$. Then x_j is admissible and $x_{j+i} = 0$ for every $i \geq 1$.
Indeed, write $x_j = \frac{1}{2^{h+1}}$ for some $h \in \mathbb{N}$. Then $h = a_\ell + 1$ for some $\ell \in \mathbb{N}$ by Claim 1. Now, write $t_{a_\ell}(x) = 0 + \frac{1}{2^{a_\ell+1}} + 0 + \frac{b_{a_\ell+2}}{2^{a_\ell+2}} + \ldots$. Then:

$$\xi_{a_\ell-1}(x) = \xi_{a_\ell-1}(t_{a_\ell}(x)) \equiv_1 \frac{1}{2^2} + \frac{b_{h+2}}{2^4} + \frac{b_{h+3}}{2^5} + \ldots = \frac{1}{4} + \varepsilon,$$

with $\varepsilon \in [0, \frac{1}{8})$. Observe that $a_\ell - 1 \in J_1$ by Claim 7.13, hence $x \in Q_1$ if and only if $\varepsilon = 0$, i.e., $x_{j+i} = 0$ for every $i \geq 1$.

In the next lemma we give necessary condition on the first block of every $x \in Q_1$.

Lemma 7.21 *Let $x = x_1 + x_2 + \ldots$. If $x \in Q_1$, then x_1 can have one of the following three forms:*

- *x_1 is an initial admissible block;*
- *x_1 is a non-degenerate admissible block;*
- *x_1 is a degenerate admissible block.*

Proof. Suppose that $x \in Q_1$, and write $x_1 = \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \ldots + \frac{1}{2^m}$. By Claim 1, there exists $\ell \in \mathbb{N}$ such that $m = a_\ell + 1$. If $n = 0$, then x_1 is an initial admissible block. So assume $n > 0$. If $n+1 = m$ then x_1 is degenerate admissible. Otherwise, we can apply Claim 2 to get $n = a_g + 1$ for some $g \in \mathbb{N}$ such that $g < \ell$, therefore $x_1 = c_{g,\ell}$ is a non-degenerate admissible block. QED

Corollary 7.22 *Assume that $a_{n+1} - a_n > 1$ for every $n \in \mathbb{N}$, and let $x = x_1 + x_2 + \ldots$. If $x \in Q_1$, then x_1 is either initial admissible or non-degenerate admissible.*

Proof. We only have to note that x_1 cannot be degenerate admissible. Indeed, in this case Claim 3 applies and, in particular, $x_2 = 0$ — a contradiction. QED

The counterpart of Corollary 7.22 for "internal" blocks is the following.

Lemma 7.23 *Assume that $a_{n+1} - a_n > 1$ for every $n \in \mathbb{N}$, and let $x = x_1 + x_2 + \ldots$ be in Q_1. Consider a block x_j with $j \geq 2$ such that $x_{j+1} \neq 0$. Then x_j is a non-degenerate admissible block.*

Proof. Write $x_j = \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \ldots + \frac{1}{2^m}$. By Claim 1, there exists $\ell \in \mathbb{N}$ such that $m = a_\ell + 1$. Now observe that x is non-degenerate, i.e., $n+1 < m$; indeed, otherwise Claim 3 would imply $x_{j+1} = 0$, a contradiction. So we can apply Claim 2 to get $n + 1 = a_g + 2$ for some $g \in \mathbb{N}$, with $g < \ell$ since $n + 1 < m$. In other words, $x = c_{n,m}$ is a non-degenerate admissible block. QED

If $x \in Q_1$ is a finite sum of blocks, then the last block admits two forms, as the following lemma states.

Lemma 7.24 *Let $t > 1$, and let $x = x_1 + \ldots + x_t$ be in Q_1. Then x_t is either degenerate admissible or non-degenerate admissible.*

Proof. Write $x_t = \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \ldots + \frac{1}{2^m}$. According to Claim 1, $m = a_\ell + 1$ for some $\ell \in \mathbb{N}$. Now, suppose that x_t is degenerate, i.e., $n + 1 = m$. Then x_t is degenerate admissible. If x_t is non-degenerate (i.e., $n + 1 < m$), then there exists $g \in \mathbb{N}$ such that $n + 1 = a_g + 2$ by Claim 2. Observe that $g < \ell$ because $n + 1 < m$, hence $x = c_{n,m}$ is a non-degenerate admissible block. QED

Now we give a complete characterization of those $x \in Q_1$ that are a sum of infinitely many blocks.

Proposition 7.25 (Infinite sum of blocks) *Assume that $a_{n+1} - a_n > 1$ for every $n \in \mathbb{N}$, and let $x = x_1 + x_2 + \ldots$. Then $x \in Q_1$ if and only if:*

- *x_1 is either an initial admissible block or a non-degenerate admissible block;*
- *x_j is a non-degenerate admissible block, for every $j \geq 2$.*

Proof. The necessity follows from Corollary 7.22 and Lemma 7.23. So let us verify the sufficiency.
Suppose that x_1 is an initial admissible block, and write $x_1 = i_\ell$ for some $\ell \in \mathbb{N}$. Moreover, write $x_j = c_{e_j,f_j}$ for every $j \geq 2$. In order to show that $x \in Q_1$, fix $k \in J_1$. We have to show that $\xi_k(x) \in \mathbb{T}_+$. According to Claim 7.13, $k \neq \{a_\ell, a_{e_1}, a_{f_1}, a_{e_2}, a_{f_2}, \ldots\}$. We distinguish two cases:

(-) k "hits" a block, i.e., $k < a_\ell$ or $a_{e_j} < k < a_{f_j}$ for some $j \geq 1$. In this case $\xi_k(x)$ contains the term $\frac{1}{2} + \frac{1}{4}$ and eventually a sum ε of a geometric progression whose greatest term is $\frac{1}{8}$, so $\varepsilon \in [0, \frac{1}{4})$;

(-) k falls in a "gap", i.e., $a_\ell < k < a_{e_1}$ or $a_{f_j} < k < a_{e_{j+1}}$ for some $j \geq 1$, then $\xi_k(x)$ is a sum of a geometric progression whose greatest term is $\frac{1}{8}$, therefore $\xi_k(x) \in [0, \frac{1}{4})$.

The situation is resumed as follows:

$$\xi_k(x) \equiv_1 \begin{cases} \frac{1}{2} + \frac{1}{4} + \varepsilon & \text{if } k < a_\ell \text{ or } a_{e_j} < k < a_{f_j} \text{ for some } j \geq 1; \\ \delta & \text{if } a_\ell < k < a_{e_1} \text{ or } a_{f_j} < k < a_{e_{j+1}} \text{ for some } j \geq 1; \end{cases}$$

with $\delta, \varepsilon \in [0, \frac{1}{4})$. Therefore, $\xi_k(x) \in \mathbb{T}_+$.
Now suppose that x_1 is a non-degenerate admissible block: $x_1 = c_{g,\ell}$. Fix $k \in J_1$. Then again we distinguish whether k hits a block or k falls in a gap, hence $\xi_k(x)$ admits two possibilities:

$$\xi_k(x) \equiv_1 \begin{cases} \frac{1}{2} + \frac{1}{4} + \varepsilon & \text{if } a_g < k < a_\ell \text{ or } a_{e_j} < k < a_{f_j},\ j \geq 1; \\ \delta & \text{if } k < a_g,\ a_\ell < k < a_{e_1} \text{ or } a_{f_j} < k < a_{e_{j+1}},\ j \geq 1; \end{cases}$$

with $\delta, \varepsilon \in [0, \frac{1}{4})$. Therefore, $\xi_k(x) \in \mathbb{T}_+$. QED

If $x \in Q_1$ is formed by a finite number of blocks (i.e., $x \in \mathbb{Z}(2^\infty)$, see Example 7.7), then x can be characterized as follows.

Proposition 7.26 (Finite sum of blocks) *Assume that $a_{n+1} - a_n > 1$ for every $n \in \mathbb{N}$, and let $x = x_1 + \ldots + x_t$. Then $x \in Q_1$ if and only if*

- *x_1 is either an initial admissible block or a non-degenerate admissible block;*
- *x_j is a non-degenerate admissible block, for every $j \neq 1, t$;*
- *x_t is either a degenerate admissible block or a non-degenerate admissible block.*

Proof. The necessity follows from Corollary 7.22, Lemma 7.23 and Lemma 7.24. So let us show the sufficiency.
For every $1 < j < t$, write $x_j = c_{e_j,f_j}$.
[case 1] Suppose that x_1 is an initial admissible block and that x_t is a degenerate admissible block. Write $x_1 = i_\ell$ and $x_t = d_h$. Fix $k \in J_1$. If $k \geq a_h+1$ then $\xi_k(x_t) = 0$ by Corollary 7.10. So suppose that $k \leq a_h-1$. We distinguish whether k hits a block, (i.e., $k < a_\ell$, $a_{e_1} < k < a_{f_1}, \ldots, a_{e_{t-1}} < k < a_{f_{t-1}}$) or k falls in a gap (i.e., $a_\ell < k < a_{e_1}, a_{f_1} < k < a_{e_2}, \ldots, a_{f_{t-1}} < k < a_h$). Then:

$$\xi_k(x) \equiv_1 \begin{cases} \frac{1}{2} + \frac{1}{4} + \varepsilon & \text{if } k < a_\ell,\ a_{e_1} < k < a_{f_1}, \ldots, a_{e_{t-1}} < k < a_{f_{t-1}}; \\ \delta & \text{if } a_\ell < k < a_{e_1}, a_{f_1} < k < a_{e_2}, \ldots, a_{f_{t-1}} < k < a_h; \end{cases}$$

with $\delta, \varepsilon \in [0, \frac{1}{4})$, and $\xi_k(x) \in \mathbb{T}_+$.
[case 2] Suppose that x_1 is an initial admissible block and that x_t is a non-degenerate block. Write $x_1 = i_\ell$ and $x_t = c_{h,d}$. Fix $k \in J_1$. If $k \geq a_d + 1$ then $\xi_k(x_t) = 0$ by Corollary 7.10. So suppose that $k \leq a_d - 1$. Then:

$$\xi_k(x) \equiv_1 \begin{cases} \frac{1}{2} + \frac{1}{4} + \varepsilon & \text{if } k < a_\ell,\ a_{e_1} < k < a_{f_1}, \ldots, a_h < k < a_d; \\ \delta & \text{if } a_\ell < k < a_{e_1}, a_{f_1} < k < a_{e_2}, \ldots, a_{f_{t-1}} < k < a_h; \end{cases}$$

with $\delta, \varepsilon \in [0, \frac{1}{4})$, and $\xi_k(x) \in \mathbb{T}_+$.
[case 3] Suppose that x_1 is a non-degenerate admissible block and that x_t is a degenerate admissible block. Write $x_1 = c_{g,\ell}$ and $x_t = d_h$. Fix $k \in J_1$. If $k \geq a_h + 1$ then $\xi_k(x_t) = 0$ by Corollary 7.10. So suppose that $k \leq a_h - 1$. Then:

$$\xi_k(x) \equiv_1 \begin{cases} \frac{1}{2} + \frac{1}{4} + \varepsilon; \\ \delta; \end{cases}$$

if, respectively, $a_g < k < a_\ell$, $a_{e_1} < k < a_{f_1}, \ldots, a_{e_{t-1}} < k < a_{f_{t-1}}$ or $k < a_g$, $a_\ell < k < a_{e_1}, \ldots, a_{e_{t-1}} < k < a_h$, with $\delta, \varepsilon \in [0, \frac{1}{4})$. Hence, $\xi_k(x) \in \mathbb{T}_+$.
[case 4] Suppose that x_1 and x_t are non-degenerate admissible blocks. Write $x_1 = c_{g,\ell}$ and $x_t = c_{h,d}$. Fix $k \in J_1$. If $k \geq a_d + 1$ then $\xi_k(x_t) = 0$ by Corollary 7.10. So suppose that $k \leq a_d - 1$. Then:

$$\xi_k(x) \equiv_1 \begin{cases} \frac{1}{2} + \frac{1}{4} + \varepsilon & \text{if } a_g < k < a_\ell,\ a_{e_1} < k < a_{f_1}, \ldots, a_h < k < a_d; \\ \delta & \text{if } k < a_g,\ a_\ell < k < a_{e_1},\ \ldots,\ a_{e_{t-1}} < k < a_h; \end{cases}$$

with $\delta, \varepsilon \in [0, \frac{1}{4})$, and $\xi_k(x) \in \mathbb{T}_+$. QED

7.2.2 Proofs of Theorem 7.2 and Theorem 7.3

Claim 7.27 *Let x be a non-degenerate admissible block. Then $x \notin Q_3$.*

Proof. Write $x = c_{g,\ell}$, for some $g < \ell \in \mathbb{N}$. Then

$$3\xi_{a_g - 1}(x) = 3\left(\frac{1}{2^3} + \frac{1}{2^4} + \ldots\right) = \frac{9}{16} + 3\varepsilon$$

with $\varepsilon \in [0, \frac{1}{2^4})$. This implies that $3\xi_k(x) \notin \mathbb{T}_+$. Since $a_g - 1 \in J_3$, we conclude $x \notin Q_3$ by Claim 7.13. QED

Lemma 7.28 *Let $x = x_1 + \ldots$ be in $Q_1 \cap Q_3$ such that x_1 is not a degenerate block. Then $x \in K_{\underline{a}}$.*

Proof. According to Proposition 7.25 and Proposition 7.26, x_1 is either initial admissible (non-degenerate by our hypothesis) or non-degenerate admissible. Hence, one has either $x_1 = i_\ell$ or $x_1 = c_{g,\ell}$. In both cases,

$$3\xi_{a_\ell-1}(x) = 3\xi_{a_\ell-1}(t_{a_\ell}(x)) \equiv_1 3\left(\frac{1}{2}+\frac{1}{4}+\varepsilon\right) \equiv_1 \frac{1}{4}+3\varepsilon,$$

with $\varepsilon \in [0, \frac{1}{8})$. Since $a_\ell - 1 \in J_3$ (and $x \in Q_3$), $\varepsilon = 0$, i.e., $x = x_1$. Then, according to Claim 7.27, x is necessarily an initial admissible block. In particular, $x \in K_{\underline{a}}$. QED

Now we are in position to prove Theorem 7.2.

Proof of Theorem 7.2. It is clear from (7.2) that

$$K_{\underline{a}} \subseteq Q_{\mathbb{T}}(K_{\underline{a}}) \subseteq Q_1 \cap Q_3.$$

To prove that $K_{\underline{a}} = Q_{\mathbb{T}}(K_{\underline{a}})$ we need to verify that

$$K_{\underline{a}} \supseteq Q_1 \cap Q_3.$$

Fix $x = x_1 + \ldots \in Q_1 \cap Q_3$. According to Proposition 7.25 and Proposition 7.26, x_1 is either initial admissible (necessarily non-degenerate, by Claim 1) or non-degenerate admissible. Hence we apply Lemma 7.28 to deduce that $x \in K_{\underline{a}}$ and we are done.

Now we are going to prove Theorem 7.3. First consider the following example.

Example 7.29 *If $a_0 = 0$, then $\frac{1}{2} + \frac{1}{2^{a_n+1}} \in Q_1$ for every $n \in \mathbb{N}$.*

Proof. Fix $k \in J_1$. Then $\frac{1}{2^{1-k}} \in \mathbb{Z}$ since $k \neq 0$, and, hence, $\xi_k(x) = \frac{1}{2^{1-k}} + \frac{1}{2^{a_n+1-k}} \equiv_1 \frac{1}{2^{a_n+1-k}} \in \mathbb{T}_+$ since $k \neq a_n$. QED

We show now that if $a_0 = 0$, then $\frac{1}{2} + \frac{1}{2^{a_n+1}} \in Q_m$ for every $n \in \mathbb{N}$ and every $m \in M$, i.e., $\frac{1}{2} + K_{\underline{a}} \subseteq Q_{\mathbb{T}}(K_{\underline{a}})$. This will prove Theorem 7.3 stating that $Q_{\mathbb{T}}(K_{\underline{a}})$ is precisely the union of $K_{\underline{a}}$ and $\frac{1}{2} + K_{\underline{a}}$.

Proof of Theorem 7.3. We have to prove that $Q_{\mathbb{T}}(K_{\underline{a}}) = K_{\underline{a}} \cup (1/2 + K_{\underline{a}})$.

$\supseteq$: Clearly, $K_{\underline{a}} \subseteq Q_{\mathbb{T}}(K_{\underline{a}})$, so we only need to show that $1/2 + K_{\underline{a}} \subseteq Q_{\mathbb{T}}(K_{\underline{a}})$. To this aim, fix $x \in 1/2 + K_{\underline{a}}$. Since $Q_{\mathbb{T}}(K_{\underline{a}})$ is symmetric, we can assume wlog that $x = \frac{1}{2} + \frac{1}{2^{a_h+1}}$ for some $h \in \mathbb{N}$ (because $\frac{1}{2} - \frac{1}{2^{a_h+1}} = -(\frac{1}{2} + \frac{1}{2^{a_h+1}})$). According to (7.2), it suffices to show that $m\xi_k(x) \subseteq \mathbb{T}_+$ for every $m\xi_k \in K_{\underline{a}}^{\triangleright}$. So, fix $m\xi_k \in K_{\underline{a}}^{\triangleright}$, i.e., $m \in J_m$. Then $k \in J_m \subseteq J_1 = \mathbb{N} \setminus \{a_n \mid n \in \mathbb{N}\}$ (see Lemma 7.15 and Claim 7.13), in particular $k > 0$, so $\frac{m}{2^{1-k}} \in \mathbb{Z}$ and

$$m\xi_k(x) = \frac{m}{2^{1-k}} + \frac{m}{2^{a_h+1-k}}.$$

It remains to observe that $\frac{m}{2^{a_h+1-k}} \in m\xi_k(K_{\underline{a}}) \subseteq \mathbb{T}_+$ by the choice of $m\xi_k$.

$\supseteq$: By (7.2), it suffices to prove that $Q_1 \cap Q_3 \subseteq K_{\underline{a}} \cup (1/2 + K_{\underline{a}})$. So, fix $0 \neq x \in Q_1 \cap Q_3$.
If x_1 is a non-degenerate block, then $x \in K_{\underline{a}}$ by Lemma 7.28. So, suppose that x_1 is degenerate. Then, according to Proposition 7.25 and Proposition 7.26, x_1 is initial admissible and degenerate, i.e. $x_1 = \frac{1}{2}$. If $x = x_1$ then we are clearly done. So we assume $x = x_1 + x_2 + \ldots$ with $x_2 \neq 0$. We want to show that x_2 is a degenerate block. Indeed, observe that $x_1 = \frac{1}{2^{a_0+1}}$ by Claim 1. Now, suppose that x_2 is non-degenerate and write $x_2 = c_{n,m} = \frac{1}{2^{a_n+2}} + \ldots + \frac{1}{2^{a_m+1}}$. Then $a_n + 2 \geq a_1 + 2 \geq a_0 + 3$ by the lacunarity assumption on $\underline{a}$. This implies that $b_{a_n} = b_{a_n+1} = 0$. Now,

$$3\xi_{a_n-1}(x) = 3\xi_{a_n-1}(t_{a_n}(x)) =$$

$$= 3\xi_k\left(\ldots + 0 + 0 + \frac{1}{2^{a_n+2}} + \ldots + \frac{1}{2^{a_m+1}} + \ldots\right) \equiv_1 \frac{3}{8} + 3\varepsilon,$$

with $\varepsilon \in [0, \frac{1}{8})$, therefore $3\xi_k(x) \notin \mathbb{T}_+$. Since $a_n - 1 \in J_3$ by Claim 7.13, this contradicts the fact that $x \notin Q_3$.
So we have shown that $x = \frac{1}{2} + x_2 + \ldots$ with x_2 degenerate. Now, from Proposition 7.25 and Proposition 7.26 it is clear that x cannot be the sum of more than two (necessarily degenerate) blocks, i.e. $x = \frac{1}{2} + x_2$ (with x_2 degenerate). To conclude, just observe that x_2 is degenerate admissible by Lemma 7.24.

7.3 Possible generalizations

We give some tips to indicate how one can extend our new technique of factorization of the quasi-convex hull to more general contexts.

7.3.1 The general setting of factorization

Here is briefly the idea of *factorization* of the quasi-convex hull in the case of a (necessarily torsion) locally finite precompact abelian group (G, τ) such that the dual is (algebraically) $\mathbb{Z}$ (recall that, as far as the computation of the quasi-convex hull is concerned, the assumption that the topology is precompact is irrelevant; see Remark 5.4).

As every locally cyclic group, G is countable and has a sequence (g_n) of generators such that $\langle g_n \rangle \leq \langle g_n + 1 \rangle$ for all $n \in \mathbb{N}$. For brevity, call G_n the finite cyclic group $\langle g_n \rangle$, so that

$$G_0 \leq G_1 \leq \ldots \leq G_n \leq \ldots \tag{7.3}$$

and this chain of subgroups of G has as union G. For $n \in \mathbb{N}$:

- let ζ_n be the positive generator of $G_n^\perp \leq \mathbb{Z} = G^\wedge$, and
- put $q_n := |G_{n+1}|/|G_n|$.

Then $\langle \zeta_n \rangle = G_n^\perp \leq \mathbb{Z}$ and $\bigcap_n G_n^\perp = 0$. Therefore, for every non-zero $n \in \mathbb{Z}$ there exists a uniquely determined k such that $n \in \langle \zeta_k \rangle \setminus \langle \zeta_{k+1} \rangle$. Therefore, $n = m\zeta_k$ and $q_k \not| m$, otherwise $n \in \langle \zeta_{k+1} \rangle$, a contradiction. Now let $\Lambda = \{\zeta_k \mid k \in \mathbb{N}\}$. This gives a partition

$$\Lambda = \bigcup_{k=0}^{\infty} \{m\zeta_k \mid m \not| q_k\}. \tag{7.4}$$

In case all q_k coincide with some q (i.e., ζ_n is a geometric progression with ratio ζ_0), then (7.4) simplifies to

$$\Lambda = \bigcup_{k=0}^{\infty} \{m\zeta_k \mid m \not| q\} = \bigcup_{m \not| q} m\Lambda. \tag{7.5}$$

In case $q = p$ is prime, (7.5) can be written also as

$$\Lambda = \bigcup_{(m,p)=1} m\Lambda. \tag{7.6}$$

The partition (7.6) (or, more generally, (7.3)) allows for a partition of any polar set $K^\triangleright$, for a subset K of G, as $K^\triangleright = \bigcup_{(m,p)=1} K^\triangleright \cap m\Lambda$. In order to ease notation, we introduce the set $J_m := \{k \in \mathbb{N} \mid m\zeta_k \in K^\triangleright\}$. Clearly, this is the inverse image of the part $K^\triangleright \cap m\Lambda$ of the polar $K^\triangleright$ under the bijection $\mathbb{N} \to m\Lambda$ given by $k \mapsto m\zeta_k$. At this point we define

$$Q_m = (K^\triangleright \cap m\Lambda)^\triangleleft = \{m\zeta_k \mid m\zeta_k \in K^\triangleright\}^\triangleleft.$$

It turns out that for certain m one has $J_m = \mathbb{N}$ (this occurs precisely when $K^\triangleright \subseteq m\Lambda$). Then $Q_m = (m\Lambda)^\triangleleft$. Note that this set *does not depend on* K at all. The sets of the form $(m\Lambda)^\triangleleft$ are certainly quasi-convex and $\bigcap_m \{(m\Lambda)^\triangleleft \mid K^\triangleright \subseteq m\Lambda\}$ always contains the quasi-convex hull $Q_G(K)$ of K.

7.3.2 The triadic case

We give without proof the following result, which is clearly the counterpart of Theorem 7.2 in the triadic case.

Theorem 7.30 *Let* $\underline{a} = (a_n)_n$ *be a sequence of positive integers, and suppose that* $a_{n+1} - a_n > 2$ *for every* $n \in \mathbb{N}$. *Put*

$$K_{\underline{a},3} := \{0\} \cup \{\pm 3^{-(a_n+1)} \mid n \in \mathbb{N}\} \subseteq \mathbb{T}.$$

Then $K_{\underline{a},3}$ *is quasi-convex in* $\mathbb{T}$.

Note that the lacunarity condition $a_{n+1} - a_n > 2$ for every $n \in \mathbb{N}$ cannot be omitted neither in Theorem 7.30. In fact, if $a_n = n$ for every $n \in \mathbb{N}$, then we obtain a sequence which behaves in the opposite way: see Example 5.27.

Consider the characters $\eta_k \in \mathbb{T}^\wedge$ defined by $x \mapsto 3^k \cdot x$ for all $x \in \mathbb{T}$, for every nonnegative integer k. Since every integer $n \neq 0$ can be written uniquely as $m \cdot 3^{\nu_3(n)}$, where $\nu_3(n)$ is the biggest nonnegative integer k such that $3^k | n$ and $m \geq 1$ has the form $m = 3k \pm 1$, the characters $m\xi_k$ describe all non-trivial characters of $\mathbb{T}$. Hence

$$\mathbb{T}^\wedge = \bigcup_{m \in 3\mathbb{Z} \pm 1} \pm m\Gamma, \tag{7.7}$$

where $\Gamma := \{3^k \mid k \in \mathbb{N}\}$ (compare the previous formula with (7.1)).

It is clear that we can consider the counterpart of the sets J_i's and Q_i's; use the notation $\mathcal{J}_i$ (for $i \in 3\mathbb{Z} \pm 1$) and $\mathcal{Q}_i$.
In the triadic case it is impossible to obtain the quasi-convex hull of $K_{\underline{a},3}$ as a finite intersection of sets of the form $\mathcal{Q}_i$ (like in the proof of Theorem 7.2). In any case, it is possible to have a "blockwise" description of $\mathcal{Q}_1$ and $\mathcal{Q}_1 \cap \mathcal{Q}_2$ similarly to what we did in § 7.2.1. The proof of Theorem 7.30 is based on a useful technique that produces characters from the polar of $K_{\underline{a},3}$ necessary to eliminate specific elements $x \in Q_1 \cap Q_2$ that do not belong to $K_{\underline{a},3}$.

7.4 Applications

Let $0 \in K \subseteq \mathbb{T}$. For every $m \geq 1$, put

$$W_m := W(K, \mathbb{T}_m) = \{z \in \mathbb{Z} \mid z(K) \subseteq \mathbb{T}_m\}.$$

Remark 7.31 Observe that:

- $W_1 = K^\triangleleft$ (hence, $(W_1)^\triangleright = Q_\mathbb{T}(K)$) and $W_m = (\underbrace{K + \ldots + K}_{m})^\triangleleft$;
- $W_{n+1} \subseteq W_n$, for every $n \geq 1$.

It turns out that $\{W_m \mid m \geq 1\}$ is a base of neighborhood at 0 of a metrizable group topology on $\mathbb{Z}$. Denote this topology by γ_K.

Remark 7.32 The following holds:

1. If $0 \in K \subseteq K'$, then $\gamma_K \leq \gamma_{K'}$.
2. For every $0 \in K \subseteq \mathbb{T}$, $\gamma_K = \gamma_{Q_\mathbb{T}(K)}$. In particular, we can restrict the study of the topologies γ_K to those subsets of $\mathbb{T}$ that are quasi-convex.

Proposition 7.33 *For every* $0 \in K \subseteq \mathbb{T}$*, we have:*

(1) If either K is infinite or $K \not\subseteq \mathbb{Q}/\mathbb{Z}$ (i.e., K contains an irrational $\alpha \in \mathbb{T}$), then γ_K is Hausdorff;

(2) γ_K is locally quasi-convex.

Proof. *(1)* It suffices to show that $\bigcap_m W_m = \{0\}$. So, fix $z \in \bigcap_m W_m$; then $z(K) \subseteq \mathbb{T}_m$ for every m. Since $\bigcap_m \mathbb{T}_m = \{0\}$, $z(K) = \{0\}$, and this yields $z = 0$ by our hypothesis on K.
(2) It follows from the definition of the W_m's, since every polar set is quasi-convex. QED

It is easy to produce examples in which γ_K is discrete.

Example 7.34 *If $K = \mathbb{T}$ or $K = \mathbb{T}_m$ for some $m \geq 1$, then γ_K is discrete.*

Proof. If $K = \mathbb{T}$, then clearly $W_m = \{0\}$ for every $m \geq 1$. In the case $K = \mathbb{T}_m$, for some fixed $m \geq 1$, observe that, for example, $W_{2m} = \{0\}$. QED

The conditions on K in the previous example are not necessary, as we show below.

Example 7.35 *We have that:*

(1) If $K = K_{\underline{a}}$, where $\underline{a}$ is the sequence defined by $a_n = n$ for every $n \geq 0$ (see also Example 5.24), then γ_K is discrete.

(2) If $K = K_{\underline{a}}$, where $\underline{a}$ is the sequence defined by $a_n = 2n$ for every $n \geq 0$, then γ_K is discrete.

Proof. *(1)* It follows from the fact that $W_1 = K^{\triangleright}$, and this coincides with $\{0\}$ according to Claim 5.25.
(2) Observe that $K + K \supseteq \{0\} \cup \{\pm 2^{-(n+1)} \mid n \geq 0\}$, therefore $W_2 \subseteq (\{0\} \cup \{\pm 2^{-(n+1)} \mid n \geq 0\})^{\triangleright} = \{0\}$, according to Claim 5.25. QED

The previous example can be generalized as follows:

Proposition 7.36 *Let $\underline{a}$ be a sequence of nonnegative integers such that $\underline{a}$ contains an arithmetic progression. Then $\gamma_{K_{\underline{a}}}$ is discrete.*

Proof. Put $K := K_{\underline{a}}$. By our hypothesis on $\underline{a}$, it is possible to find $m \geq 1$ such that $\underbrace{K + \ldots + K}_{m \text{ times}} \supseteq \{0\} \cup \{\pm 2^{-(n+1)} \mid n \geq 0\}$, so $(\underbrace{K + \ldots + K}_{m \text{ times}})^{\triangleright} \subseteq (\{0\} \cup \{\pm 2^{-(n+1)} \mid n \geq 0\})^{\triangleright} = \{0\}$ by Claim 5.25. QED

On the other hand, a weak hypothesis on the asymptotic behavior of the sequence $\underline{a}$ guarantees that $\gamma_{K_{\underline{a}}}$ is not discrete.

Notation 7.37 *From now on,* $\underline{a}$ *will be a sequence* $(a_n)_n$ *of positive integers such that* $a_{n+1} - a_n \nearrow \infty$*,* $K := K_{\underline{a}}$ *and* $G := (\mathbb{Z}, \gamma_K)$*. Put also* $k_n := 2^{a_n+1}$*.*

Proposition 7.38 *The sequence* $(k_n)_{n\in\mathbb{N}}$ *converges to* 0 *in* G*.*

Proof. Fix $m \geq 1$. By our assumption on $\underline{a}$ it is possible to find n_m such that $a_{n+1} - a_n \geq m+1$ for every $n \geq n_m$. Now, we claim that $k_j \in W_m$ for every $j \geq n_m$. Indeed, fix $j \geq n_m$ and $x = \frac{1}{k_s} \in K$. Then

$$k_j x = \frac{2^{a_j+1}}{2^{a_s+1}} = \frac{1}{2^{a_s-a_j}}.$$

Now, if $j \geq s$, then, clearly, $k_j x \equiv_1 0$. If $j < s$, then $k_j x \leq \frac{1}{2^{a_{j+1}-a_j}} \leq \frac{1}{2^{m+1}} \leq \frac{1}{4m}$. QED

So, G possesses a non-trivial convergent sequence, hence:

Corollary 7.39 *The group* G *is not discrete.*

Our aim now is to characterize the converging sequences in γ_K. We first show that every γ_K-converging sequence is also τ_2-converging, where τ_2 denotes the 2-adic topology on $\mathbb{Z}$.

Lemma 7.40 *Let* $(\ell_j)_j$ *be a sequence in* $\mathbb{Z}$*. Then:*

$$\ell_j \to 0 \text{ } w.r.t. \text{ } \gamma_K \implies \ell_j \to 0 \text{ } w.r.t. \text{ } \tau_2.$$

In particular, γ_K *is finer then the* 2*-adic topology* τ_2 *and, hence,* $G^\wedge \supseteq \mathbb{Z}(2^\infty)$*.*

Proof. Let us suppose that $\ell_j \to 0$ in γ_K, i.e.,

$$\text{for every } m \text{ there exists } j_0 \text{ s.t. } \ell_j(K) \subseteq \mathbb{T}_m \ \forall\, j \geq j_0. \tag{7.8}$$

We need to show that for every $m \geq 1$ there exists j_0 such that $k_m | \ell_j$ for every $j \geq j_0$. So, fix $m \in \mathbb{N}$. By (7.8), we can find a j_0 such that $\ell_j(K) \in \mathbb{T}_{k_m}$ for every $j \geq j_0$. In particular, this implies that

$$\frac{\ell_j}{k_m}, 2\frac{\ell_j}{k_m}, \ldots, k_m\frac{\ell_j}{k_m} \in \mathbb{T}_+.$$

Since, obviously, $k_m\frac{\ell_j}{k_m} \equiv_1 0$, this means that the finite cyclic group $\langle \frac{\ell_j}{k_m} \rangle$ (of order k_m) is contained in $\mathbb{T}_+$, and this is possible only if $\frac{\ell_j}{k_m} \equiv_1 0$, i.e., $k_m | \ell_j$ and we are done. QED

Now we give a characterization of the converging sequences in G. According to Lemma 7.40, every γ_k-converging sequence $(\ell_j)_j$ is necessarily converging w.r.t. τ_2. So, fix a sequence $(\ell_j)_j$ in G such that $\ell_j \to 0$ w.r.t.

τ_2. Fix $n \in \mathbb{N}$ and let j_n be the smallest index such that $k_n | \ell_j$ for every $j \geq j_n$. Observe that $(j_n)_n$ is an increasing sequence. For every $n > 0$, put

$$M_n := \begin{cases} \{j_n, j_n + 1, \dots, j_{n+1} - 1\} & \text{if } j_n < j_{n+1} \\ \{j_n\} & \text{if } j_n = j_{n+1} \end{cases}$$

and

$$M_0 := \{1, 2, \dots, j_1 - 1\}.$$

Put also

$$s_n := \max_{j \in M_n} \frac{\ell_j}{k_{n+1}} \in \mathbb{T}.$$

Note that the family $\{M_n \mid n \in \mathbb{N}\}$ covers the set of indices $\{j \in \mathbb{N} \mid j \geq 1\}$.

Theorem 7.41 *For $\ell_j \to 0$ in τ_2, the following are equivalent:*

(1) $\ell_j \to 0$ in G;

(2) $s_n \to 0$ in $\mathbb{T}$.

Proof. *(1)* $\Longrightarrow$ *(2)*: Let us suppose that $s_n \not\to 0$, and let us show that $\ell_j \not\to 0$, i.e.

$$\text{there exists } m \text{ such that } \forall\, j_0\ \exists\, j \geq j_0 \text{ with } \ell_j(K) \not\subseteq \mathbb{T}_m. \tag{7.9}$$

By our assumption on s_n, we can find $m > 0$ such that $s_n > \frac{1}{4m}$ for infinitely many n's. So, given j_0, we can pick n such that $s_n > \frac{1}{4m}$ and also $M_n \ni j$ for some $j \geq j_0$. In particular, $\frac{\ell_j}{k_{n+1}} \notin \mathbb{T}_m$ with $j \geq j_0$ and (7.9) holds.
(2)$\Longleftarrow$*(1)*: Now assume $s_n \to 0$ and let us show that (7.8) holds. Fix $m \in \mathbb{N}$. By our hypothesis, we can find an index n_0 such that

$$s_n \leq \frac{1}{4m} \tag{7.10}$$

for every $n \geq n_0$. Now, put $j_0 := \max M_{n_0}$, and fix ℓ_j with $j > j_0$. So, $\ell_j \in M_n$ for some $n > n_0$. Then

$$\ell_j \frac{m}{k_{n+1}} \in \ell_j m \cdot \mathbb{T}_{k_{n+1}-1},$$

and this is contained, according to (7.10), into $m\mathbb{T}_m = \mathbb{T}_+$. Therefore, $\ell_j(K) \subseteq \mathbb{T}_m$ and we are done. QED

Example 7.42 *In Proposition 7.38 we proved that for $\ell_n := k_n$, $\ell_n \to 0$ in G. Let us see that $s_n \to 0$.*
First observe that j_n is simply equal to n, so $M_n = \{n\}$ for every n. Now,

$$s_n = \frac{\ell_n}{k_{n+1}} = \frac{2^{a_n+1}}{2^{a_{n+1}+1}} = \frac{1}{2^{a_{n+1}-a_n}}$$

for every n, therefore $s_n \to 0$ since we have assumed that $a_{n+1} - a_n \nearrow \infty$.

The last result of this section states that γ_K is not precompact whenever K is an infinite subset of $\mathbb{T}$. Observe that it was wrongly affirmed in [17, II, Exercise 2, page 24] that every non-discrete group topology on $\mathbb{Z}$ must be totally bounded. The author is obliged to V. Tarieladze for letting him know the historical background of this fact.

Proposition 7.43 *For every infinite* $0 \in K \subseteq \mathbb{T}$*,* γ_K *is not precompact.*

Proof. Put $G := (\mathbb{Z}, \gamma_K)$. Fix $K \subseteq \mathbb{T}$. Since $K^{\triangleright} \in \mathcal{N}_0(G)$, it is well known that $K^{\triangleright\triangleleft}$ is compact in $G^{\wedge}$ (see also Fact 8.15). Now, suppose that G is precompact. Since G is also metrizable, we have that $G^{\wedge} = (\widetilde{G})^{\wedge}$ by [5, 22], where $\widetilde{G}$ denotes the completion of G; consequently, $G^{\wedge}$ is discrete. Therefore, $K^{\triangleright\triangleleft}$ is necessarily finite. The contradiction follows from the fact that $K^{\triangleright\triangleleft}$ contains the infinite set K. QED

Remark 7.44 It is not hard to prove that $|(\mathbb{Z}, \gamma_K)^{\wedge}| = \mathfrak{c}$ (a proof is given in [7]). In particular, this shows that $(\mathbb{Z}, \gamma_K)^{\wedge} \neq (\mathbb{Z}, \tau_2)^{\wedge}$, where τ_2 denotes the 2-adic topology.

7.5 Additional remarks and open problems

Remark 7.45 We observe that if $\underline{a} = (a_n)_n$ is a sequence of positive integers such that $a_{n+1} - a_n > 1$ for every n, then every subsequence of $\underline{a}$ verifies the same condition. Therefore, Theorem 7.2 implies that every symmetric closed subset of $K_{\underline{a}}$ that contains $0_{\mathbb{T}}$ is still quasi-convex in $\mathbb{T}$, i.e., $K_{\underline{a}}$ is *hereditarily quasi-convex* in $\mathbb{T}$.

We have already commented at the beginning of this chapter that the lacunarity condition $a_{n+1} - a_n > 1$ in Theorem 7.2 appears to be necessary. Actually one can conjecture that it cannot be essentially improved in the following sense:

Conjecture 7.46 *Let* $\underline{a} = (a_n)$ *be such that* $a_0 > 0$*, and suppose that* $\underline{a}$ *contains (infinitely many) adjacent pairs* (a_n, a_{n+1}) *(i.e.,* $a_{n+1} = a_n + 1$*), then* $K_{\underline{a}}$ *is not quasi-convex in* $\mathbb{T}$*.*

The version in brackets admits possibly a positive answer, but maybe even the presence of a single adjacent pair is enough to ruin quasi-convexity.

In Theorem 7.30 we suppose $a_0 > 0$ (i.e., $\frac{1}{3} \notin K_{\underline{a},3}$). We conjecture that, analogously to the dyadic case, if we add the term $\frac{1}{3}$ to $K_{\underline{a}}$, then the quasi-convexity of $K_{\underline{a}}$ is lost.

Conjecture 7.47 *Let $\underline{a} = (a_n)_n$ be a sequence of integers such that $a_0 = 0$ and $a_{n+1} - a_n > 2$ for every $n \in \mathbb{N}$. Then the quasi-convex hull $Q_{\mathbb{T}}(K_{\underline{a},3})$ of $K_{\underline{a},3}$ is given by:*

$$Q_{\mathbb{T}}(K_{\underline{a},3}) = K_{\underline{a},3} \cup (1/3 + K_{\underline{a},3}) \cup (2/3 + K_{\underline{a},3}).$$

Towards another generalization of Theorem 7.2, we propose:

Problem 7.48 *Characterize for which sequences of natural numbers $(c_n)_n$ such that $x_n := \frac{c_n}{2^{a_n+1}} \to 0$ in $\mathbb{T}$ the set $\{0\} \cup \{x_n \mid n \in \mathbb{N}\}$ is quasi-convex.*

Problem 7.49 *Characterize for which sequences $x_n \to 0$ of $\mathbb{T}$ the set $\{0\} \cup \{\pm x_n \mid n \in \mathbb{N}\}$ is quasi-convex in $\mathbb{T}$.*

Moreover,

Question 7.50 *Is it true that every symmetric sequence $x_n \to 0$ in $\mathbb{T}$ (or at least in $\mathbb{Z}(2^\infty)$) possesses a subsequence x_{n_k} such that $K := \{0\} \cup \{x_{n_k} \mid n \in \mathbb{N}\}$ is quasi-convex?*

A positive answer to the weaker version in brackets of the previous question would imply that the 2-adic topology τ_2 on $\mathbb{Z}$ is Mackey (in the sense of Definition 8.6). Indeed, since locally quasi-convex compatible topologies τ on $(\mathbb{Z}, \tau_2)$ can be obtained as topologies of uniform convergence on the sets of a family $\mathfrak{S}$ of quasi-convex compact subsets of the dual (see § 8.1.1), for any non-precompact such topology τ on $\mathbb{Z}$ the family $\mathfrak{S}$ will contain at least one infinite quasi-convex compact set C. Since the dual of $(\mathbb{Z}, \tau_2)$ is $\mathbb{Z}(2^\infty)$ equipped with the topology induced by $\mathbb{T}$, one can use the positive answer to Questions 7.50 to claim that C will certainly contain a sequence converging to 0 that forms, along with 0, a quasi-convex subset K of C. Now, $\gamma_K \leq \tau$, so τ cannot be compatible with τ_2 by Remark 7.44. Thus, τ_2 is Mackey.

In Proposition 7.36 we showed a sufficient condition on $\underline{a}$ in order to get $\gamma_{K_{\underline{a}}}$ discrete: namely, $\underline{a}$ contains an arithmetic progression. On the other hand, in Corollary 7.39 we have shown that if $\underline{a} = (a_n)_n$ is such that $a_{n+1} - a_n \nearrow \infty$, then $\gamma_{K_{\underline{a}}}$ is not discrete. Clearly, there is a gap between these two situations. So we propose the following

Problem 7.51 *Give a complete description of the discreteness of $\gamma_{K_{\underline{a}}}$ depending on $\underline{a}$. For example, is the condition $a_{n+1} - a_n \nearrow \infty$ also necessary to get $\gamma_{K_{\underline{a}}}$ discrete?*

Chapter 8

The Mackey topology for abelian groups

8.1 The definition of the Mackey topology

We recall here very briefly the definition of the Mackey topology of a locally convex vector space and the most substantial result, the Mackey-Arens Theorem, in order to have a guiding line for similar notions in topological groups. We first fix the notation and facts on locally convex spaces needed for this aim.

For a topological vector space E, denote by E^* the vector space of all continuous linear forms on E, also called the *dual space of* E. We denote by $\sigma(E, E^*)$ the weak topology on E, that is, the smallest topology on E which makes continuous the elements of E^*. Dually, the topology denoted by $\sigma(E^*, E)$ in E^* is the weakest topology that makes continuous the linear forms obtained by evaluation on points of E, provided E is identified with a subspace of E^{**}.

The Mackey topology for a locally convex space E is the topology of uniform convergence on the family $\mathfrak{S}$ of all the $\sigma(E^*, E)$-compact convex and circled[1] subsets of E^*. It is usually denoted by $\tau(E, E^*)$. The Mackey-Arens Theorem states that (see, for example, [67, Chapter III, 3.2]) *if E is a locally convex space, then:*

1) $(E, \tau(E, E^*))^* = E^*$, and

2) The topology $\tau(E, E^*)$ is the finest among all the locally convex topologies on E whose dual is again E^*.

We prefer the following (equivalent) formulation of the Mackey-Arens Theorem which fits better our purpose:

[1]Given a vector space E, a subset $C \subseteq E$ is called *circled* if $\lambda C \subseteq C$ for every λ such that $|\lambda| \leq 1$

Theorem 8.1 *Let E be a locally convex space. Then:*

1) *there exists a topology $\tau(E,E^*)$ which is the finest among all the locally convex topologies on E whose dual is again E^*;*

2) *$\tau(E,E^*)$ is characterized as the topology of uniform convergence on the family $\mathfrak{S}$ of all the $\sigma(E^*,E)$-compact convex and circled subsets of E^*.*

The Mackey topology for abelian groups in its full generality was introduced in [23]. Here we obtain several advances. First we give the definitions and needed results of [23] which support our work.

The notion of linear system in the context of vector spaces admits an analogue for groups, namely the *dual pairing* of groups which is defined as follows. If G is an abelian group and $H \leq \mathrm{Hom}(G,\mathbb{T})$, we will say that (G,H) is a *dual pair of groups* (or *dual pairing*). It is obvious that G always separates the points of H. If H separates the points of G, then we say that the dual pairing is *separating.*

Example 8.2 *If G is a topological group, then there exists a standard dual pairing for G, namely $(G,G^\wedge)$. Clearly, G is MAP if and only if $(G,G^\wedge)$ is a separating dual pairing.*

Given a dual pairing (G,H), $\sigma(G,H)$ denotes the initial topology on G with respect to H and $\sigma(H,G)$ denotes the topology on $G^\wedge$ of pointwise convergence on the elements of G. Clearly, $(G,\sigma(G,G^\wedge))$ has the same meaning as G^+, but in the context of dualities this notation is more appropriated. Observe that $\sigma(G,H)$ is Hausdorff whenever (G,H) is a separating pair.

Remark 8.3 For a dual pairing (G,H), it has been proved in [27] that $(G,\sigma(G,H))^\wedge = H$ (see also [23, Theorem 3.7]). In particular, every dual pairing (G,H) can be considered as a pair of the form $(G,G^\wedge)$: in fact, one can take the topological group $G = (G,\sigma(G,H))$ and $G^\wedge = (G,\sigma(G,H))^\wedge = H$.
For this reason we will always consider dual pairings of the form $(G,G^\wedge)$.

Definition 8.4 *Let G be a topological group. A group topology ν on G is said to be* compatible for G *or* with $(G,G^\wedge)$ *if $(G,\nu)^\wedge = G^\wedge$.*

Clearly, for a topological group G we have that $\sigma(G,G^\wedge)$ is compatible for G. Furthermore, $\sigma(G,G^\wedge)$ is the minimum of the set of all compatible topologies for G. As we will see later, there are some instances for which $\sigma(G,G^\wedge)$ is the only compatible topology (see § 8.3).

The major open question faced in this chapter is whether, for a MAP group G, there exists a maximum in the set of all locally quasi-convex compatible topologies for G. We formulate it as follows:

Problem 8.5 ([23]) *For a MAP group G, is there a maximum element in the family of all the topologies that are locally quasi-convex and compatible for G?*

In analogy with the vector space theory, the maximum element should be called the Mackey topology for G. Formally, let us define:

Definition 8.6 *Let G be a MAP topological group. The Mackey topology $\tau(G, G^\wedge)$ is the finest locally quasi-convex topology on G compatible with $(G, G^\wedge)$, provided it exists.*

If the Mackey topology of G exists, then we will say that G is *pre-Mackey* (observe that this notion is different from the definition of "pre-Mackey group" given in [23]).

Let us show that we can restrict the study of the Mackey topology to the class of locally quasi-convex groups. Given a MAP group G, we consider on G the weak topology τ_{LQC} with respect to the class LQC (i.e., it is the topology induced on G by the family of all continuous homomorphisms from G to all locally quasi-convex groups). Observe that such a topology exists since LQC is closed under products and subgroups, and τ_{LQC} is the finest locally quasi-convex topology on G which is coarser then the original one (see § 1.3.1). Obviously, a topology τ on G is locally quasi-convex if and only if $\tau = \tau_{LQC}$.

Remark 8.7 Let (G, τ) be a MAP group. Since $\sigma(G, G^\wedge)$ is locally quasi-convex, $\sigma(G, G^\wedge) \leq \tau_{LQC} \leq \tau$. Consequently, τ_{LQC} is compatible with $(G, G^\wedge)$.

Denote (G, τ_{LQC}) by G_{LQC}. What is interesting now is that, by means of the previous remark, G is pre-Mackey if and only if G_{LQC} is pre-Mackey, and $\tau(G, G^\wedge)$ coincides with $\tau(G_{LQC}, (G_{LQC})^\wedge)$. For this reason we can restrict to the class of locally quasi-convex groups; in particular, Problem 8.5 can be reformulated as follows:

Problem 8.8 *Is it true that every locally quasi-convex group is pre-Mackey?*

In the next definition we call a locally quasi-convex group G a *Mackey group* if it carries the Mackey topology:

Definition 8.9 *A locally quasi-convex group (G, ν) is a* Mackey group *if ν coincides with the Mackey topology $\tau(G, G^\wedge)$.*

Clearly, a Mackey group (G, ν) is characterized by the property that if τ is another locally quasi-convex topology on G with the same dual group as (G, ν), then $\tau \leq \nu$.

Observe that the definition of (pre-)Mackey group is inspired by 1) of Theorem 8.1. However, the explicit definition given in item 2) of the same theorem by means of uniform convergence on a family of subsets of the dual could also be imitated for groups. We deal with both these points of view in the next section.
We anticipate here that in § 8.2 we will present a class of groups (namely, the class of *Arens groups*) for which a precise counterpart of Theorem 8.1 holds (see Remark 8.26 and Definition 8.27). Nevertheless, we will show in Theorem 8.61 that there are groups that admit the Mackey topology but the counterpart of item 2) of Theorem 8.1 fails. This permits to claim that *the Mackey-Arens Theorem 8.1 cannot be completely imitated in the class of topological groups.* Essentially, this is due to the fact that the $\sigma(G^\wedge, G)$-compact and quasi-convex subsets of $G^\wedge$ do not play the same role as the $\sigma(E^*, E)$-compact convex circled subsets of E^*.

8.1.1 The topologies $\tau_g(G, G^\wedge)$ and $\tau_{qc}(G, G^\wedge)$

The topologies mentioned in the title were introduced in [23]. They are the natural candidates to be the Mackey topology for a group G.

Definition 8.10 ([23]) *The topology $\tau_g(G, G^\wedge)$ is the least upper bound of the family of all locally quasi-convex topologies on G compatible with $(G, G^\wedge)$.*

So, if (G, τ) il a locally quasi-convex group, then $\tau \leq \tau_g(G, G^\wedge)$.
By the definition of supremum topology, $(G, \tau_g(G, G^\wedge))$ can be embedded in a product of locally quasi-convex groups, as we state now:

Fact 8.11 *If $\{\tau_i \mid i \in I\}$ denotes the family of all locally quasi-convex topologies on a group G that are compatible with the dual pairing $(G, G^\wedge)$, then $(G, \tau_g(G, G^\wedge)) \hookrightarrow \prod_{i \in I}(G, \tau_i)$ by means of the diagonal mapping.*

This yields the following:

Corollary 8.12 ([23]) *For every topological group G, $\tau_g(G, G^\wedge)$ is a locally quasi-convex topology.*

It is clear that if $\tau_g(G, G^\wedge)$ is compatible with $(G, G^\wedge)$, then $\tau_g(G, G^\wedge) = \tau(G, G^\wedge)$. More precisely:

Theorem 8.13 ([23]) *A topological group G is pre-Mackey if and only if $\tau_g(G, G^\wedge)$ is a compatible topology for G. In such case, the Mackey topology $\tau(G, G^\wedge)$ coincides with $\tau_g(G, G^\wedge)$.*

We now deal with the possibility of describing the Mackey topology of a locally quasi-convex group G by uniform convergence on a certain family of subsets of the dual $G^\wedge$, as done in the context of locally convex spaces (see Theorem 8.1). To this aim, it will be convenient to recall the definition of *equicontinuous subset of the dual group* $G^\wedge$. Observe that this notion comes from the context of uniform spaces, but for homomorphisms of topological groups it can be formulated as follows:

Definition 8.14 *Let G be a topological group. A subset $M \subseteq G^\wedge$ is equicontinuous if there exists $V \in \mathcal{N}_G(0)$ such that $M \subseteq V^\triangleright$.*

Note that $U^\triangleright$ is equicontinuous for all $U \in \mathcal{N}_G(0)$. Moreover, for future purpose let us fix the following:

Fact 8.15 *The following properties hold:*

1) *For every $U \in \mathcal{N}_G(0)$, $U^\triangleright$ is $\sigma(G^\wedge, G)$-compact. In particular, every equicontinuous subset $M \subseteq G^\wedge$ is relatively $\sigma(G^\wedge, G)$-compact.*

2) *If $M \subseteq G^\wedge$ is equicontinuous, $\sigma(G^\wedge, G) \restriction_M = \tau_{co} \restriction_M$.*

Consequently, for every $U \in \mathcal{N}_G(0)$, $U^\triangleright$ is τ_{co}-compact.

Given a topological group G, if $\mathfrak{S}$ is a family of non-empty subsets of $G^\wedge$, then we can consider the $\mathfrak{S}$*-topology* $\tau_{\mathfrak{S}}(G, G^\wedge)$ on G, namely the topology of uniform convergence on the sets $A \in \mathfrak{S}$. For instance, the smallest compatible topology for G is given by uniform convergence on the family of all finite subsets of $G^\wedge$:

Example 8.16 *For a topological group G, $\sigma(G, G^\wedge) = \tau_{\mathfrak{F}(G^\wedge)}$, where $\mathfrak{F}(G^\wedge)$ denotes the family of all the finite subsets of $G^\wedge$.*

It follows from the definition that $\tau_{\mathfrak{S}}(G, G^\wedge)$ is a locally quasi-convex topology on G, for every family $\mathfrak{S}$. The converse also holds, that is, any locally quasi-convex topology on a group G is an $\mathfrak{S}$-topology in G for a certain family $\mathfrak{S}$ (see Corollary 8.22); this will be deduced from the fact that the topology τ_{LQC} is an $\mathfrak{S}$-topology.

Fact 8.17 *Given two families $\mathfrak{S}_1, \mathfrak{S}_2$ we have:*

- *if $\mathfrak{S}_1 \subseteq \mathfrak{S}_2$, then $\tau_{\mathfrak{S}_1}(G, G^\wedge) \leq \tau_{\mathfrak{S}_2}(G, G^\wedge)$;*
- *if $\mathfrak{S}_1$ is cofinal in $\mathfrak{S}_2$, then $\tau_{\mathfrak{S}_2}(G, G^\wedge) \leq \tau_{\mathfrak{S}_1}(G, G^\wedge)$.*

Remark 8.18 It has been observed in [23] that if $\mathfrak{S}$ verifies the following properties:

a) for $B_1, B_2 \in \mathfrak{S}$, there exists $B_3 \in \mathfrak{S}$ such that $B_1 \cup B_2 \subseteq B_3$;

b) for $B \in \mathfrak{S}$ and $n \in \mathbb{N}$, there exists $A \in \mathfrak{S}$, such that $B^{(n)} := \{y^n; y \in B\} \subseteq A$;

then $\{B^{\triangleleft} \mid B \in \mathfrak{S}\}$ is a base of neighborhoods of 0_G in $\tau_{\mathfrak{S}}(G, G^\wedge)$.
A family $\mathfrak{S}$ that verifies a) and b) is called *well-directed.*

We are interested in some concrete $\mathfrak{S}$-topologies, namely, those given by the following families:

Notation 8.19 *For a locally quasi-convex group G:*

- *$\mathfrak{S}_e$ denotes the family of all the equicontinuous subsets of $G^\wedge$;*
- *$\mathfrak{S}_{qc}$ denotes the family of all $\sigma(G^\wedge, G)$-compact quasi-convex subsets of $G^\wedge$.*

Moreover, put $\mathfrak{S}_0 := \mathfrak{S}_e \cap \mathfrak{S}_{qc}$.

Clearly, $\mathfrak{S}_0 \subseteq \mathfrak{S}_e$. Moreover:

Lemma 8.20 ([23]) *The family $\mathfrak{S}_0$ is a cofinal subfamily of $\mathfrak{S}_e$. Consequently, $\tau_{\mathfrak{S}_e}(G, G^\wedge) = \tau_{\mathfrak{S}_0}(G, G^\wedge)$.*

Proof. Fix an equicontinuous subset $M \subseteq G^\wedge$. Then there exists $V \in \mathcal{N}_G(0)$ such that $M \subseteq V^{\triangleright}$. Now, $V^{\triangleright} \subseteq G^\wedge$ is equicontinuous and quasi-convex in $\sigma(G^\wedge, G)$. Moreover, $V^{\triangleright}$ is $\sigma(G^\wedge, G)$-compact by Fact 8.15 *1).*
The last assertion follows from Fact 8.17. QED

The topology $\tau_e := \tau_{\mathfrak{S}_e}(G, G^\wedge)$ is a locally quasi-convex compatible topology for (G, τ) which is coarser than τ and it is called *the locally quasi-convex modification of* τ. It turns out that τ_e is the finest among all locally quasi-convex compatible topologies coarser than τ, so it coincides with τ_{LQC}. For historical reasons, we maintain the notation τ_e.
From our considerations on the topology τ_{LQC}, we deduce:

Proposition 8.21 *Let (G, τ) be a topological group. Then:*

(1) $\sigma(G, G^\wedge) \leq \tau_e \leq \tau$. In particular, τ_e is a compatible topology for (G, τ);

(2) (G, τ) is locally quasi-convex if and only if $\tau = \tau_e$;

Corollary 8.22 *Let (G, τ) be a locally quasi-convex group. Then τ is the topology of uniform convergence on the equicontinuous subsets of $G^\wedge$.*

Proof. Just observe that $\tau = \tau_e$ by Proposition 8.21 *(2)* and use the definition of τ_e. QED

Now we deal with the topology $\tau_{qc}(G, G^\wedge)$.

Lemma 8.23 *For every topological group* (G, τ)*:*

1) $\tau_e \leq \tau_{qc}(G, G^\wedge)$*;*

2) $\tau_{qc}(G, G^\wedge)$ *is finer than any other locally quasi-convex compatible topology.*

Proof. Item *1)* follows from 7.2 Fact 8.17 and Lemma 8.20. For *2)*, take a locally quasi-convex compatible topology ν on G. Then $\nu = \nu_e$ by Corollary 8.22. Now apply *1)*. QED

The picture is as follows (see also [23, Proposition 3.13]):

Proposition 8.24 ([23]) *Let* (G, τ) *be a locally quasi-convex topological group. Then:*

$$\sigma(G, G^\wedge) \leq \tau \leq \tau_g(G, G^\wedge) \leq \tau_{qc}(G, G^\wedge).$$

Proof. The fact that $\sigma(G, G^\wedge) \leq \tau \leq \tau_g(G, G^\wedge)$ is clear. To prove that $\tau_g(G, G^\wedge) \leq \tau_{qc}(G, G^\wedge)$, use Lemma 8.23 *2)* and recall that $\tau_g(G, G^\wedge)$ is the least upper bound of the family of all locally quasi-convex topologies on G compatible with $(G, G^\wedge)$. QED

In particular, if $\tau_{qc}(G, G^\wedge)$ is compatible, then $\tau_g(G, G^\wedge)$ is compatible as well and these two topologies coincide. More precisely,

Corollary 8.25 *If* $\tau_{qc}(G, G^\wedge)$ *is compatible with* $(G, G^\wedge)$*, then the Mackey topology of* G *exists and it coincides with* $\tau_{qc}(G, G^\wedge)$*.*

Proof. Since $\tau_{qc}(G, G^\wedge)$ is compatible and locally quasi-convex, clearly $\tau_{qc}(G, G^\wedge) \leq \tau_g(G, G^\wedge)$. Hence $\tau_{qc}(G, G^\wedge) = \tau_g(G, G^\wedge)$, according to Proposition 8.24. In particular, $\tau_g(G, G^\wedge)$ is compatible and, hence, $\tau_g(G, G^\wedge) = \tau(G, G^\wedge)$ by Theorem 8.13. QED

In § 8.3.1 we will show that the converse implication does not hold in general. Indeed, we prove in Theorem 8.61 the existence of a class of locally quasi-convex groups G such that every G is a Mackey group but $\tau_{qc}(G, G^\wedge)$ is not compatible for every G.

By means of Corollary 8.25, it is important to note that:

Remark 8.26 The class of those locally quasi-convex groups G such that $\tau_{qc}(G, G^\wedge)$ is compatible is precisely the class of groups for which the counterpart of Theorem 8.1 holds.

Since Arens was the first who described the Mackey topology as an $\mathfrak{S}$-topology, we give the following definition.

Definition 8.27 *A locally quasi-convex group* G *such that* $\tau_{qc}(G, G^\wedge)$ *is compatible with* $(G, G^\wedge)$ *is said to be an* Arens group*.*

Observe that the property "being an Arens group" is actually a property of the dual pair $(G, G^\wedge)$ in the following sense. Given (G, ν), consider the family $\{\tau_i \mid i \in I\}$ of all the topologies that are compatible with $(G, G^\wedge)$. Then (G, ν) is Arens if and only if (G, τ_i) is Arens for every $i \in I$.

Corollary 8.25 states that if G is an Arens group, then $\tau_{qc}(G, G^\wedge)$ is its Mackey topology. In particular,

Remark 8.28 Every Arens group is pre-Mackey.

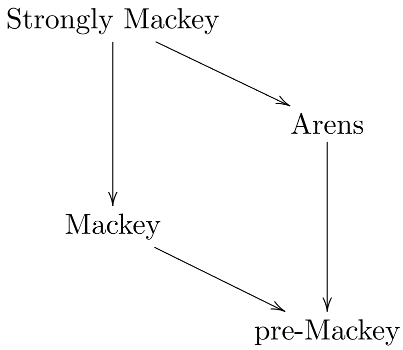
If the topology $\tau_{qc}(G, G^\wedge)$ coincides with the original topology of G, then we say that G is *strongly Mackey*:

Definition 8.29 *A locally quasi-convex group* (G, τ) *is* strongly Mackey *if* $\tau_{qc}(G, G^\wedge) = \tau$.

By means of Corollary 8.25, we easily deduce the following:

Remark 8.30 A group G is strongly Mackey if and only if it is Mackey and Arens.

Given a MAP group, so far we have the following implications:



Both the implications starting from "Strongly Mackey" cannot be inverted: indeed, there are plenty of Arens groups that are not Mackey (hence, nor strongly Mackey). For instance, consider a group of the form $G^\#$. Then $\tau_{qc}(G, G^\wedge)$ is discrete (since $G^\wedge$ is compact, so it is an element of $\mathfrak{S}_{qc}$ which polar is trivial) and, hence, it is compatible with $(G, G^\wedge)$. So, $\tau_{qc}(G, G^\wedge)$ is the Mackey topology by Corollary 8.25, but it does not coincide with the topology of $G^\#$.

In § 8.3.1 we present an example of a precompact group G in which

$$\sigma(G, G^\wedge) = \tau_g(G, G^\wedge) = \tau(G, G^\wedge) \lneq \tau_{qc}(G, G^\wedge), \tag{8.1}$$

and $\tau_{qc}(G, G^\wedge)$ is discrete and not compatible with $(G, G^\wedge)$ (see Theorem 8.61). In particular, this shows that a Mackey group need not be Arens.
According to Theorem 8.13, the first equality in (8.1) for an arbitrary MAP group G imposes that G is pre-Mackey. Moreover, if G is locally quasi-convex, then G is necessarily precompact.

8.2 The class of g-barrelled groups

The notion of *g-barrelled* group was introduced in [23]. Our interest on this class of groups stems from the fact that every locally quasi-convex g-barrelled group is strongly Mackey (see Theorem 8.34). However, they do not exhaust the class. Indeed, in Theorem 8.62 we deal with a class of strongly Mackey groups which are not g-barrelled.

Definition 8.31 ([23]) *A topological group* (G, τ) *is* g-barrelled *if every* $\sigma(G^\wedge, G)$*-compact subset of* $G^\wedge$ *is equicontinuous.*

In other words, for a g-barrelled group we have that $\mathfrak{S}_{qc} = \mathfrak{S}_0$. Then, Lemma 8.20 immediately yields

Proposition 8.32 *If* G *is g-barrelled, then* $\tau_{qc}(G, G^\wedge) = \tau_e$*.*

As a consequence of Proposition 8.32 and Proposition 8.21 *(1)*, we obtain

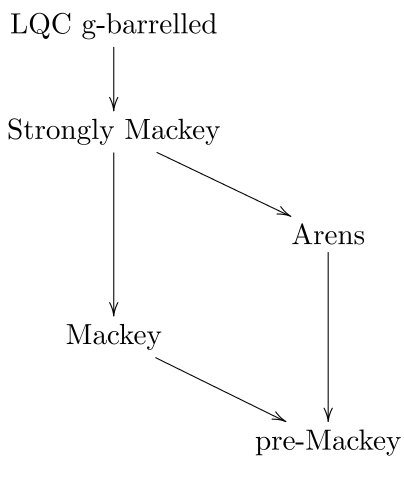
Corollary 8.33 *Every locally quasi-convex g-barrelled group is an Arens group.*

In the next result we show a stronger property.

Theorem 8.34 *If* G *is a locally quasi-convex g-barrelled group, then* G *is strongly Mackey.*

Proof. Let ν denote the topology of G. Since (G, ν) is locally quasi-convex, $\nu = \nu_e$ by Proposition 8.21 *(2)*. Hence, $\nu = \tau_{qc}(G, G^\wedge)$ by Proposition 8.32. QED

Let us update our diagram of implications:



As a matter of fact, the implication "strongly Mackey $\Longrightarrow$ g-barrelled" does not hold in general. In order to give an example (see Theorem 8.62), we need to develop some additional properties of locally quasi-convex g-barrelled groups.
Let us prove that every locally quasi-convex g-barrelled group is a dual group, namely:

Proposition 8.35 *Let G be a locally quasi-convex g-barrelled group. Then G is topologically isomorphic to $(G^\wedge, \sigma(G^\wedge, G))^\wedge$.*

Proof. Put $X := (G^\wedge, \sigma(G^\wedge, G))$. Since the continuous characters on X are evaluations on points of G, $X^\wedge$ can be algebraically identified with G. In order to check that this identification is also topological, we must see that any zero-neighborhood in G corresponds to a zero-neighborhood in the compact open topology τ_{co} in $X^\wedge$ and conversely.
Fix a quasi-convex $V \in \mathcal{N}_G(0)$. Then by Fact 8.15 *1)*, $V^\triangleright$ is a $\sigma(G^\wedge, G)$-compact subset of $G^\wedge$, and so $(V^\triangleright)^\triangleright$ is a neighborhood of zero in the compact open topology of $X^\wedge$. Observe now that by the identification above mentioned, $(V^\triangleright)^\triangleright \equiv (V^\triangleright)^\triangleleft = V$. Thus, $V \in \mathcal{N}_{X^\wedge}(0)$.
Conversely, if $W \subseteq X^\wedge$ is a τ_{co}-neighborhood of zero, then $W \supseteq L^\triangleright$ for some $\sigma(G^\wedge, G)$-compact subset L of $G^\wedge$. Since G is g-barrelled, L is equicontinuous and this means that $L^\triangleleft$ is a neighborhood of zero in the original topology of G. Observe now that $L^\triangleright$ and $L^\triangleleft$ may be identified (since $X^\wedge \equiv G$), therefore W is a neighborhood of zero in the original topology of G. QED

Corollary 8.36 ([6], Proposition 5.3) *If G is a countable locally quasi-convex g-barrelled group, then G is discrete.*

Proof. Put $X := (G^\wedge, \sigma(G^\wedge, G))$. Clearly, $X \hookrightarrow \mathbb{T}^G$, and hence X is metrizable (since G is countable). Let $\widetilde{X}$ denote the Bohr-completion of X. Then X and $\widetilde{X}$ have the same dual group by [5, 22], therefore $X^\wedge$ is discrete. By Proposition 8.35, G is discrete as well. QED

The following result collects all the subclasses of g-barrelled groups known so far:

Theorem 8.37 ([23]) *The class of g-barrelled groups includes:*

- *all metrizable hereditarily Baire groups;*
- *all separable Baire groups;*
- *all Čech-complete groups.*

Corollary 8.38 *Every complete metrizable group and every locally compact (in particular, compact) group is g-barrelled.*

Observe that all the groups that appear in the previous theorem are uncountable. This property may be necessary for every g-barrelled group. Indeed, in Corollary 8.36 we showed that if G is countable and locally quasi-convex, then it is not g-barrelled unless it is discrete. See also Question 8.83.

One of the main results of this section consists in showing the existence of a new class of g-barrelled groups (see Remark 8.48). Before proving this result, let us point out a big class of abelian groups which are not g-barrelled. To this aim, consider the following proposition.

Lemma 8.39 *If G is a topological locally quasi-convex group such that $(G, \sigma(G, G^\wedge))$ is g-barrelled, then $G = (G, \sigma(G, G^\wedge))$ is precompact.*

Proof. It is clear since $(G, \sigma(G, G^\wedge))$ is Mackey by Theorem 8.34 and Remark 8.30, hence $\sigma(G, G^\wedge)$ is the only locally quasi-convex compatible topology on G. QED

In particular, for a topological locally quasi-convex group G which is not precompact, $(G, \sigma(G, G^\wedge))$ is not g-barrelled. Nevertheless, there are examples of (necessarily precompact) groups such that $(G, \sigma(G, G^\wedge))$ is g-barrelled (for instance, every compact group; see Corollary 8.38). In the next result we give a characterization of this fact by means of a simple condition on the $\sigma(G^\wedge, G)$-compact subsets of $G^\wedge$.

Theorem 8.40 *Let (G, τ) be a topological abelian group. Then the following assertions are equivalent:*

1) Any $\sigma(G^\wedge, G)$-compact subset of $G^\wedge$ is finite;

2) The group $(G, \sigma(G, G^\wedge))$ is g-barrelled.

Proof. *1) ⇒ 2)*: This is clear since every finite subsets of $G^\wedge$ is equicontinuous with respect to any group topology of G, in particular with respect to $\sigma(G, G^\wedge)$.
2) ⇒ 1): Take any $K \subset G^\wedge$ which is $\sigma(G^\wedge, G)$-compact. By *2)* there is a zero-neighborhood U in $(G, \sigma(G, G^\wedge))$ such that $U \subseteq K^\triangleleft$. In particular, there exists a finite subset $F \subseteq G^\wedge$ such that $F^\triangleleft \subseteq U$ (according to Example 8.16). Thus, $F^\triangleleft \subseteq K^\triangleleft$, and this implies that $(K^\triangleleft)^\triangleright \subseteq (F^\triangleleft)^\triangleright$. Since $G^\wedge$ is MAP, we deduce from Theorem 0.4 *(2)* that $(F^\triangleleft)^\triangleright$ is finite and, consequently, $K \subseteq (K^\triangleleft)^\triangleright \subseteq (F^\triangleleft)^\triangleright$ is also finite. QED

We introduce now a class of groups which happens to be precompact and g-barrelled, therefore satisfying *2)* (and, equivalently, *1)*) of Theorem 8.40.

Definition 8.41 *An abelian topological group G is ω-bounded if every countable subset of G is contained in a compact subgroup of G.*

Clearly, every compact group is ω-bounded. If $\mathcal{K}$, Ω and $\mathcal{CK}$ denote respectively the classes of compact, ω-bounded and countably compact abelian groups, the following relationships hold among these classes:

$$\mathcal{K} \subset \Omega \subset \mathcal{CK}$$

Fact 8.42 *a) Every ω-bounded group is pseudocompact and, hence, precompact;*

b) any separable ω-bounded topological group must be compact.

Let us give an example of a group which is ω-bounded non-compact.

Example 8.43 *Let $\{(K_i, \tau_i) \mid i \in I\}$ be a family of compact groups with $|I| \geq \mathfrak{c}$, and consider the Σ-product $\Sigma := \{x \in \prod_{i\in I} K_i \ : \ |\, supp\ x| \leq \omega\}$ equipped with the topology induced from $\prod_{i\in I} K_i$. Then Σ is ω-bounded and non-compact.*

The main result about ω-bounded groups is the following one.

Theorem 8.44 *If G is ω-bounded, then G is g-barrelled.*

The proof of Theorem 8.44 consists on several steps. So let us consider the following auxiliary results.

Lemma 8.45 *Let G be a MAP topological group and let $N \leq G$ be a compact subgroup. Then the Bohr topology of the quotient group $G^\wedge/N^\perp$ is precisely the quotient topology of $(G^\wedge, \sigma(G^\wedge, G))$ with respect to $N^\perp$.*

Proof. For brevity, let us write $X := G^\wedge$ and $X_\sigma := (X, \sigma(X, G))$.
Since N is compact, $N^\perp$ is open in X and $(X/N^\perp)^\wedge \cong (N^\perp)^\perp$ ([12]), where the second annihilator is taken in $G^{\wedge\wedge}$. Let us see now that $(N^\perp)^\perp$ may be identified with N. Denote by $i : N \to G$ the inclusion mapping. Its dual $i^\wedge : G^\wedge \to N^\wedge$ is continuous and onto since N is dually embedded in G ([21]), and it induces a continuous isomorphism $G^\wedge/N^\perp \cong N^\wedge$ which, in fact, is a topological isomorphism since both are discrete. Taking duals, we get $(G^\wedge/N^\perp)^\wedge \cong N^{\wedge\wedge} \cong N$ (the last isomorphism is due to compactness of N). Now, $(X/N^\perp)^\wedge \cong (N^\perp)^\perp$ together with $(X/N^\perp)^\wedge \cong N$ yield the identification of N with $(N^\perp)^\perp$.
Next, we check that $X_\sigma/N^\perp$ admits the same continuous characters as $X/N^\perp$. By the definition of a quotient topology, a homomorphism $\kappa : X_\sigma/N^\perp \to \mathbb{T}$ is continuous if and only if κp is continuous where $p : X_\sigma \to X_\sigma/N^\perp$ is the canonical projection. Now, if κp is a continuous character in X_σ, it must be the evaluation on a point, say $x \in G$. On the other hand, κp is null in $N^\perp$, thus $x \in N$.

To finish the proof take into account that $X_\sigma/N^\perp$ is precompact, as a quotient of the precompact group X_σ. Hence, its topology is determined by its continuous characters. QED

With the same assumptions and notation as in the previous lemma we obtain:

Corollary 8.46 *The compact subsets of $X_\sigma/N^\perp$ are finite.*

Proof. Indeed, by the previous lemma the quotient topology of $X_\sigma/N^\perp$ coincides with the Bohr topology of the discrete quotient group $X/N^\perp$, i.e., $X_\sigma/N^\perp \cong (X/N^\perp)^\#$. By [49], all compact sets of $(X/N^\perp)^\#$ are finite. QED

In the next proposition, we assume that G is ω-bounded. Note that such a group is necessarily precompact (see *a)* of Fact 8.42) and, hence, MAP. We denote by T the completion of G (observe that T is compact). Then the dual group $X = G^\wedge$ is algebraically isomorphic to the discrete group $T^\wedge$. Note that since the group G is precompact, its topology coincides with the weak topology $\sigma(G, X)$. In other words, $(G, \sigma(G, X)) = G$, so we shall also simply write G most often.

Proposition 8.47 *If the group G is ω-bounded, then the compact subsets of X_σ are finite.*

Proof. Assume for a contradiction that K is an infinite compact set in X_σ and fix a countably infinite subset D of K. Let L denote the countable subgroup of X generated by D, and consider the annihilator $L^\perp$ of L in T. Then $L^\perp$ is a compact subgroup of T such that $L^\wedge$ is isomorphic to $T/L^\perp$. Since L is countable, $T/L^\perp$ is metrizable, hence separable (being compact). Let S be a dense countable subgroup of $T/L^\perp$. By the pseudocompactness of G, we can claim that G meets every G_δ-subset of T. Since $T/L^\perp$ is metrizable, the subgroup $L^\perp$ is a G_δ-subset of T (as well as all its cosets). This means that G meets every coset $x + L^\perp$. In other words, the restriction of the quotient map $q : T \to T/L^\perp$ to G is still surjective. In particular, we can find a countable subset S_1 of G such that $q(S_1) = S$. Now exploit the ω-boundedness of G to find a compact subgroup N of G containing S_1. Clearly, N contains the closure $\overline{S_1}$. Then $q(N) \supseteq \overline{S} = T/L^\perp$. This means that $N + L^\perp = T$. Now taking once again the annihilators we get

$$0 = T^\perp = (N + L^\perp)^\perp = N^\perp \cap (L^\perp)^\perp = N^\perp \cap L. \qquad (8.2)$$

Let $r : X \to X/N^\perp$ be the quotient homomorphism. By (8.2), the restriction of r on L is injective, in particular $r(K)$ is infinite.
So far no topology on X was involved. The annihilator $N^\perp$ is closed in X_σ, so we can consider on $X/N^\perp$ the quotient topology. By Corollary 8.46, the set $r(K)$ must be finite, and we get a contradiction. QED

Now we are in position to prove Theorem 8.44.

Proof of Theorem 8.44. Observe that G is precompact by *a)* of Fact 8.42, so G carries the weak topology $\sigma(G, G^\wedge)$. Now recall Proposition 8.47 and apply Theorem 8.40. The theorem is proved.

It is important to note that:

Remark 8.48 The class of ω-bounded noncompact groups is a class of g-barrelled groups that was not known before.

Indeed, the only known classes of g-barrelled groups are those mentioned in Theorem 8.37. Now just consider the following

Proposition 8.49 *An ω-bounded non-compact group is not metrizable, nor separable and neither Čech-complete.*

Proof. Let $G \in \Omega \setminus \mathcal{K}$. Recall that for every metrizable group, compactness and numerably compactness coincide. In particular, G is not metrizable since it is numerably compact but not compact. Moreover, G is not separable according to Fact 8.42 *b)*. The fact that G is not Čech-complete follows from *a)* of Fact 8.42 jointly with the following claim.

Claim. Every pseudocompact Čech-complete group is compact.

Let us prove the claim. Fix a pseudocompact Čech-complete group G. By definition, G is Čech-complete if and only if it is a G_δ-set of its Stone-Čech compactification. Since G is pseudocompact, its Stone-Čech compactification coincides with its compact completion K and G is G_δ-dense in K (i.e., G hits every non-empty G_δ-set of K). Assume for a contradiction that G is not compact. Then G is a proper subgroup of K, therefore there exists a coset xG of G in K that is (obviously) disjoint with G. This contradicts the G_δ-density of G, since xG (as a translate of G) is a G_δ-set of K. The claim is proved. QED

To conclude this section, let us present an important consequence of Proposition 8.32.

Theorem 8.50 *Given a topological group G, then there exists at most one locally quasi-convex compatible topology τ such that (G, τ) is g-barrelled. In particular, there is at most one locally quasi-convex compatible topology which is in the union of the following classes of topological groups: metrizable hereditarily Baire, separable Baire, Čech-complete and ω-bounded.*

Proof. Suppose that there exists τ such that (G, τ) is locally quasi-convex and g-barrelled. Then (G, τ) is strongly Mackey by Theorem 8.34, hence

$\tau = \tau_{qc}(G, G^\wedge)$. Since the topology $\tau_{qc}(G, G^\wedge)$ is the same for every locally quasi-convex topology which is compatible with τ, this implies that τ is the only topology on G with this combination of properties.
The last assertion of the theorem follows from Theorem 8.37 and Theorem 8.44. QED

8.3 On the set of LQC compatible topologies for a topological group

[On the set of LQC compatible topologies for a top. grp.]

Let G be a topological group. Consider all the topologies on G compatible with $(G, G^\wedge)$. It is natural to ask how they are related to each other. Clearly, all the compatible topologies on G have the same Bohr topology. Thus, we can claim that there is a minimum in the class of all compatible topologies, which happens to be locally quasi-convex. It is not clear under which conditions on a MAP group G the locally quasi-convex compatible topologies will form a chain, or even a lattice.

Fact 8.51 *If G is a pre-Mackey group, then the family of locally quasi-convex compatible topologies on G is a complete lattice, having the weak topology (i.e., the Bohr topology) as bottom element and the Mackey topology as top element.*

We are going to study some families of topological groups for which the only locally quasi-convex compatible topology is the Bohr topology.

Definition 8.52 *A locally quasi-convex group G is said to be* ULQC *if G admits only one locally quasi-convex compatible topology, namely $\sigma(G, G^\wedge)$.*

Remark 8.53 If G is ULQC, then G is precompact and Mackey. Moreover, if G is precompact, then G is Mackey if and only if G is ULQC.

Examples of ULQC groups can be found in the class of ω-bounded groups. Indeed, every ω-bounded is precompact by Fact 8.42 *a)*, and it is also g-barrelled by Theorem 8.44. Now recall that every locally quasi-convex (in particular, every precompact) g-barrelled group is strongly Mackey by Theorem 8.34, hence Mackey. Now apply Remark 8.53.

In the next section we show another class of ULQC groups. Moreover, we show that the topology $\tau_{qc}(G, G^\wedge)$ can either coincide with $\sigma(G, G^\wedge)$ (in particular, G is strongly Mackey; see Theorem 8.62) or not (indeed, $\tau_{qc}(G, G^\wedge)$ can be even discrete and non-compatible; see Theorem 8.61). From this we deduce that in the class of precompact groups, the property of possessing only one compatible topology does not imply "being an Arens group" and, hence, "being strongly Mackey" (compare with Remark 8.53).

8.3.1 Another class of ULQC groups

We define a class of topological groups (which we have called BTM-groups inspired by [16]) by some conditions on their dual groups. It turns out that every locally quasi-convex BTM group is ULQC (Corollary 8.59). In particular, in this class we obtain examples of strongly Mackey groups which are not g-barrelled (Theorem 8.61) and of Mackey groups that are not strongly Mackey (Theorem 8.62).

Definition 8.54 *A MAP topological group G is a* BTM-group *if:*

1. *there exists an integer $m \geq 2$ such that $m\chi = 1$ for every $\chi \in G^\wedge$;*
2. *$|G^\wedge| < \mathfrak{c}$.*

Example 8.55 *Let $G := \bigoplus_{k=1}^{\infty} \mathbb{Z}_{n_k}$ where $\{n_k\}_{k=1}^{\infty}$ is a bounded sequence of positive integers. Equip G with the topology μ induced from the product $\prod_{k=1}^{\infty} \mathbb{Z}_{n_k}$. Then (G, μ) is a BTM-group.*
Indeed, this is clear since G is dense in the metrizable group $\prod_{k=1}^{\infty} \mathbb{Z}_{n_k}$, therefore $G^\wedge \cong (\prod_{k=1}^{\infty} \mathbb{Z}_{n_k})^\wedge \cong G$.

We show that if G is BTM, then $\sigma(G, G^\wedge)$ is the only locally quasi-convex compatible topology on G. To this aim, we first show that every BTM group is precompact. This will implies that a locally quasi-convex BTM group is ULQC.
Consider the following auxiliary result:

Proposition 8.56 *Let G be a topological BTM-group. Then, for every $U \in \mathcal{N}_G(0)$ we have that $U^\triangleright \subseteq G^\wedge$ is finite.*

Some notation is now convenient: put $(A, B) = \{\chi \in G^\wedge \mid \chi(A) \subseteq B\}$ for nonempty subsets $A \subseteq G$ and $B \subseteq G^\wedge$.

Proof. Fix $U \in \mathcal{N}_0(G)$. We will prove that $U^\triangleright$ is contained in a $\sigma(G^\wedge, G)$-compact subgroup of $G^\wedge$, which must be finite.
Assume that $o(G^\wedge) = m$. Fix a positive integer n_0 such that $n_0 > m/4$ and let $V \in \mathcal{N}_0(G)$ be such that $\underbrace{V + V + \ldots + V}_{n_0 \text{ times}} \subseteq U$. If $\chi \in (V, \mathbb{T}_{n_0})$, then $\chi(V) = \{0\}$. Indeed, $o(\chi) = m$ and $\chi(V) \subseteq \mathbb{Z}(m) \cap \mathbb{T}_{n_0}$. Now observe that $\mathbb{Z}(m) \cap \mathbb{T}_{n_0} = \{0\}$ by our choice of n_0. Thus, $(V, \mathbb{T}_{n_0})$ is the subgroup $V^\perp \subseteq G^\wedge$, which clearly is $\sigma(G^\wedge, G)$-compact. Since any non-finite compact group must have cardinality at least $\mathfrak{c}$ and $G^\wedge$ is countable, we obtain that $(V, \mathbb{T}_{n_0})$ is finite.
We show now that $U^\triangleright$ is contained in $(V, \mathbb{T}_{n_0})$. Take $x \in V$, and observe that

$$hx \in U \text{ for every } h \in \{1, 2, \ldots, n_0\}. \tag{8.3}$$

Let $\chi \in U^{\triangleright}$. By (8.3) we get that $\chi(hx) = h\chi(x) \in \mathbb{T}_+$ for every $h \in \{1, \dots, n_0\}$, in particular $\chi(x) \in \mathbb{T}_{n_0}$ according to Lemma 5.22. Therefore $\chi \in (V, \mathbb{T}_{n_0})$. So we have proved that $U^{\triangleright} \subseteq (V, \mathbb{T}_{n_0})$, in particular $U^{\triangleright}$ is finite. QED

Here we show some consequences.

Proposition 8.57 *Let (G, τ) be a locally quasi-convex BTM-group. Then $\tau = \sigma(G, G^\wedge)$ and G is precompact.*

Proof. Let U be a quasi-convex τ-neighborhood of 0_G. By Proposition 8.56, $U^{\triangleright}$ is finite, therefore $(U^{\triangleright})^{\triangleleft} = U$ is a neighborhood of 0 in $(G, \sigma(G, G^\wedge))$. QED

This permits to characterize those BTM groups that are locally quasi-convex:

Theorem 8.58 *For a topological abelian group G, the following properties are equivalent:*

(a) G is a locally quasi-convex BTM group;

(b) G is bounded, precompact and $w(G) < c$.

Proof. Fix a topological group G, and let K denote the compact completion of G.
$(a) \implies (b)$ By Proposition 8.57, G is precompact. Then, $G^\wedge = K^\wedge$ as abstract abelian groups. As $G^\wedge$ is bounded by hypothesis, also $K^\wedge$ is bounded, and K is bounded as well; so, G is obviously bounded. Finally, $|G^\wedge| = |K^\wedge| = w(K)$, therefore the hypothesis $|G^\wedge| < \mathfrak{c}$ yields $w(G) = w(K) < \mathfrak{c}$.
$(b) \implies (a)$ Assume that G is bounded. Then also K and, hence, $K^\wedge$ are bounded as well. As $G^\wedge = K^\wedge$, we conclude that $G^\wedge$ is bounded. Finally, $|G^\wedge| = |K^\wedge| = w(K) = w(G) < \mathfrak{c}$. So G is a BTM group. QED

Another consequence of Proposition 8.57 is the following one:

Corollary 8.59 *Let G be a topological BTM-group. Then $\sigma(G, G^\wedge)$ is the unique locally quasi-convex compatible topology on G. In particular, it is the Mackey topology of G.*

Proof. Let ν be a locally quasi-convex compatible topology in G and set $G_\nu = (G, \nu)$. Then $G^\wedge = \mathcal{C}\mathrm{Hom}(G_\nu, \mathbb{T})$, therefore G_ν is a BTM-group. So Proposition 8.57 applies to G_ν and we are done.
The last assertion is Remark 8.53. QED

Proposition 8.57 and Corollary 8.59 immediately yield:

Corollary 8.60 *If G is BTM and locally quasi-convex, then G is ULQC. In particular, G is precompact and Mackey.*

The following result shows the existence of a class of ULQC groups in which $\tau_{qc}(G, G^\wedge) \neq \sigma(G, G^\wedge)$ (in fact, $\tau_{qc}(G, G^\wedge)$ is discrete and non-compatible). In particular, this shows that a Mackey group is not necessarily Arens and, hence, strongly Mackey. It is based on (and generalizes) Example 4.2 in [16].

Theorem 8.61 *Let $G = \bigoplus_{k=1}^{\infty} \mathbb{Z}_{n_k}$, where $\{n_k\}_{k=1}^{\infty}$ is a bounded sequence of odd integers with $n_k > 3$. If G is endowed with the topology induced by the product topology in $\prod_{k=1}^{\infty} \mathbb{Z}_{n_k}$, then the following assertions hold:*

1) G is a precompact Mackey group;

2) τ_{qc} is discrete and non-compatible; in particular, G is not an Arens group.

Proof. *1)* Observe that G is a precompact BTM group by Example 8.55. Moreover, G is locally quasi-convex, so Corollary 8.60 applies.
Now let us prove *2)*. Observe that $(G, \sigma(G, G^\wedge)) = (G^\wedge, \sigma(G^\wedge, G))$. Moreover, the set K defined as $K := \{x \in G \,:\, |\mathrm{supp}(x)| \leq 1\}$ is compact in G. Let us consider the following:

Claim. Let $C_k \subseteq \mathbb{Z}_{n_k}$, then $(K \cap \bigoplus_{k=1}^{\infty} C_k)^{\triangleleft} = \bigoplus_{k=1}^{\infty} C_k^{\triangleleft}$.

Define $M_1 := K \cap \bigoplus_{k=1}^{\infty} (\mathbb{T}_+ \cap \mathbb{Z}_{n_k})$ and $M_2 := K \cap \bigoplus_{k=1}^{\infty} V_k$, where, for any k, $V_k = \{0, \pm v_k\}$ and $\pm v_k$ are the elements of $\mathbb{Z}_{n_k} \subseteq \mathbb{T}$ with maximum norm (i.e., $v_k = \frac{n_k - 1}{2n_k}$ for every k) and put $F_1 := Q_G(M_1)$ and $F_2 := Q_G(M_2)$.
It is evident that F_1 and F_2 are quasi-convex. Let us calculate the polars: $F_1^{\triangleleft} = M_1^{\triangleleft}$, and this is equal — using the claim — to $\bigoplus (\mathbb{T}_+ \cap \mathbb{Z}_{n_k})^{\triangleleft} = \bigoplus \{0, \pm\frac{1}{n_k}\}$; again by the claim, $F_2^{\triangleleft} = \bigoplus V_k^{\triangleleft}$. We conclude noting that $\pm\frac{1}{n_k} \notin V_k^{\triangleleft}$, therefore $F_1^{\triangleleft} \cap F_2^{\triangleleft} = \{0\}$.
Now let us show that F_1 and F_2 are compact in $(G^\wedge, \sigma(G^\wedge, G))$. To this aim, observe that $G^\wedge$ coincides algebraically with $H^\wedge$, where $H := \prod_{k=1}^{\infty} \mathbb{Z}_{n_k}$, and $\sigma(G^\wedge, G) = \sigma(G^\wedge, H) = \sigma(H^\wedge, H)$. Moreover, $(G, \sigma(G, G^\wedge))$ is dually embedded in $(H, \sigma(H, H^\wedge))$, so

$$F_1 = Q_{(G, \sigma(G, G^\wedge))}(M_1) = Q_{(H, \sigma(H, H^\wedge))}(M_1)$$

and

$$F_2 = Q_{(G, \sigma(G, G^\wedge))}(M_2) = Q_{(H, \sigma(H, H^\wedge))}(M_2)$$

by Lemma 5.17. Now, M_1 is compact since K is compact and $\bigoplus_{k=1}^{\infty} (\mathbb{T}_+ \cap \mathbb{Z}_{n_k})$ is closed in G; the same argument proves that M_2 is compact as well. The group $(H, \sigma(H, H^\wedge))$ has the *quasi-convex property* (briefly, *qcp*), i.e.,

the quasi-convex hull of any compact subset is still compact (this follows from Proposition 6.3.13 in [19]), therefore F_1, F_2 are $\sigma(H, H^\wedge)$-compact in H and, hence, they are compact in $(G, \sigma(G, G^\wedge))$ as well.
So we have found a pair F_1, F_2 of $\sigma(G^\wedge, G)$-compact and quasi-convex subsets of $G^\wedge$ such that $F_1^\triangleleft \cap F_2^\triangleleft = \{0\}$, and this implies that $\tau_{qc}(G, G^\wedge)$ is discrete.
To deduce that G is not Arens, observe that

$$(G,\tau)^\wedge = \bigoplus_{k=1}^{\infty} \mathbb{Z}_{n_k} \lneqq \prod_{k=1}^{\infty} \mathbb{Z}_{n_k} = (G, \tau_{qc}(G, G^\wedge))^\wedge,$$

therefore $\tau_{qc}(G, G^\wedge)$ is non-compatible with τ and $\tau \lneqq \tau_{qc}(G, G^\wedge)$. QED

A slight modification of Theorem 8.61 gives an example of a class of ULQC groups that are strongly Mackey but not g-barrelled.

Theorem 8.62 *Let $L \cong \mathbb{Z}_2$ or $L \cong \mathbb{Z}_3$. Put $G := \bigoplus_\omega L$ equipped with the topology induced from the product $\prod_\omega L$. Then:*

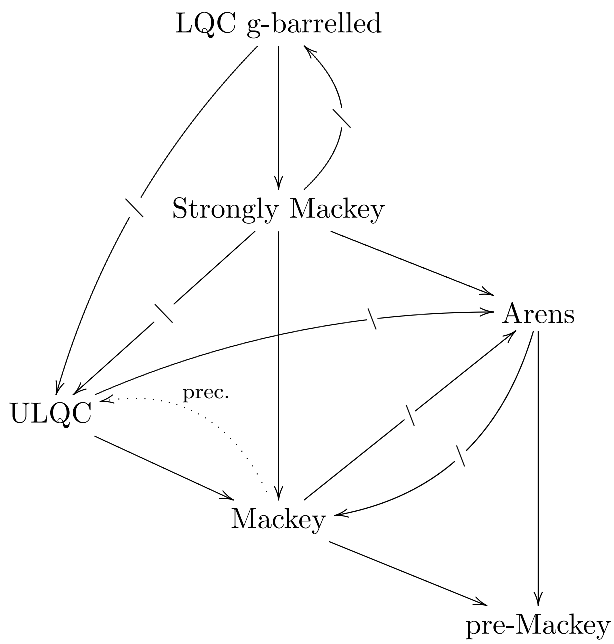
- *G is strongly Mackey;*
- *G is not g-barrelled.*

Proof. The fact that G is not g-barrelled is a consequence of Corollary 8.36. Now let us show that G is strongly Mackey.
Observe first that $G^\wedge$ has the same support as G, and the topology $\sigma(G^\wedge, G)$ on $G^\wedge$ coincides with the original topology of G. Now, let K be a $\sigma(G^\wedge, G)$-compact quasi-convex subset of $G^\wedge$. Then K is a subgroup of $G^\wedge$, according to the following claim:

Claim. Let G be an abelian group of exponent 2 (or 3). Then every quasi-convex subset of G is a subgroup.
Indeed, observe that for every $\chi \in \mathrm{Hom}(G, \mathbb{T})$ we have that $\chi(G) \leq \mathbb{Z}_2 \leq \mathbb{T}$ (respectively, $\chi(G) \leq \mathbb{Z}_3 \leq \mathbb{T}$), so for every $K \subseteq G$ it holds that $\chi(K) \subseteq \mathbb{T}_+$ iff $\chi(K) = \{0\}$.

Since $G^\wedge$ is countable and K is a $\sigma(G^\wedge, G)$-compact subgroup of $G^\wedge$, K is finite. This yields $\tau_{qc}(G, G^\wedge) \leq \sigma(G, G^\wedge)$ (use Example 8.16). Now apply Proposition 8.24 to deduce that $\tau_{qc}(G, G^\wedge) = \sigma(G, G^\wedge)$. To finish the proof, recall that G is precompact by Example 8.55 and Proposition 8.57, therefore its original topology coincides with $\sigma(G^\wedge, G) = \tau_{qc}(G, G^\wedge)$. QED

Now we are in position to give a complete diagram of implications, including those forbidden implications that we have obtained in Theorem 8.61 and Theorem 8.62.



8.4 A categorical approach to Mackey topologies

Following [13], we consider a general categorical approach to Mackey topologies. In particular, we prove a new and more complete version of the main result of [13] (see Theorem 8.75). This is part of the bigger project [8].

For every full subcategory $\mathcal{X}$ of the category $\mathcal{MAP}$ of all the maximally almost periodic groups, we will define the $\mathcal{X}$-Mackey topology of $G \in \mathcal{X}$ as the finest $\mathcal{X}$-topology on G among those $\mathcal{X}$-topologies that have the same dual group as G (see Definition 8.64). Then, the "classical" Mackey topology of a topological group G is — roughly speaking — the $\mathcal{X}$-Mackey topology when $\mathcal{X}$ is the category $\mathcal{LQC}$ of all the locally quasi-convex groups.

The following full categories of $\mathcal{MAP}$ will also be relevant in the sequel: $\mathcal{LPK}$ = {locally precompact groups}, $\mathcal{PK}$ = {precompact groups}, $\mathcal{LK}$ = {locally compact groups} and $\mathcal{K}$ = {compact groups}.

For every full subcategory $\mathcal{X}$ of $\mathcal{MAP}$ such that $\mathbb{T} \in \mathcal{X}$, we give the following definition (which is somewhat more precise than Definition 8.4):

Definition 8.63 *Let $G \in \mathcal{X}$, and let ν be a topology on G such that $(G, \nu) \in \mathcal{X}$. Then ν is said to be* compatible *with $(G, G^\wedge)$ if $(G, \nu)^\wedge = G^\wedge$.*

Definition 8.64 *A group $G \in \mathcal{X}$ is $\mathcal{X}$-pre-Mackey if there exists a $\mathcal{X}$-topology τ such that:*

(M1) τ is compatible with $(G, G^\wedge)$;

(M2) $\tau \geq \gamma$ *for every* $\mathcal{X}$*-topology* γ *which is compatible with* $(G, G^{\wedge})$*.*

A $\mathcal{X}$*-topology* τ *that verifies (M1) and (M2) will be called the* $\mathcal{X}$*-Mackey topology of* G*. Moreover,* G *is said to be* $\mathcal{X}$*-Mackey if its original topology coincides with the* $\mathcal{X}$*-Mackey topology.*

We will write $\tau_{\mathcal{X}}(G, G^{\wedge})$ to denote the $\mathcal{X}$-Mackey topology of $(G, G^{\wedge})$. We also will simply write $\tau_{G^{\wedge}}$ if no confusion is possible. Clearly, for every $\mathcal{X}$-pre-Mackey (G, τ), $\tau \leq \tau_{\mathcal{X}}(G, G^{\wedge})$ (i.e., $\mathcal{X}$-Mackey topology is always finer then the original one).

Example 8.65 *Here are some examples and non-examples.*

(a) As shown in [23], given $G \in \mathcal{MAP}$ *the least upper bound of the family of all compatible MAP group topologies on* G *need not exist.*

(b) It is well known that two locally compact group topologies on the same underlying abelian group G *have the same continuous characters if and only if they coincide (this is a consequence of Glicksberg Theorem [49]). The same property holds in the class of precompact topologies ([27]). This implies that if* $\mathcal{X} = \mathcal{LK}$ *or* $\mathcal{X} = \mathcal{PK}$*, then every* $G \in \mathcal{X}$ *is* $\mathcal{X}$*-Mackey since they have only one compatible topology in* $\mathcal{X}$*.*

(c) Consider the group $G = (\mathbb{R}, \sigma(\mathbb{R}, \mathbb{R}))$*. Then* G *is* $\mathcal{PK}$*-Mackey by (b). Moreover,* G *is also* $\mathcal{LQC}$*-pre-Mackey; indeed,* $\mathbb{R}$ *equipped with the usual topology* τ_u *is* $\mathcal{LQC}$*-Mackey (by Theorem 8.37).*

Analogously to what we did in § 8.1.1, we introduce another notion in order to describe the $\mathcal{X}$-Mackey topology.
Given $G \in \mathcal{X}$, let $\{\tau_i \mid i \in I\}$ be the family of all the $\mathcal{X}$-topologies on G that are compatible with $(G, G^{\wedge})$. Put $\tau_g^{\mathcal{X}}(G, G^{\wedge}) := \sup\{\tau_i \mid i \in I\}$. If no confusion is possible, we will omit the upper index and we will simply write $\tau_g(G, G^{\wedge})$.

Remark 8.66 (1) Observe that, in general, $(G, \tau_g^{\mathcal{X}}(G, G^{\wedge})) \notin \mathcal{X}$ (for example, if $\mathcal{X} = \mathcal{LPK}$).

(2) The topology $\tau_g^{\mathcal{X}}(G, G^{\wedge})$ coincides with the topology that the product group $\prod_{i \in I}(G, \tau_i)$ induces on G via the diagonal map $x \mapsto (x, x, \ldots)$, where $\{\tau_i \mid i \in I\}$ is the family of all compatible $\mathcal{X}$-topologies on G. Consequently, if $\mathcal{X}$ is closed under products and subgroups (e.g., $\mathcal{X} = \mathcal{LQC}$), then $(G, \tau_g^{\mathcal{X}}(G, G^{\wedge})) \in \mathcal{X}$ for every $(G, \tau) \in \mathcal{X}$.

(3) The following property of monotonicity holds: if $\mathcal{X}$ is a subcategory of $\mathcal{Y}$, then $\tau_g^{\mathcal{X}}(G, G^{\wedge}) \leq \tau_g^{\mathcal{Y}}(G, G^{\wedge})$.

The topology $\tau_g^{\mathcal{X}}(G, G^\wedge)$ is used to characterize the existence of the $\mathcal{X}$-Mackey topology whenever $\mathcal{X}$ is closed under products and subgroups, as the following proposition states.

Proposition 8.67 *Suppose that $\mathcal{X}$ is closed under products and subgroups, and let $G \in \mathcal{X}$. Then G is $\mathcal{X}$-pre-Mackey if and only if $\tau_g^{\mathcal{X}}(G, G^\wedge)$ is compatible.*

Proof. First observe that $\tau_g(G, G^\wedge)$ is a $\mathcal{X}$-topology by Remark 8.66 (2). So, clearly, if $\tau_g(G, G^\wedge)$ is compatible, then it coincides with the $\mathcal{X}$-Mackey topology of G (since every topology γ which is compatible with $(G, G^\wedge)$ is coarser then $\tau_g(G, G^\wedge)$ by definition). Conversely, suppose that G is $\mathcal{X}$-pre-Mackey, i.e. the topology $\tau_{\mathcal{X}}(G, G^\wedge)$ exists. Then $\tau_{\mathcal{X}}(G, G^\wedge)$ necessarily coincides with $\tau_g(G, G^\wedge)$. Indeed, since $\tau_{\mathcal{X}}(G, G^\wedge)$ is finer then every $\mathcal{X}$-topology on G which is compatible with $G^\wedge$, then $\tau_{\mathcal{X}}(G, G^\wedge)$ is also finer then the supremum of all of them, i.e., $\tau_g(G, G^\wedge) \leq \tau_{\mathcal{X}}(G, G^\wedge)$. On the other hand, $\tau_{\mathcal{X}}(G, G^\wedge)$ is compatible with $G^\wedge$ by definition, so $\tau_{\mathcal{X}}(G, G^\wedge)$ is coarser than the supremum of all compatible $\mathcal{X}$-topologies, i.e., $\tau_{\mathcal{X}}(G, G^\wedge) \leq \tau_g(G, G^\wedge)$. Therefore, $\tau_{\mathcal{X}}(G, G^\wedge) = \tau_g(G, G^\wedge)$. In particular, $\tau_g(G, G^\wedge)$ is compatible with $(G, G^\wedge)$. QED

As already observed in Remark 8.66 (1), if the hypothesis on $\mathcal{X}$ in the previous proposition does not hold, then $\tau_g^{\mathcal{X}}(G, G^\wedge)$ can fail to be a $\mathcal{X}$-topology. But $\tau_g^{\mathcal{X}}(G, G^\wedge)$ still can be compatible with $G^\wedge$. For example, $\tau_g^{\mathcal{LPK}}(G, G^\wedge)$ is compatible with $(G, G^\wedge)$ for every $G \in \mathcal{LPK}$.

The following property of *coreflectivity* of pre-Mackey groups in a category $\mathcal{X}$ can be considered:

Definition 8.68 *Let (G, τ) and (H, γ) be two $\mathcal{X}$-pre-Mackey groups in a category $\mathcal{X}$. We say that* the pair G, H has the CR-property *if whenever $f : (G, \tau) \to (H, \gamma)$ is a continuous homomorphism, then the corresponding homomorphism $\mu f : (G, \tau_{G^\wedge}) \to (H, \tau_{H^\wedge})$ (algebraically coinciding with f) in the following diagram*

$$\begin{array}{ccc} (G, \tau) & \xrightarrow{\ f\ } & (H, \gamma) \\ {\scriptstyle id_G}\uparrow & & \uparrow{\scriptstyle id_H} \\ (G, \tau_{G^\wedge}) & \xrightarrow{\ \mu f\ } & (H, \tau_{H^\wedge}) \end{array} \tag{8.4}$$

is continuous.

Definition 8.69 *Let $\mathcal{X}$ be a full subcategory of the category $\mathcal{MAP}$. We say that $\mathcal{X}$ has a* Mackey subcategory *if:*

(MS 1) every group in $\mathcal{X}$ is a $\mathcal{X}$-pre-Mackey group;

(MS 2) every pair (G,τ) *and* (H,γ) *in* $\mathcal{X}$ *has the CR-property.*

The above definition says that the full subcategory of $\mathcal{X}$ having for object all $\mathcal{X}$-Mackey groups is a coreflective subcategory of $\mathcal{X}$.

An elegant characterization of the categories admitting a Mackey subcategory is given in [13, Theorem 4.1].

Theorem 8.70 ([13]) *Let* $\mathcal{X}$ *be a category that is closed under finite products. Let* $SP\mathcal{X}$ *denote the closure of* $\mathcal{X}$ *with respect to (infinite) products and subobjects. Then TFAE:*

(1) $\mathbb{T}$ *is injective with respect to inclusions in* $\mathcal{X}$*;*

(2) $\mathbb{T}$ *is injective with respect to inclusions in* $SP\mathcal{X}$*;*

(3) $\mathcal{X}$ *admits a Mackey subcategory, i.e.,* $\mu : \mathcal{X} \to \mathcal{X}_{\mathcal{M}}$ *is a coreflection.*

The authors give in [13, 4.3 Examples] also an important example, namely the category $\mathcal{NUC}$ of nuclear groups admits a Mackey subcategory, i.e., the $\mathcal{NUC}$-topology is a strong topology in $\mathcal{NUC}$.

Remark 8.71 It is easy to see that the injectivity of the torus is inherited by subobjects (this holds because, if $K \leq H$ is dually embedded in H and $H \leq G$ is dually embedded in G, then K is dually embedded in G), so in principle it is not necessary to consider the closure of $\mathcal{X}$ with respects to subobjects. By the equivalence between *(1)* and *(2)* in the previous result, also the fact that $\mathcal{X}$ is not closed under taking infinite products is irrelevant in the present framework. On the other hand, this hypothesis on $\mathcal{X}$ is necessary to assure that every $G \in \mathcal{X}$ is $\mathcal{X}$-pre-Mackey (as already commented before).

So the stronger categorical form of the problem of existence of a Mackey topology for a topological group $(G,\tau) \in \mathcal{X}$ (such that $\mu : \mathcal{X} \to \mathcal{X}_M$ is a coreflection) is completely translated in global (categorical) terms, i.e. the "Mackey problem" is equivalent to characterize those categories in which $\mathbb{T}$ is an injective object.

8.4.1 New (more precise) version of Theorem 8.70

We offer a more complete version of Theorem 8.70 with the aim of clarifying which is the role of the hypothesis "injectivity of $\mathbb{T}$" on the category $\mathcal{X}$ in order to assure the existence of a Mackey subcategory. Indeed, we show that the injectivity of $\mathbb{T} \in \mathcal{X}$ is not necessary for the simpler condition "every group in $\mathcal{X}$ is $\mathcal{X}$-pre-Mackey". Our motivating question is the following one:

Question 8.72 *It is well-known that $\mathbb{T}$ is not injective in $\mathcal{LQC}$ (for an example of a pair $H \leq G \in \mathcal{LQC}$ such that H is not dually embedded in G, see [5, Remark 5.27]), so $\mathcal{LQC}$ does not admit a Mackey subcategory. But, is it true (at least) that every $G \in \mathcal{LQC}$ is $\mathcal{LQC}$-pre-Mackey?*

Remark 8.73 Since $\mathcal{LQC}$ does not admit a Mackey subcategory, we deduce that the Mackey strong topology does not exist in $\mathcal{LQC}$, even if the previous question admits a positive answer. By means of Example 1.10, this represents a deep difference between $\mathcal{LQC}$ and the category of all locally convex vector space.

We will prove that a weaker level of injectivity is sufficient to assure that every $G \in \mathcal{X}$ is $\mathcal{X}$-pre-Mackey, provided $\mathcal{X}$ is closed under arbitrary products and subobjects (see Theorem 8.75).

For a category $\mathcal{X}$, we say that the property (SUP) holds in $\mathcal{X}$ if

(SUP) for every $G \in \mathcal{X}$, $\tau_g^{\mathcal{X}}(G, G^\wedge)$ is compatible.

In particular, Proposition 8.67 can be reformulated as follows: *If $\mathcal{X}$ is closed under products and subgroups, then every group in $\mathcal{X}$ is $\mathcal{X}$-pre-Mackey if and only if (SUP) holds.*

We consider also another instance of injectivity, which is motivated by the following consideration. Take a homomorphism $f : G \to H$ and the monomorphism $j : G \to H \times G$ defined by $j(g) = (f(g), g)$ for every $g \in G$ (i.e., j is the "graph" of f). Then $j(G)$ is a very special subgroup of $H \times G$: it is a direct summand of $H \times G$, being $H \times \{0_G\}$ its complement. Now we consider the same situation for topological groups (G, τ), (H, γ) and a continuous homomorphism $f : (G, \tau) \to (H, \gamma)$.
Let $(G, \tau), (H, \gamma)$ be two objects in $\mathcal{X}$. Suppose that (H, γ) is $\mathcal{X}$-pre-Mackey, and let $\tau_{H^\wedge}$ denote its $\mathcal{X}$-Mackey topology. Let $f : (G, \tau) \to (H, \gamma)$ be a continuous homomorphism, and let $j : G \to (H, \tau_{H^\wedge}) \times (G, \tau)$ be its graph. Then we say that the property (GRAPH) holds if

(GRAPH) if $j(G)$ is dually embedded in $(H, \tau_{H^\wedge}) \times (G, \tau)$.

Observe that if $\mathbb{T}$ is an injective object in $\mathcal{X}$, then (SUP) and (GRAPH) hold. While the second property is clear, the first one requires some explication.

Proposition 8.74 *If $\mathbb{T}$ is injective in $\mathcal{X}$, then (SUP) holds.*

Proof. Let $(G, \tau) \in \mathcal{X}$, and consider the family $\{\tau_i \mid i \in I\}$ of compatible $\mathcal{X}$-topologies on G. According to Remark 8.66 (2), $\tau_g^{\mathcal{X}}(G, G^\wedge)$ coincides precisely with the topology μ that the product group $\prod_i (G, \tau_i)$ induces on G via the diagonal map $x \mapsto (x, x, \ldots)$. In particular, $\mu \geq \tau$. We want to show that μ is compatible with $(G, \tau)^\wedge$. The inclusion $(G, \tau)^\wedge \subseteq (G, \mu)^\wedge$ is clear since μ is finer than τ, so let us check the other one. Fix $\chi \in (G, \mu)^\wedge$. By our hypothesis, there exists an extension $\widetilde{\chi} \in (\prod_i (G, \tau_i))^\wedge$ of χ. Now, $\widetilde{\chi}$ is of the form $\widetilde{\chi} = \prod_{i \in F} \chi_i \circ p_i$ for some finite set F, where $\chi_i \in (G, \tau_i)^\wedge = (G, \tau)^\wedge$ and $p_i : \prod_{j \in I} (G, \tau_j) \to (G, \tau_i)$ is the canonical projection. Now, $\widetilde{\chi} \restriction_G = \chi$, so χ is a finite product of elements of $(G, \tau)^\wedge$ and, therefore, $\chi \in (G, \tau)^\wedge$. QED

Theorem 8.75 *Let $\mathcal{X}$ be a category which is closed under finite products. Consider the following list of items:*

(1) (SUP) holds in $\mathcal{X}$;

(2) every object $(G, \tau) \in \mathcal{X}$ is $\mathcal{X}$-pre-Mackey;

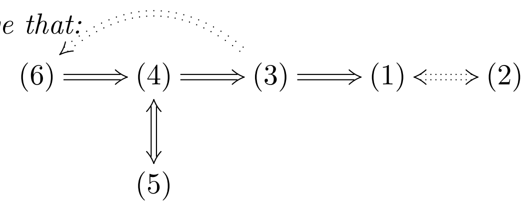
(3) (SUP) and (GRAPH) hold;

(4) $\mathbb{T}$ is injective with respect to inclusions in $\mathcal{X}$;

(5) $\mathbb{T}$ is injective with respect to inclusions in $SP\mathcal{X}$;

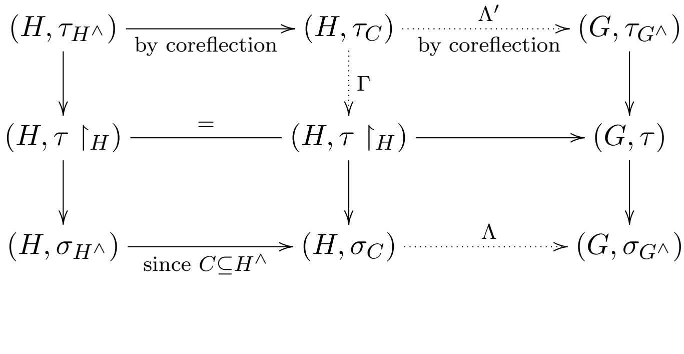
(6) $\mathcal{X}$ admits a Mackey subcategory.

Then we have that:



where the dotted arrows hold with the additional hypothesis that $\mathcal{X}$ is closed under arbitrary products and subobjects.

Proof. (6) $\implies$ (4): Let (G, τ) be an object in $\mathcal{X}$, and let $(H, \tau \restriction_H)$ be a subobject (G, τ) (i.e., $(H, \tau \restriction_H)$ is a topological subgroup of (G, τ)). Put $G^\wedge := (G, \tau)^\wedge$ and $H^\wedge := (H, \tau \restriction_H)^\wedge$. Consider the inclusion map $\iota : H \to G$ and the dual map $\iota^\wedge : H^\wedge \to G^\wedge$. Put $C := \iota^\wedge(G^\wedge) \subseteq H^\wedge$. We want to show that $C = H^\wedge$ (clearly, it suffices to show that $H^\wedge \subseteq C$).
In order to clarify the situation, we consider the following diagram:



In the diagram above, like in the other ones that follow, "every arrow is continuous" (solid arrows are used for maps that are continuous for some more or less obvious reason, and dotted arrows are used to denote those arrows that appear on some second stage).
We are interested in proving the existence of arrow Γ in the diagram above. To this aim, we first consider Λ (and Λ', that will automatically follow from Λ by our hypothesis).

Λ: just observe that for every $\chi \in G^\wedge$, we have that $\chi \circ \iota \in C$:

$$(H, \sigma_C) \xrightarrow{\ \iota\ } (G, \sigma_{G^\wedge}) \xrightarrow{\ \chi\ } \mathbb{T}, \qquad (H, \sigma_C) \xrightarrow{\ \chi\circ\iota\ } \mathbb{T}$$

Γ: we have that

$$\begin{array}{ccc} (H, \tau_C) & \xrightarrow{\ \Lambda'\ } & (G, \tau_{G^\wedge}) \\ & & \downarrow \\ & & (G, \tau) \end{array}$$

and $(H, \tau_C) \to (H, \tau \restriction_H)$ follows recalling that $(H, \tau \restriction_H)$ is a subobject of (G, τ).

So we get that (with abuse of notation)

$$(H, \tau_C) \xrightarrow{\ \Gamma\ } (H, \tau \restriction_H) \xrightarrow{\ \chi\ } \mathbb{T}, \qquad (H, \tau_C) \xrightarrow{\ \chi\ } \mathbb{T}$$

for every $\chi \in H^\wedge$, thus $H^\wedge \subseteq (H, \tau_C)^\wedge = C$ and we are done.

(4) $\iff$ (5): This is *(1)* $\iff$ *(2)* of Theorem 8.70.

(4) $\implies$ (3): This has already been discussed (see also Proposition 8.74).

(3) $\implies$ (1): Trivial.

From now on, $\mathcal{X}$ will be a category which is closed under arbitrary products and subobjects.

(1) $\iff$ (2): This is Proposition 8.67.

(3) $\implies$ (6): The existence of the $\mathcal{X}$-Mackey topology is guaranteed by (1) $\implies$ (2).

[CR-property] Fix a pair of objects (G,τ) and (H,γ) in $\mathcal{X}$, and put $G^\wedge := (G,\tau)^\wedge$, $H^\wedge := (H,\gamma)^\wedge$. Consider a continuous homomorphism $f:(G,\tau)\to(H,\gamma)$. We need to show that $\mu f:(G,\tau_{G^\wedge})\to(H,\tau_{H^\wedge})$ is continuous (this is the situation described in (8.4)).
Denote by ρ the topology that $(H,\tau_{H^\wedge})\times(G,\tau)\in\mathcal{X}$ induces on G via the injective map

$$G \longrightarrow (H,\tau_{H^\wedge})\times(G,\tau)$$

defined by the formula $g\mapsto(f(g),g)$. Obviously,

$$\rho\geq\tau. \tag{8.5}$$

We claim that $(G,\rho)^\wedge = G^\wedge$. Indeed, to see that $(G,\rho)^\wedge\subseteq G^\wedge$, fix a character $\chi:(G,\rho)\to\mathbb{T}$. Then, by (3), there exists a continuous extension $\widetilde{\chi}:(H,\tau_{H^\wedge})\times(G,\tau)\to\mathbb{T}$. Now, $\widetilde{\chi}\restriction_G=\chi(x)$ is of the form $\chi(x)=\widetilde{\chi}_1(f(x))\cdot\widetilde{\chi}_2(x)$ for every $x\in G$, with $\widetilde{\chi}_1\in(H,\tau_{H^\wedge})^\wedge$ and $\widetilde{\chi}_2\in G^\wedge$. So it suffices to show that $\widetilde{\chi}_1\circ f\in G^\wedge$. Consider $f^\wedge:(H,\gamma)^\wedge\to G^\wedge$. Then we have that $\widetilde{\chi}_1\circ f=f^\wedge(\widetilde{\chi}_1)\in f^\wedge((H,\tau_{H^\wedge})^\wedge)=f^\wedge((H,\gamma)^\wedge)\subseteq G^\wedge$. The other inclusion follows from (8.5).
From the compatibility of (G,ρ) we deduce that $\rho\leq\tau_{G^\wedge}$. Now just observe that f can be obtained as the composition of the following continuous functions:

$$\begin{array}{ccccccc}(G,\rho) & \longrightarrow & (H,\tau_{H^\wedge}) & \times & (G,\tau) & \longrightarrow & (H,\tau_{H^\wedge})\\ x & \longmapsto & (f(x) & , & x) & \longmapsto & f(x)\end{array}$$

QED

8.5 Remarks and open problems

We have obtained some other results related to the Mackey topology which are briefly collected in this section. Our aim is mainly illustrative, so many results are leaved without proof.
We start with permanence properties with respect to subgroups, quotients and products.

8.5.1 Permanence properties

Theorem 8.76 *Let (G,τ) be a topological group which contains an open subgroup H. Then:*

- *(G,τ) is a Mackey group if and only if $(H,\tau\restriction_H)$ is a Mackey group;*
- *(G,τ) is g-barrelled if and only if $(H,\tau\restriction_H)$ is g-barrelled.*

In this theorem the assumption that H is an open subgroup of G cannot be weakened to "dually closed and dually embedded" subgroup. In fact, take $G := \mathbb{R} \times \mathbb{T}$, where the first factor is equipped with the Bohr topology associated to the usual topology of $\mathbb{R}$, say $\sigma(\mathbb{R}_u, \mathbb{R}^\wedge)$, and the second factor with the usual topology. Clearly $\{0\} \times \mathbb{T}$ is a compact subgroup, therefore it is g-barrelled by Corollary 8.38 and, hence, it carries the Mackey topology. However, G is not a Mackey group: the product of the usual topology of $\mathbb{R}$ with the usual of $\mathbb{T}$ gives a locally quasi-convex compatible topology on G which is strictly finer than the original one, with the same dual group.

As a matter of fact, dense subgroups do not determine the property of "being a Mackey group":

Theorem 8.77 *Let G be the group ℓ_∞ equipped with its Mackey topology $\tau(\ell_\infty, \ell_1)$, and let $H \leq G$ be the dense subgroup c_0. Then the induced topology $\tau(\ell_\infty, \ell_1) \restriction_{c_0}$ on H is not Mackey. So, a dense subgroup of a Mackey group is not necessarily a Mackey group.*

The diversity between $\tau(\ell_\infty, \ell_1) \restriction_{c_0}$ and the norm topology τ_0 of c_0 can be explained through the fact that there exist subsets of $c_0^* = \ell_1$ that are $\sigma(\ell_1, c_0)$-compact but not $\sigma(\ell_1, \ell_\infty)$-compact. Take, for instance, the closed unit ball $B \subseteq c_0^*$ (see [58, 20.9 (5), 22.4 (3)]).

We now study the converse situation, that is, whether a dense Mackey subgroup implies that the whole group is Mackey.

Proposition 8.78 *Let (G, τ) be a locally quasi-convex group, and let H be a dense finite-index subgroup of G. Suppose that $\tau \restriction_H$ is the Mackey topology of H. Then τ is the Mackey topology of G.*

We wonder if it is possible to remove the hypothesis "finite-index subgroup" in Proposition 8.78:

Conjecture 8.79 *Let H be a dense subgroup of (G, τ). Suppose that τ_H is the Mackey topology of H. Then τ is the Mackey topology of G.*

As far as quotient groups is concerned, a positive result is the following one:

Theorem 8.80 *Let G be a Mackey group. If $H \leq G$ is a closed and dually embedded, then the quotient topology on G/H is the corresponding Mackey topology for G/H, provided it is locally quasi-convex.*

Question 8.81 *Is it possible to drop the assumption that H is dually embedded in the previous theorem?*

Given an arbitrary family $\{G_i \mid i \in I\}$ of topological groups, it can be proved that if the product $G := \prod_{i\in I} G_i$ is a Mackey group, then G_i is Mackey for every $i \in I$. On the other hand, the problem of determining whether the product of Mackey groups is again a Mackey group is still open, even in the case of a finite product. So we propose the following:

Problem 8.82 *Let $\{G_i \mid i \in I\}$ be a (finite) family of Mackey groups. Determine under which conditions the group $G := \prod_{i\in I} G_i$, equipped with the product topology, is a Mackey group.*

8.5.2 g-barrelled groups

As already observed in Corollary 8.36, every countable group G which is locally quasi-convex and g-barrelled is necessarily discrete. We ask if the hypothesis "locally quasi-convex" on G can be weakened to "MAP". In other words,

Question 8.83 *Does there exist a MAP g-barrelled group which is countable and non-discrete?*

In Theorem 8.44 we show the every ω-bounded group is g-barrelled. We do not know whether it can be extended to the wider class of countably compact groups, namely

Question 8.84 *Is it true that every countably compact abelian group is g-barrelled?*

or even

Question 8.85 *Is it true that every pseudocompact abelian group is g-barrelled?*

Since pseudocompact groups are Baire, the previous question admits a partial answer in the case of separable pseudocompact groups by means of Theorem 8.37. For this reason, and also for the fact that compact groups are g-barrelled (see Corollary 8.38), we propose the following more precise formulation:

Question 8.86 *Are non-separable non-compact pseudocompact groups g-barrelled?*

Moreover, since countably compact groups are hereditarily Baire (as every closed subgroup of a countably compact group is still countably compact, hence pseudocompact and consequently Baire), a positive answer to the following question solves also Question 8.84:

Question 8.87 *Must a hereditarily Baire group be g-barrelled?*

Clearly, this is equivalent to ask whether the hypothesis "metrizable" can be dropped in the first item of Theorem 8.37.

8.5.3 G-groups

J. Galindo dealt in [45] with the problem of determining when a totally bounded topology on a group G is the Bohr modification of a locally compact topology on G (this is Question 1 in the above cited paper). We propose the following definition:

Definition 8.88 *A precompact group* (H, τ) *is said to be a* G-group *if one of the following properties holds:*

- *either* H *is compact, or*
- *there exists no locally compact topology* τ_1 *on* H *such that* $\tau_1^+ = \tau$.

For example, let us show that every countable precompact abelian group is a G-group.

Lemma 8.89 *Every countable precompact abelian group is a G-group.*

Proof. Suppose that H is a countable precompact abelian group. If H is finite, then H is compact, so a G-group. Assume that H is infinite. Since no locally compact abelian group can be countably infinite, H is necessarily a G-group. QED

Proposition 8.90 *Let* H *be a ULQC group. Then* H *is a G-group.*

Proof. If H is compact then it is a G-group. So, let us suppose that it is not compact. Let τ_1 be a locally compact topology (in particular, locally quasi-convex) on H such that $\tau_1^+ = \tau$. Then $(H, \tau_1)^\wedge = (H, \tau_1^+)^\wedge = (H, \tau)^\wedge$. By our hypothesis, $\tau_1 = \tau$. In particular, τ is locally compact (hence, complete) and precompact, therefore it is compact and we get a contradiction. QED

From Remark 8.53, we immediately deduce that the class of precompact Mackey groups is contained in the class of all G-groups:

Corollary 8.91 *Every precompact Mackey group is a G-group.*

Observe also that a precompact Arens non-Mackey group need not to be a G-group. To see it, let us consider a locally compact non-compact abelian group H. Then H^+ is precompact Arens and not Mackey. Indeed, H is g-barrelled and locally quasi-convex, so it is strongly Mackey and, therefore the original topology τ of H coincides with $\tau_{qc}(H, H^\wedge) = \tau(H, H^\wedge)$. Now, since H^+ has the same dual group as H and H is Arens, H^+ is Arens as well. Since $\tau \neq \tau^+$, we obtain that H^+ is not Mackey. To conclude, just observe that the Bohr modification of a locally compact non-compact abelian group is not a G-group.

8.5.4 Miscellanea

In § 8.3 we discussed some example of topological groups that posses only one locally quasi-convex compatible topology. In the more general situation of groups with many locally quasi-convex compatible topologies, we are interested in knowing — roughly speaking — how many they are:

Question 8.92 *Given a topological group G, which is the cardinality of the family of all locally quasi-convex topologies on G that are compatible with $(G, G^\wedge)$?*

Moreover,

Problem 8.93 *It is not clear under which conditions on a MAP group G the locally quasi-convex compatible topologies will form a chain, or even a lattice.*

Question 8.92 and Problem 8.93 can be formulated in a more general setting, namely by dropping the assumption "locally quasi-convex".

We present now two problems that may be considered the fundamental questions of the Mackey topology for locally quasi-convex groups. As already mentioned in Question 8.72, $\mathcal{LQC}$ does not admit a Mackey subcategory. So we propose the following questions:

Question 8.94 *Is it true that every locally quasi-convex group is $\mathcal{LQC}$-pre-Mackey?*

Since $\mathcal{LQC}$ is closed under products and subgroups, the previous question is equivalent to ask wether (SUP) holds in $\mathcal{LQC}$ by Theorem 8.75.

It can be shown that the precompact (hence, locally quasi-convex) 2-adic topology τ_2 on $\mathbb{Z}$ is not an Arens topology. Indeed, the dual $\mathbb{Z}(2^\infty)$ of $G := (\mathbb{Z}, \tau_2)$ has an infinite compact quasi-convex set K (see § 7) that gives rise to a locally quasi-convex topology γ_K on $\mathbb{Z}$ which is coarser than $\tau_{qc}(G, G^\wedge)$ and not compatible, so $\tau_{qc}(G, G^\wedge)$ is not compatible either (see [7] for details). Thus $(\mathbb{Z}, \tau_2)$ is not Arens. This shows that the previous question admits a negative answer if we replace "$\mathcal{LQC}$-pre-Mackey" by "Arens".

If Question 8.94 admits a positive answer, then necessarily there exists a pair of $\mathcal{LQC}$-pre-Mackey groups that does not verify the CR-property. So we propose the following

Problem 8.95 *Give an example of a pair of $\mathcal{LQC}$-pre-Mackey groups that does not verify the CR-property.*

The following example offers a potential solution to both these problems.

Example 8.96 *Let $(G,\tau) \in \mathcal{LQC}$ be the group ℓ_2 equipped with the usual norm topology. Then G is locally quasi-convex and g-barrelled by Corollary 8.38, hence $\mathcal{LQC}$-Mackey. Let H be the discrete subgroup $\langle e_n \mid n \in \mathbb{N}\rangle \cong \mathbb{Z}^{(\mathbb{N})}$ (to see that H is discrete just observe that $H \cap B_1 = \{0\}$, where $B_1 = \{x \in G \,:\, ||x|| < 1\}$ is the unitary ball). Then H is not dually embedded in G by [5] (in particular, this shows that $\mathbb{T}$ is not injective in $\mathcal{LQC}$).*

Now, let $C \subsetneq H^\wedge$ the set of all the characters of H that admits a continuous extensions to G. Consider the group $(H, \sigma(H,C))$. Observe that $(H,\sigma(H,C))^\wedge = C$ (in particular, $\sigma(H,C)$ is not discrete).
Two possibilities can hold:

1. *$(H,\sigma(H,C))$ is not $\mathcal{LQC}$-pre-Mackey;*

2. *$(H,\sigma(H,C))$ is $\mathcal{LQC}$-pre-Mackey.*

If 1. holds, then we have an example of group which is not $\mathcal{LQC}$-pre-Mackey. If 2. holds, then let us see that the property of coreflectivity does not hold. Indeed, take the (continuous) inclusion $\iota : (H,\sigma(H,C)) \to (G,\sigma(G,G^\wedge)$, and suppose that $\mu\iota : (H,\mu_C) \to (G,\mu_{G^\wedge}) = (G,\tau)$ is continuous. Then $\iota^{-1}(B_1) = \{0\}$ is open in (H,μ_C), hence (H,μ_C) is discrete. But $(H,\mu_C)^\wedge = (H,\sigma(H,C))^\wedge = C \subsetneq H^\wedge = (H,\tau_{disc})^\wedge$, so we get a contradiction. Therefore $\mu\iota$ is not continuous and the CR-property does not hold.

So the solution of the following problem offers an important tip towards the solution of Question 8.94 and Problem 8.95.

Problem 8.97 *Solve the dichotomy 1. – 2. in the previous example.*

To conclude, consider the following question.

Question 8.98 *Is a locally quasi-convex metrizable group necessarily Mackey?*

This is motivated by the well-known fact that *every metrizable locally convex vector space is a Mackey space*. Observe that for complete metrizable group, this question admits a positive answer according to Theorem 8.37. A class of examples of metrizable non-complete Mackey groups is given in Theorem 8.61.

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