# ČECH COHOMOLOGY OF ATTRACTORS OF DISCRETE DYNAMICAL SYSTEMS

FRANCISCO R. RUIZ DEL PORTAL AND JAIME JORGE SÁNCHEZ-GABITES

Abstract. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a homeomorphism and $K$ an asymptotically stable attractor for $f$. The aim of this paper is to study when the inclusion of $K$ in its basin of attraction $\mathcal{A}(K)$ induces isomorphisms in Čech cohomology. We show that (i) this is true if coefficients are taken in $\mathbb{Q}$ or $\mathbb{Z}_p$ ($p$ prime) and (ii) it is true for integral cohomology if and only if the Čech cohomology of $K$ or $\mathcal{A}(K)$ is finitely generated. We compute the Čech cohomology of periodic point free attractors of volume-contracting $\mathbb{R}^3$-homeomorphisms and present applications to quite general models in population dynamics.

## 1. Introduction

Let $K$ be a compact attractor of a flow and $\mathcal{A}(K)$ its basin of attraction. There are many papers in the literature relating the homotopy properties of $\mathcal{A}(K)$ and $K$. Since $K$ may have a very complicated topological structure, the homotopy theory that best suits the study of this problem is shape theory, which can be thought of as a sort of Čech homotopy theory (see [2], [1], [3] or [15]). If the flow is defined in a nice space (a manifold or more generally an ANR) the main conclusions are that the inclusion $i$ of $K$ in $\mathcal{A}(K)$ is a shape equivalence and that $K$ has the shape of a finite polyhedron (see for instance [7], [4], [6] or [12]). In particular, $i$ induces isomorphisms in Čech cohomology. The proofs of these facts depend in an essential way on the homotopies that a flow provides for free.

In the case of discrete dynamical systems few results are known about the homotopical relationship between $K$ and $\mathcal{A}(K)$, and —due to the absence of the homotopies which a flow would naturally provide— they require strong conditions on the homeomorphism or on the attractor (see [5], [16], [19]). Such conditions are not useful in practice because *a priori* it is not known how strange the attractor can be, although in low dimensions the situation is slightly more tractable ([19]).

Given the situation just described it is profitable to be less ambitious and concentrate on the relation between the Čech cohomology of $K$ and $\mathcal{A}(K)$. One of the difficulties that arises is related to the fact that, unless the attractor has some kind of movability property (which is, again, difficult to check), information may be lost when the whole inverse sequence used to compute $\check{H}^*(K)$ is replaced by its inverse limit. In this paper we show that when coefficients are taken in $\mathbb{Q}$ or $\mathbb{Z}_p$ ($p$ prime) this issue disappears and the inclusion of $K$ in $\mathcal{A}(K)$ induces isomorphisms

*Date*: March, 2013.
1991 *Mathematics Subject Classification.* 54H20, 55N05, 55P55.
*Key words and phrases.* Čech cohomology, attractor, basin of attraction, homeomorphism.
The authors have been supported by MINECO, grant MTM2012-30719.

in Čech cohomology (Theorem 1); we also characterize when the same holds true for integral cohomology (Theorem 2):

**Theorem 1.** *Let $K \subseteq \mathbb{R}^n$ be an attractor for a homeomorphism $f$. Then the inclusion $i : K \longrightarrow \mathcal{A}(K)$ induces isomorphisms $i^* : H^d(\mathcal{A}(K);\mathbb{Q}) \longrightarrow \check{H}^d(K;\mathbb{Q})$. Moreover, both $\check{H}^d(K;\mathbb{Q})$ and $H^d(\mathcal{A}(K);\mathbb{Q})$ are finite dimensional vector spaces.*

*The same holds true when coefficients are taken in $\mathbb{Z}_p$ with $p$ prime.*

**Theorem 2.** *Let $K \subseteq \mathbb{R}^n$ be an attractor for a homeomorphism $f$. The following are equivalent:*

1. *the inclusion $i : K \subseteq \mathcal{A}(K)$ induces isomorphisms in Čech cohomology with $\mathbb{Z}$ coefficients,*
2. *$K$ has finitely generated Čech cohomology with $\mathbb{Z}$ coefficients,*
3. *$\mathcal{A}(K)$ has finitely generated cohomology with $\mathbb{Z}$ coefficients.*

As an application we consider volume contracting homeomorphisms of $\mathbb{R}^3$ and compute the cohomology of attractors having no fixed or periodic points (Theorem 17). The reader may find in [9] a complete exposition of the fixed point index and Lefschetz theory. We then use this to study attractors of some periodic equations in $\mathbb{R}^3$ (Theorem 18), in particular quite general 3-dimensional models of population dynamics.

**Acknowledgments**. The authors want to express their gratitude to professors S. Mardešić, A. Murillo, R. Ortega and A. Viruel for several comments and ideas which have helped to substantially improve this manuscript.

**Background definitions and notation.** Unless otherwise stated, $f$ will always denote a homeomorphism of $\mathbb{R}^n$. An *attractor* for $f$ is a compact set $K$ which the following properties:

1. $f(K) = K$ ($K$ is *invariant*),
2. $K$ has a neighbourhood $U$ such that for every compact set $P \subseteq U$ and every neighbourhood $V$ of $K$ there exists $k_0$ with the property that $f^k(P) \subseteq V$ for every $k \geq k_0$ ($K$ *attracts* compact subsets of $U$).

The maximal $U$ such that (2) holds is called the *basin of attraction* of $K$ and denoted $\mathcal{A}(K)$. It is always an invariant, open subset of $\mathbb{R}^n$. The usual way of proving that $f$ has an attractor is by finding a compact set $N \subseteq \mathbb{R}^n$ such that $f(N) \subseteq \text{int } N$, for then it can be shown that $f$ has an attractor $K \subseteq \text{int } N$ whose basin of attraction $\mathcal{A}(K)$ contains $N$. In fact

$$K = \bigcap_{k=0}^{\infty} f^k(N) \quad \text{and} \quad \mathcal{A}(K) = \bigcup_{k=0}^{\infty} f^{-k}(N).$$

We will make frequent use of the following fact: if $K$ is an attractor and $N$ is a compact neighbourhood of $K$ contained in $\mathcal{A}(K)$ then by (2) there exists a power $k_0$ such that $f^k(N) \subseteq \text{int } N$ for every $k \geq k_0$. In order to keep notation as simple as possible we shall usually rename $f^{k_0}$ again as $f$ and simply assume that $f(N) \subseteq \text{int } N$. This is legitimate because $f^{k_0}$ is a homeomorphism still having $K$ as an attractor with basin of attraction $\mathcal{A}(K)$ [19].

**Proposition 3.** *There exists a neighbourhood $N$ of $K$ contained in $\mathcal{A}(K)$ which has finitely generated integral cohomology in all dimensions. In fact, $N$ can be chosen to be a compact manifold with boundary.*

*Proof.* We give two different proofs for the sake of generalisation later on.

(1) Let $P$ be a compact neighbourhood of $K$ contained in $\mathcal{A}(K)$. For each $p = (p_1, \ldots, p_n) \in P$ let $Q_p$ be a cube $\prod_{i=1}^n [p_i - \varepsilon, p_i + \varepsilon]$ centered at $p$, with $\varepsilon > 0$ so small that $Q_p \subseteq \mathcal{A}(K)$. The interiors of these cubes cover the compact set $P$, and so a finite number of them also cover $P$. Let $N$ be their union. Then $N$ is a compact neighbourhood of $K$ contained in $\mathcal{A}(K)$ with finite dimensional cohomology in all dimensions. Replacing $N$ by a regular neighbourhood of itself (in the sense of piecewise linear topology) it can also be assumed to be a compact manifold with boundary.

(2) There exists a differentiable function $\theta : \mathbb{R}^n \longrightarrow [0,1]$ such that $\theta|_K \equiv 1$ and $\theta \equiv 0$ outside $\mathcal{A}(K)$. By Sard's theorem $\theta$ has a regular value $c \in (0,1)$. Letting $N := \theta^{-1}([0,c])$ we obtain a compact manifold that is a neighbourhood of $K$ contained in $\mathcal{A}(K)$. Since compact manifolds have finitely generated cohomology in all dimensions, the proof is finished. ☐

Summing up,

**Notation 4.** *$K$ is an attractor for a homeomorphism $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, with basin of attraction $\mathcal{A}(K)$. We denote $i : K \longrightarrow \mathcal{A}(K)$ the inclusion. $N$ is a compact neighbourhood of $K$ contained in $\mathcal{A}(K)$ and with finitely generated cohomology. We will always assume that $f(N) \subseteq N$.*

## 2. Proof of Theorem 1

Consider a set $Z$ and a map $u : Z \longrightarrow Z$. In this section $Z$ will be a topological space and $u$ a continuous map or $Z$ will be a vector space and $u$ a homomorphism. In the next section we will also consider the case when $Z$ is just an abelian group. Using these elements we construct a biinfinite sequence

$$\mathcal{S}: \quad \ldots \xleftarrow{u} Z \xleftarrow{u} Z \xleftarrow{u} Z_0 \xleftarrow{u} Z \xleftarrow{u} Z \xleftarrow{u} \ldots$$

The middle $Z_0$ is just another copy of $Z$, but one that we want to distinguish for reference purposes. Sometimes we also need to label the remaining copies of $Z$, and then we use the notation

$$\mathcal{S}: \quad \ldots \xleftarrow{u} Z_2 \xleftarrow{u} Z_1 \xleftarrow{u} Z_0 \xleftarrow{u} Z_{-1} \xleftarrow{u} Z_{-2} \xleftarrow{u} \ldots$$

We should also write $u_k : Z_k \longrightarrow Z_{k+1}$ (rather than simply $u$) for the bonding maps, but such care will not be necessary.

$\mathcal{S}$ can be split at the distinguished $Z_0$, giving rise to an inverse sequence

$$\mathcal{S}^*: \quad Z_0 \xleftarrow{u} Z \xleftarrow{u} Z \xleftarrow{u} \ldots$$

and a direct sequence

$$\mathcal{S}_*: \quad \ldots \xleftarrow{u} Z \xleftarrow{u} Z \xleftarrow{u} Z_0$$

We denote $S^*$ the inverse limit of $\mathcal{S}^*$ and call it the *inverse limit* of $\mathcal{S}$. This inverse limit comes equipped with natural maps from $S^*$ to each of the terms in the sequence $\mathcal{S}^*$, but we are only interested in the one whose target is $Z_0$, which we denote $s^* : S^* \longrightarrow Z_0$. The inverse limit $S^*$ may be described as the set

$$S^* = \left\{ (z_k)_{k \leq 0} : z_k \in Z_k \text{ and } u(z_k) = z_{k+1} \right\}$$

and then

$$s^*((z_k)) = z_0.$$

Similarly we denote $S_*$ the direct limit of $\mathcal{S}_*$ and call it the *direct limit* of $\mathcal{S}$. Again, this comes equipped with natural maps from each of the terms in the sequence $\mathcal{S}_*$ to $S_*$; we are only interested in the one from $Z_0$ to $S_*$, which we denote $s_* : Z_0 \longrightarrow S_*$. The direct limit $S_*$ may be described as the quotient

$$S_* = (\coprod_{k \geq 0} Z_k)/\sim,$$

where $\sim$ is the smallest equivalence relation in $\coprod_{k\geq 0} Z_k$ that is compatible with the vector space structure and such that $z_k \sim u(z_k)$ for every $z_k \in Z_k$ and any $k \geq 0$. With this notation,

$$s_*(z_0) = [z_0].$$

We call $s^*$ and $s_*$ the *canonical maps* associated with $\mathcal{S}$.

**Proposition 5.** *Let $K$ be an attractor for a homeomorphism $f$ and let $N$ be a compact neighbourhood of $K$ contained in $\mathcal{A}(K)$ such that $f(N) \subseteq N$. Then for the sequence*

$$\mathcal{S}: \quad \ldots \xleftarrow{f} N \xleftarrow{f} N \xleftarrow{f} N \xleftarrow{f} \ldots$$

*we have*

1. *$S^* = K$ and $s^*$ is the inclusion of $K$ into $N$,*
2. *$S_* = \mathcal{A}(K)$ and $s_* \circ s^*$ is the inclusion of $K$ into $\mathcal{A}(K)$.*

Part (1) should be interpreted as meaning that $S^*$ can be identified with $K$ and, under this identification, the map $s^*$ becomes the inclusion of $K$ into $N$. An analogous reading should be made of part (2). The proof of Proposition 5 is easy and hence we omit it.

**Proposition 6.** *Let $V$ be a vector space of finite dimension $m$ and $\varphi : V \longrightarrow V$ an endomorphism. Then for the sequence*

$$\mathcal{S}: \quad \ldots \xleftarrow{\varphi} V \xleftarrow{\varphi} V \xleftarrow{\varphi} V \xleftarrow{\varphi} \ldots$$

*we have*

1. *$s^*$ is an isomorphism onto* $\mathrm{im}\ \varphi^m$*,*
2. *$s_* \circ s^* : S^* \longrightarrow S_*$ is an isomorphism.*

*Proof.* Denote $W := \mathrm{im}\ \varphi^m$. It is a well known fact from linear algebra that $\varphi|_W : W \longrightarrow W$ is an isomorphism.

(1) Let $(u_k) \in S^*$. Then $s^*((u_k)) = u_0 = \varphi^m(u_m) \in W$, and so $\mathrm{im}\ s^* \subseteq W$.

Now we show that $s^*$ is a bijection onto $W$. Pick $u_0 \in W$. We need to find $u_1, u_2, \ldots$ with $(u_k) \in S^*$, so that $s^*((u_k)) = u_0$. Since all the $u_k$ have to belong to $W$ because $u_k = \varphi^m(u_{k+m}) \in W$, the only possible choices for the $u_k$ are $u_k := (\varphi|_W)^{-k}(u_0)$. These indeed give rise to an element $(u_k)$ belonging to $S^*$, so $s^*$ is bijective.

(2) Now we prove that $s_*|_W : W \longrightarrow S_*$ is a bijection. Coupled with (1), this establishes (2).

Injectivity is easy: assuming that $s_*(u) = s_*(v)$ for $u, v \in W$, there exists $k$ such that $\varphi^k(u) = \varphi^k(v)$, and the injectivity of $\varphi|_W$ implies that $u = v$. To prove surjectivity, choose an element $[u] \in S_*$ represented by a vector $u \in V_k$. Then $u \sim \varphi^m(u) \in V_{k+m}$. Since $\varphi^m(u) \in W$ and $\varphi^{k+m}|_W : W \longrightarrow W$ is an isomorphism, there exists $v \in W$ such that $\varphi^{m+k}(v) = \varphi^m(u)$. Thus $u \sim \varphi^m(u) = \varphi^{m+k}(v) \sim v$ so $[u] = [v] = s_*(v)$, which shows that $s_*$ is surjective. $\square$

*Proof of Theorem 1.* We prove the theorem for coefficients in $\mathbb{Q}$, the $\mathbb{Z}_p$ case being entirely analogous.

Fix a dimension $d \geq 0$. Consider the sequence

$$\ldots \xleftarrow{f} N \xleftarrow{f} N \xleftarrow{f} N \xleftarrow{f} \ldots$$

which, according to Proposition 5, has direct limit $\mathcal{A}(K)$ and inverse limit $K$. Moreover, the composition of its canonical maps equals the inclusion $i : K \longrightarrow \mathcal{A}(K)$. Passing to cohomology and denoting $\varphi : H^d(N;\mathbb{Q}) \longrightarrow H^d(N;\mathbb{Q})$ the map induced by $f|_N : N \longrightarrow N$ we get the biinfinite sequence

$$\mathcal{S} : \quad \ldots \xrightarrow{\varphi} H^d(N;\mathbb{Q}) \xrightarrow{\varphi} H^d(N;\mathbb{Q}) \xrightarrow{\varphi} H^d(N;\mathbb{Q}) \xrightarrow{\varphi} \ldots$$

The direct limit $S_*$ of $\mathcal{S}$ is precisely $\check{H}^d(K;\mathbb{Q})$ by the very definition of Čech cohomology. And, because coefficients are taken in $\mathbb{Q}$, the inverse limit $S^*$ of $\mathcal{S}$ is $H^d(\mathcal{A}(K);\mathbb{Q})$. This is an easy consequence of the universal coefficient theorem and the fact that homology commutes with direct limits [8, Proposition 3F.8, p. 312]. Finally, it is clear that $s_* \circ s^*$ is precisely the inclusion induced homomorphism $i^* : H^d(\mathcal{A}(K);\mathbb{Q}) \longrightarrow \check{H}^d(K;\mathbb{Q})$. Since $H^d(N;\mathbb{Q})$ has finite dimension $m$, Proposition 6.(2) applies to show that $i_*$ is an isomorphism. Also, Proposition 6.(1) implies that both $\check{H}^d(K;\mathbb{Q})$ and $H^d(\mathcal{A}(K);\mathbb{Q})$ are isomorphic to im $\varphi^m$, which is a subspace of $H^d(N;\mathbb{Q})$ and therefore has finite dimension too. $\square$

In fact we can be more precise about the cohomology of the attractor and the basin of attraction:

**Remark 7.** *Let $K$ be an attractor for a homeomorphism $f$. Let $N$ be a compact neighbourhood of $K$ such that $f(N) \subseteq N$ and assume that $m := \dim H^d(N;\mathbb{Q})$ is finite. Denote $m(0)$ the algebraic multiplicity of the eigenvalue $0$ for the homomorphism $(f|_N)^* : H^d(N;\mathbb{Q}) \longrightarrow H^d(N;\mathbb{Q})$. Then $\dim \check{H}^d(K;\mathbb{Q}) = m - m(0)$.*

The condition that $m$ should be finite is met whenever $N$ is any reasonable neighbourhood of $K$. For instance, it holds if $N$ is a manifold, which in applications will be a common situation.

*Proof of Remark 7.* We saw in the proof of Theorem 1 that $\dim \check{H}^d(K;\mathbb{Q}) = \dim \operatorname{im} \varphi^m$, where $\varphi = (f|_N)^*$. Then it is a standard fact from linear algebra that this dimension is precisely $m - m(0)$. $\square$

## 3. Proof of Theorem 2

Although cohomology with coefficients in $\mathbb{Q}$ is especially useful for explicit computations, as illustrated by Remark 7, it is well known that coefficients in $\mathbb{Z}$ convey more information. Therefore it would be desirable to have a "$\mathbb{Z}$ coefficients version" of Theorem 1. Unfortunately, the analogue of Theorem 1 for cohomology with $\mathbb{Z}$ coefficients is false. In order to show this we first prove a necessary condition for the theorem to be true and then present Example 9, where it is shown that this condition is not always met.

**Proposition 8.** *Let $K \subseteq \mathbb{R}^n$ be an attractor for a homeomorphism $f$. If the inclusion $i : K \subseteq \mathcal{A}(K)$ induces isomorphisms in Čech cohomology with $\mathbb{Z}$ coefficients, then $\check{H}^d(K;\mathbb{Z})$ is finitely generated for every $d \geq 0$.*

*Proof.* Let $N$ be a compact neighbourhood of $K$ contained in $\mathcal{A}(K)$ and having finitely generated Čech cohomology in all dimensions, as the one constructed in the proof of Theorem 1. The inclusion $i$ can be written as $i = k \circ j$, where $j$ and $k$ are the inclusions $K \xrightarrow{j} N \xrightarrow{k} \mathcal{A}(K)$, and on passing to cohomology we see that the composition

$$H^d(\mathcal{A}(K); \mathbb{Z}) \xrightarrow{k^*} H^d(N; \mathbb{Z}) \xrightarrow{j^*} \check{H}^d(K; \mathbb{Z})$$

equals $i^*$ which is an isomorphism by assumption. Hence $j^*$ is surjective and, since $H^d(N; \mathbb{Z})$ is finitely generated, we conclude that $\check{H}^d(K; \mathbb{Z})$ must be finitely generated as well. $\square$

For the sake of brevity we will say that a compact set $K \subseteq \mathbb{R}^n$ that satisfies the necessary conditions of Proposition 8 (that is, all of its Čech cohomology groups with $\mathbb{Z}$ coefficients are finitely generated) has *finitely generated $\mathbb{Z}$–cohomology.*

**Example 9.** *The dyadic solenoid $K$ is an attractor for a homeomorphism of $\mathbb{R}^3$. However, it does not have finitely generated Čech cohomology with $\mathbb{Z}$ coefficients, and so the inclusion $i : K \subseteq \mathcal{A}(K)$ cannot induce isomorphisms in Čech cohomology with $\mathbb{Z}$ coefficients.*

Let us revisit the proof of Theorem 1 very briefly to see where it breaks down when coefficients are taken in $\mathbb{Z}$. Fix a dimension $d$ and denote $G := H^d(N; \mathbb{Z})$ and $\varphi : G \longrightarrow G$ the homomorphism induced by $f|_N : N \longrightarrow N$ in Čech cohomology. We again have a biinfinite sequence

$$\ldots \xleftarrow{\varphi} G \xleftarrow{\varphi} G \xleftarrow{\varphi} G \xleftarrow{\varphi} \ldots \tag{1}$$

whose direct limit $G_*$ is $\check{H}^d(K; \mathbb{Z})$, but now two difficulties arise: $(a)$ the inverse limit $G^*$ does not need to be isomorphic to $H^d(\mathcal{A}(K); \mathbb{Z})$, because the cohomology of a direct limit is not necessarily isomorphic to the (inverse) limit of the cohomologies, $(b)$ there is no guarantee that $G_* = G^*$, because $G$ is not a vector space and Proposition 6 does not apply.

We address $(b)$ in the first place, because it will help us to deal with $(a)$. The following result is the appropriate analogue of Proposition 6 in the context of groups.

**Proposition 10.** *Let $G$ be a finitely generated abelian group and $\varphi : G \longrightarrow G$ an endomorphism. Then for the sequence*

$$\mathcal{S} : \quad \ldots \xleftarrow{\varphi} G \xleftarrow{\varphi} G \xleftarrow{\varphi} G \xleftarrow{\varphi} \ldots$$

*the following are equivalent:*

1. *$S_*$ is finitely generated,*
2. *there exists $m$ such that* $\mathrm{im}\ \varphi^m = \mathrm{im}\ \varphi^{m+1}$,
3. *$s_* \circ s^* : S^* \longrightarrow S_*$ is an isomorphism.*

Before giving the proof, let us recall the fact that finitely generated abelian groups are *Hopfian*; that is, they have the property that every surjective endomorphism of such a group is actually an isomorphism (as it happens for vector spaces). This is very easy to prove for $G =$ finite group and $G = \mathbb{Z}$. The general result follows using the fact that every finitely generated abelian group $G$ is a direct sum of a finite number of copies of $\mathbb{Z}$ and a torsion group which is finite.

*Proof of Proposition 10.* (1) $\Rightarrow$ (2) Assume $S_*$ has a finite set of generators. For reference purposes we write the direct limit half of $\mathcal{S}$ as

$$\mathcal{S}_* : \ldots \xleftarrow{\varphi} G_2 \xleftarrow{\varphi} G_1 \xleftarrow{\varphi} G_0$$

where as usual each $G_k$ is just a copy of $G$. We denote $s_k : G_k \longrightarrow S_*$ the canonical maps from the $G_k$ to the direct limit $S_*$.

Consider a finite family of generators for $S_*$. Each of them can be represented by an element in some $S_k$ and, pushing those representatives forward along the sequence, we may assume that all of them belong to the same $G_k$. This implies that $\mathrm{im}\ s_k = S_*$.

$G$ was assumed to be finitely generated, so it has a finite family of generators $g_1, g_2, \ldots, g_r$. Thinking for a moment of these $g_i$ as elements of $G_{k+1}$ we have that $s_{k+1}([g_i]) \in S_* = s_k(G_k)$, so there exist $g_i' \in G_k$ such that $s_{k+1}([g_i]) = s_k([g_i'])$. According to the definition of direct limit, this means that there exist $m_i$ such that $\varphi^{m_i}(g_i) = \varphi^{m_i+1}(g_i')$, where we think of the $g_i$ and $g_i'$ as elements of $G$. Taking $m := \max\ \{m_i\}$ we see that $\varphi^m(g_i) = \varphi^{m+1}(g_i') \in \mathrm{im}\ \varphi^{m+1}$ for every generator $g_i$ of $G$, and so $\mathrm{im}\ \varphi^m \subseteq \mathrm{im}\ \varphi^{m+1}$. Since the opposite inclusion is trivially true, it follows that $\mathrm{im}\ \varphi^{m+1} = \mathrm{im}\ \varphi^m$.

(2) $\Rightarrow$ (3) Let $W := \mathrm{im}\ \varphi^m$. We will just show that $\varphi|_W : W \longrightarrow W$ is an isomorphism, and then the argument of Proposition 6 applies word for word to prove that $s_* \circ s^*$ is an isomorphism.

The hypothesis that $\mathrm{im}\ \varphi^m = \mathrm{im}\ \varphi^{m+1}$ can be spelled out as

$$\varphi^m(G) = \varphi(\varphi^m(G)) \Rightarrow W = \varphi(W),$$

whence we see that $\varphi|_W : W \longrightarrow W$ is surjective. $W$ is finitely generated, because it is the image of the finitely generated group $G$ under $\varphi^m$, so it is Hopfian. Thus $\varphi|_W : W \longrightarrow W$ is an isomorphism.

(3) $\Rightarrow$ (1) This is very easy. By assumption $s_* \circ s^* : S^* \longrightarrow S_*$ is an isomorphism, so we see that $S_* = s_*(\mathrm{im}\ s^*)$. Since $\mathrm{im}\ s^*$ is a subgroup of the finitely generated abelian group $G$, it is also finitely generated. Thus $S_*$ is finitely generated too. □

**Remark 11.** *It also follows from (2) $\Rightarrow$ (3) and the proof of Proposition 6 that when* $\mathrm{im}\ \varphi^m = \mathrm{im}\ \varphi^{m+1}$ *then not only $s_* \circ s^*$ provides an isomorphism between $S^*$ and $S_*$, but also that these groups are isomorphic to $W := \mathrm{im}\ \varphi^m$.*

Now we address $(a)$; namely, when is the inverse limit of (1) isomorphic to the cohomology of the basin of attraction.

Let $N \subseteq \mathcal{A}(K)$ be a compact neighbourhood of $K$ as in Notation 4. There is an increasing sequence

$$N \subseteq f^{-1}(N) \subseteq f^{-2}(N) \subseteq \ldots$$

whose union $\bigcup_n f^{-n}(N)$ is precisely $\mathcal{A}(K)$, and on cohomology this gives rise to an inverse sequence

$$H^d(N;\mathbb{Z}) \longleftarrow H^d(f^{-1}(N);\mathbb{Z}) \longleftarrow H^d(f^{-2}(N);\mathbb{Z}) \longleftarrow \ldots$$

whose inverse limit we denote $\lim\ \{H^d(f^{-n}(N);\mathbb{Z})\}$ (unlabeled arrows denote inclusion induced homomorphisms). It is *not* true in general that this inverse limit is isomorphic to $H^d(\mathcal{A}(K);\mathbb{Z})$. The extent to which they are not isomorphic is measured by the so-called *first derived limit* $\lim^1$ of the inverse sequence

$$H^{d-1}(N;\mathbb{Z}) \longleftarrow H^{d-1}(f^{-1}(N);\mathbb{Z}) \longleftarrow H^{d-1}(f^{-2}(N);\mathbb{Z}) \longleftarrow \ldots$$

which is an abelian group that fits in the *Milnor exact sequence* [14, Lemma 2, p. 338]

$$0 \longrightarrow \lim^1 \{H^{d-1}(f^{-n}(N);\mathbb{Z})\} \longrightarrow H^d(\mathcal{A}(K);\mathbb{Z}) \longrightarrow \lim \{H^d(f^{-n}(N);\mathbb{Z})\} \longrightarrow 0$$

and therefore vanishes precisely when $H^d(\mathcal{A}(K);\mathbb{Z})$ is isomorphic to the inverse limit $\lim \{H^d(f^{-n}(N);\mathbb{Z})\}$ (Milnor's paper is set up for CW complexes, but his argument is completely general).

The first derived limit of an arbitrary sequence of abelian groups and homomorphisms

$$A_0 \overset{\psi_1}{\longleftarrow} A_1 \overset{\psi_2}{\longleftarrow} A_2 \overset{\psi_3}{\longleftarrow} \ldots$$

is defined as the cokernel of the map $d : \prod A_k \longrightarrow \prod A_k$ given by

$$d(a_0, a_1, a_2, \ldots) := (a_0 - \psi_1(a_1), a_1 - \psi_2(a_2), \ldots).$$

Thus $\lim^1\{A_k\} = 0$ precisely when $d$ is surjective, or otherwise stated when the systems of equations

$$\begin{cases} a_0 - \psi_1(a_1) & = & b_0 \\ a_1 - \psi_2(a_2) & = & b_1 \\ a_2 - \psi_3(a_3) & = & b_2 \\ \ldots & = & \ldots \end{cases}$$

has a solution $(a_k) \in \prod_{k\geq 0} A_k$ for every $(b_k) \in \prod_{k\geq 0} A_k$.

The following result is a consequence of the well known fact that any inverse sequence with the Mittag–Leffler property has vanishing first derived limit, but we include an elementary proof for completeness.

**Lemma 12.** *Let $\varphi : G \longrightarrow G$ be a homomorphism of an abelian group $G$. Assume that* im $\varphi^m =$ im $\varphi^{m+1}$ *for some $m$. Then the inverse sequence*

$$G \overset{\varphi}{\longleftarrow} G \overset{\varphi}{\longleftarrow} G \overset{\varphi}{\longleftarrow} \ldots$$

*has vanishing first derived limit.*

*Proof.* For reference purposes we will attach a subscript $k$ to each copy of $G$. We need to show that for any $(b_k) \in \prod_{k\geq 0} G_k$ the equations $a_k - \varphi(a_{k+1}) = b_k$ have a solution $(a_k) \in \prod_{k\geq 0} G_k$.

Consider the family of auxiliary equations $\varphi^{m+1}(a^*_{k+1}) = \varphi^m(a^*_k + b_k)$. These have a solution $(a^*_k) \in \prod_{k\geq 0} G_k$: choose $a^*_0$ arbitrarily and then use the condition im $\varphi^m =$ im $\varphi^{m+1}$ to find inductively $a^*_{k+1}$ such that $\varphi^{m+1}(a^*_{k+1}) = \varphi^m(a^*_k + b_k)$ for every $k \geq 0$.

The above auxiliary equations can be rewritten as $\varphi^m(a^*_k - \varphi(a^*_{k+1}) - b_k) = 0$, which inmediately shows that $a^*_k - \varphi(a^*_{k+1}) = b_k + c_k$, where $c_k \in \ker \varphi^m$. Let $a_k := a^*_k - \sum_{j=0}^{m-1} \varphi^j(c_{k+j})$. We claim that these $a_k$ satisfy our original set of equations $a_k - \varphi(a_{k+1}) = b_k$. Indeed,

$$\begin{aligned} a_k - \varphi(a_{k+1}) = & \\ = \left(a^*_k - \sum_{j=0}^{m-1} \varphi^j(c_{k+j})\right) - \left(\varphi(a^*_{k+1}) - \sum_{j=0}^{m-1} \varphi^{j+1}(c_{k+j+1})\right) = & \\ = (a^*_k - \varphi(a^*_{k+1})) - c_k + \varphi^m(c_{k+m}) = b_k, & \end{aligned}$$

where we have used that $c_{k+m} \in \ker \varphi^m$. ☐

**Theorem 13.** *Let $K \subseteq \mathbb{R}^n$ be an attractor for a homeomorphism $f$. The following are equivalent:*

1. *the inclusion $i : K \subseteq \mathcal{A}(K)$ induces isomorphisms in Čech cohomology with $\mathbb{Z}$ coefficients,*
2. *$K$ has finitely generated Čech cohomology with $\mathbb{Z}$ coefficients.*

*Proof.* (1) $\Rightarrow$ (2) This is just Proposition 8.

(2) $\Rightarrow$ (1) Let $N \subseteq \mathcal{A}(K)$ be a compact neighbourhood of $K$ as in Notation 4. As usual, we assume that $f(N) \subseteq N$ and denote $G := H^d(N;\mathbb{Z})$, $\varphi : G \longrightarrow G$ the homomorphism induced by $f|_N : N \longrightarrow N$ in cohomology, and $\mathcal{S}$ the biinfinite sequence

$$\mathcal{S} : \quad \ldots \xleftarrow{\varphi} G \xleftarrow{\varphi} G \xleftarrow{\varphi} G \xleftarrow{\varphi} \ldots$$

The direct limit $S_*$ of $\mathcal{S}$ is the cohomology of $K$, which by assumption is finitely generated. Therefore by Proposition 10 the map $s_* \circ s^* : S^* \longrightarrow S_*$ is an isomorphism, so we only need to show that $S^* = H^d(\mathcal{A}(K);\mathbb{Z})$.

Proposition 10 also guarantees that there exists $m$ such that $\operatorname{im} \varphi^m = \operatorname{im} \varphi^{m+1}$. Then by Lemma 12 the first derived limit of $\mathcal{S}^*$ vanishes. Notice that all this is true for every dimension $d$. Consider the commutative diagram

$$\begin{tikzcd}
\ldots \arrow[r, "\varphi"] & H^d(N;\mathbb{Z}) \arrow[r, "\varphi"] & H^d(N;\mathbb{Z}) \arrow[r, "\varphi"] & H^d(N;\mathbb{Z}) \\
\ldots \arrow[r] & H^d(f^{-2}(N);\mathbb{Z}) \arrow[r] \arrow[u, "(f|_N^{-2})^*"] & H^d(f^{-1}(N);\mathbb{Z}) \arrow[r] \arrow[u, "(f|_N^{-1})^*"] & H^d(N;\mathbb{Z}) \arrow[u]
\end{tikzcd}$$

where the unlabeled arrows denote inclusion induced homomorphisms. The vertical arrows are all isomorphisms, and it is very easy to check that this implies that the inverse limits of the upper and the lower rows are isomorphic, and similarly for their first derived limits. The first derived limit of the upper row is zero, as argued earlier, so the same is true of the latter. This holds for every dimension $d$, so by the Milnor exact sequence the inverse limit of the lower row is $H^d(\mathcal{A}(K);\mathbb{Z})$. Thus the inverse limit of the upper row is $H^d(\mathcal{A}(K);\mathbb{Z})$ too, which finishes the proof. $\square$

*Proof of Theorem 2.* Clearly only (2) $\Leftrightarrow$ (3) needs proof.

(2) $\Rightarrow$ (3) By Theorem 13 the cohomology of $\mathcal{A}(K)$ is isomorphic to the cohomology of $K$, hence finitely generated.

(3) $\Rightarrow$ (2) It is best to think of $\mathbb{R}^n$ as the $n$–dimensional sphere $\mathbb{S}^n$ minus the point $\infty$. $f$ can be extended to a homeomorphism $\hat{f} : \mathbb{S}^n \longrightarrow \mathbb{S}^n$ letting $\hat{f}(\infty) := \infty$, and then $K' := \mathbb{S}^n \backslash \mathcal{A}(K)$ is an attractor for $\hat{f}^{-1}$ with basin of attraction $\mathcal{A}(K') = \mathbb{S}^n \backslash K$. Clearly $K$ is still an attractor for $\hat{f}$ with basin of attraction $\mathcal{A}(K)$.

By Alexander duality $\check{H}^d(K';\mathbb{Z}) = H_{n-d}(\mathcal{A}(K);\mathbb{Z})$, so $K'$ has finitely generated Čech cohomology. It follows, as in (2) $\Rightarrow$ (3), that $\mathcal{A}(K')$ has finitely generated cohomology too. This implies that it also has finitely generated homology [8, Proposition 3F.12, p. 318], and then by Alexander duality again $K$ has finitely generated Čech cohomology. $\square$

There is an alternative way of proving Theorem 2 which roughly consists in putting together the information given by Theorem 1 for $\mathbb{Q}$ and $\mathbb{Z}_p$ coefficients to reach a conclusion about $\mathbb{Z}$ coefficients. This approach makes no use of the dynamics whatsoever; in fact, it proceeds by establishing the following lemma:

**Lemma 14.** *Let $U \subseteq \mathbb{R}^n$ be open and $K \subseteq U$ be compact. Denote $i : K \longrightarrow U$ the inclusion. Assume that $i$ induces isomorphisms in Čech cohomology with coefficients in $\mathbb{Q}$ and $\mathbb{Z}_p$ for every prime $p$, and that $U$ has finitely generated cohomology. Then $i$ induces isomorphisms in Čech cohomology with $\mathbb{Z}$ coefficients.*

*Sketch of proof.* Observe that the assertion that $i$ induces an isomorphism in Čech cohomology with $G$ coefficients is equivalent to stating that $\check{H}^*(U, K; G) = 0$. By hypothesis $\check{H}^*(U, K; G) = 0$ for $G = \mathbb{Q}$ and $G = \mathbb{Z}_p$, and we want to prove the same for $G = \mathbb{Z}$. Denote for brevity $H := \check{H}^d(U, K; \mathbb{Z})$ for some fixed dimension $d$. It is a general fact that for an abelian group $A$ the kernel of the obvious map $A \longrightarrow A \otimes \mathbb{Q}$ is precisely the torsion subgroup of $A$. Thus if we show that $(i)$ $H \otimes \mathbb{Q} = 0$ and $(ii)$ $H$ has no torsion, then $H = 0$.

Part $(i)$ follows from the universal coefficient theorem, since

$$\check{H}^d(U, K; \mathbb{Z}) \otimes \mathbb{Q} = \check{H}^d(U, K; \mathbb{Q}) = 0.$$

Some care has to be exercised, though, because the universal coefficient theorem requires that $\check{H}^d(U, K; \mathbb{Z})$ be finitely generated. This is a consequence of our hypothesis that $U$ has finitely generated cohomology.

Part $(ii)$ uses Bockstein homomorphisms. Let $p$ be a prime number. The exact sequence $0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot p} \mathbb{Z} \longrightarrow \mathbb{Z}_p \longrightarrow 0$, where $\cdot p$ denotes multiplication by $p$, induces a long exact sequence

$$\ldots \longleftarrow \check{H}^d(U, K; \mathbb{Z}) \xleftarrow{\cdot p} \check{H}^d(U, K; \mathbb{Z}) \longleftarrow \check{H}^d(U, K; \mathbb{Z}_p) \longleftarrow \ldots$$

which, because $\check{H}^d(U, K; \mathbb{Z}_p) = 0$, implies that $\cdot p : H \longrightarrow H$ is injective. Hence $H$ cannot contain elements of order $p$, and since this is true for every prime $p$, it follows that $H$ has no torsion. □

Choosing $U = \mathcal{A}(K)$ and using Theorem 1 the lemma directly proves implication (3) ⇒ (1) of Theorem 2. Proposition 8 establishes (1) ⇒ (2). Finally, (2) ⇒ (3) is proved using Alexander duality much in the same way as we already did above.

*When the phase space is a manifold $M$ other than $\mathbb{R}^n$.* There are only two stages where we have specifically used that $f$ is a homeomorphism of $\mathbb{R}^n$. The first one was in Proposition 3 of which we gave two different proofs; these can be readily translated to triangulable manifolds or differentiable manifolds respectively. Thus Proposition 3 holds when $M$ is a differentiable or triangulable manifold, and so does Theorem 1. The second one was in proving (3) ⇒ (2) in Theorem 2, because we resorted to Alexander duality. However, the alternative argument via Lemma 14 shows the validity of (3) ⇒ (2) in any manifold $M$. Hence Theorem 2 also holds true in any differentiable or triangulable manifold.

## 4. Applications (1): attractors for volume contracting homeomorphisms of $\mathbb{R}^3$

**Proposition 15.** *Let $K$ be a connected attractor for a volume contracting homeomorphism $f$ of $\mathbb{R}^3$. Then $K$ has a compact neighbourhood $\hat{N} \subseteq \mathcal{A}(K)$ which is a connected 3–manifold with $H^2(\hat{N}; \mathbb{Q}) = 0$.*

*Proof.* By Proposition 3 $K$ has a compact neighbourhood $N \subseteq \mathcal{A}(K)$ that is a 3–manifold; by discarding those components of $N$ that do not meet $K$ (if any) we may as well assume that $N$ is connected. After replacing $f$ by a suitable power we can also assume that $f(N) \subseteq N$. Notice that this operation does not alter the fact that $f$ is a volume contracting homeomorphism.

Observe that $\mathbb{R}^3 - N$ has exactly one unbounded component $V_0$ and (if any) a finite number of bounded components $V_1, \ldots, V_r$. We label the $V_i$ in such a way that

$$\text{vol } V_1 \leq \text{vol } V_2 \leq \ldots \leq \text{vol } V_r.$$

Let

$$\hat{N} := N \cup V_1 \cup \ldots \cup V_r = \mathbb{R}^3 - V_0$$

be the union of $N$ and the bounded $V_i$. Clearly $\hat{N}$ is still a compact 3–manifold which is a neighbourhood of $K$. Also, using Alexander duality we see that

$$H^2(\hat{N}; \mathbb{Q}) = \tilde{H}_0(\mathbb{R}^3 - \hat{N}; \mathbb{Q}) = \tilde{H}_0(V_0; \mathbb{Q}) = 0$$

because $V_0$ is connected. Thus it only remains to show that $\hat{N} \subseteq \mathcal{A}(K)$.

Since $f(N) \subseteq N$, we have $\mathbb{R}^3 - N \subseteq \mathbb{R}^3 - f(N)$ and therefore each component of $\mathbb{R}^3 - N$ is contained in a component of $\mathbb{R}^3 - f(N)$. The components of the latter are precisely the images under $f$ of the components of the former, so we may write

$$V_i \subseteq f(V_{j(i)})$$

for suitable indices $j(i)$. Notice that $V_{j(0)}$ has to be unbounded, so $j(0) = 0$. When $j(i) \neq 0$ (so $V_{j(i)}$ is one of the bounded components) we have

$$\text{vol } V_i \leq \text{vol } f(V_{j(i)}) < \text{vol } V_{j(i)}$$

because $f$ is volume contracting. By our labeling of the $V_i$ the above inequality implies that $i < j(i)$ and, inductively,

$$i < j(i) < j^2(i) < \ldots < j^k(i)$$

as long as $j^k(i) \neq 0$. However $j$ only takes values in $\{0, 1, \ldots, r\}$, so the above chain cannot have length greater than $r+1$ and $j^k(i) = 0$ for some $k \leq r+1$; from then on $j^{k+1}(i) = j^{k+2}(i) = \ldots = 0$ because $j(0) = 0$. Thus $j^{r+1}(i) = 0$ for every $i$, or in other terms

$$V_i \subseteq f^{r+1}(V_0).$$

Recalling that $\mathbb{R}^3 - N$ is the union of the $V_i$ and $V_0 = \mathbb{R}^3 - \hat{N}$ we then have

$$\mathbb{R}^3 - N = \bigcup_{i=0}^{n} V_i \subseteq f^{r+1}(V_0) = \mathbb{R}^3 - f^{r+1}(\hat{N}),$$

which implies $f^{r+1}(\hat{N}) \subseteq N \subseteq \mathcal{A}(K)$. Thus $\hat{N} \subseteq f^{-(r+1)}(\mathcal{A}(K)) = \mathcal{A}(K)$ because $\mathcal{A}(K)$ is invariant. $\square$

We also need the following known lemma, which can be found in [13].

**Lemma 16.** *Let $\lambda_1, \lambda_2, \ldots, \lambda_m \in \mathbb{C}$. Assume that $\sum_{i=1}^m \lambda_i^k = 1$ for every $k \geq 1$. Then (possibly after relabeling) $\lambda_1 = 1$ and $\lambda_i = 0$ for $2 \leq i \leq m$.*

**Theorem 17.** *Let $K$ be a connected attractor for a volume contracting homeomorphism $f$ of $\mathbb{R}^3$. Assume that $K$ does not contain fixed or periodic points. Then*

$$\check{H}^d(K; \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{for } d = 1, \\ 0 & \text{for } d \geq 2. \end{cases}$$

*Consequently, the same holds true for $H^d(\mathcal{A}(K); \mathbb{Q})$.*

*Addendum. The homomorphism* $(f|_K)^* : \check{H}^1(K;\mathbb{Q}) \longrightarrow \check{H}^1(K;\mathbb{Q})$ *is either the identity or minus the identity.*

*Proof.* By Proposition 15 $K$ has a compact neighbourhood $\hat{N} \subseteq \mathcal{A}(K)$ that is a connected 3–manifold with $H^2(\hat{N};\mathbb{Q}) = 0$. Notice that also $H^d(\hat{N};\mathbb{Q}) = 0$ for every $d \geq 3$ (for $d = 3$ this is a consequence of the fact that $N$ has a nonempty boundary).

Replace $f$ by a suitable power $g = f^r$ such that $g(\hat{N}) \subseteq \hat{N}$. Notice that $K$ is still an attractor for $g$ without fixed or periodic points. We want to consider the restriction $g|_{\hat{N}} : \hat{N} \longrightarrow \hat{N}$ and apply Remark 7 to compute the Čech cohomology of $K$. For dimension $d \geq 2$ the condition $H^d(\hat{N};\mathbb{Q}) = 0$ inmediately implies $\check{H}^d(K;\mathbb{Q}) = 0$. For dimension $d = 1$ we reason as follows.

Since $K$ is an attractor and $\hat{N} \subseteq \mathcal{A}(K)$, any fixed or periodic points that $g$ may have in $\hat{N}$ should be contained in $K$. Our hypothesis says that there are none of them, so we conclude that $g$ has no fixed or periodic points in $\hat{N}$. Therefore none of its powers has fixed points in $\hat{N}$ either, so their Lefschetz numbers $\Lambda(g^k|_{\hat{N}})$ are zero. Since $H^d(\hat{N};\mathbb{Q})$ vanishes for $d \geq 2$, in order to compute $\Lambda(g^k|_{\hat{N}})$ only the traces of the homomorphisms induced by $g^k|_{\hat{N}}$ in dimension zero and one need to be considered. In dimension zero the homomorphism is just the identity, and its trace is 1 because $\hat{N}$ is connected. Now let $m := \dim H^1(\hat{N};\mathbb{Q})$ and $\lambda_1, \ldots, \lambda_m$ the eigenvalues of $\varphi := (g|_{\hat{N}})^* : H^1(\hat{N};\mathbb{Q}) \longrightarrow H^1(\hat{N};\mathbb{Q})$. Then in dimension one the trace of $\varphi^k$ is given by $\sum_{i=1}^m \lambda_i^k$. Hence for every $k \geq 1$ we have

$$0 = \Lambda(g^k|_{\hat{N}}) = 1 - \sum_{i=1}^m \lambda_i^k.$$

Since this is true for all $k \geq 1$, Lemma 16 implies that (maybe after reordering the eigenvalues) $\lambda_1 = 1$ and $\lambda_2 = \ldots = \lambda_m = 0$. Hence the algebraic multiplicity of 0 is $m - 1$ and therefore by Remark 7 we conclude that $\check{H}^1(K;\mathbb{Q}) = \mathbb{Q}$.

Now we prove the addendum. Let $j : K \longrightarrow \hat{N}$ be the inclusion. From Proposition 6 it follows that

$$j_*|_{\mathrm{im}\ \varphi^m} : \mathrm{im}\ \varphi^m \longrightarrow \check{H}^1(K;\mathbb{Q})$$

is an isomorphism onto. Thus the maps

$$\varphi|_{\mathrm{im}\ \varphi^m} : \mathrm{im}\ \varphi^m \longrightarrow \mathrm{im}\ \varphi^m$$

and

$$(g|_K)^* : \check{H}^1(K;\mathbb{Q}) \longrightarrow \check{H}^1(K;\mathbb{Q})$$

are conjugate; in particular their traces are the same. The trace of $\varphi|_{\mathrm{im}\ \varphi^m}$ is easily seen to be 1, so this is also the trace of $(g|_K)^*$. Now, since we proved in the previous paragraph that $\check{H}^1(K;\mathbb{Q}) = \mathbb{Q}$, it follows that $(f|_K)^* = \alpha\,\mathrm{id}$ for some $\alpha \in \mathbb{Q}$. Thus

$$(g|_K)^* = (f^r|_K)^* = \alpha^r\ \mathrm{id},$$

so if the trace of $(g|_K)^*$ has to be 1 then $\alpha^r = 1$. This forces $\alpha = \pm 1$ and finishes the proof. □

## 5. Applications (2): periodic equations in $\mathbb{R}^3$

Consider the differential system

$$\dot{x} = X(t,x), \quad x \in \mathbb{R}^3 \tag{2}$$

where $X : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ satisfies

$$X(t + 2\pi, x) = X(t, x), \quad \text{for each } (t, x) \in \mathbb{R} \times \mathbb{R}^3.$$

We assume that the vector field $X$ is continuous and such that there is global existence and uniqueness for the initial value problem. The solution satisfying the initial condition $x(t_0) = \xi$ will be denoted by $x(t; t_0, \xi)$. It is well defined for all $t \in \mathbb{R}$.

The *Poincaré map* associated to equation (2) is the continuous map $P(\xi) = x(2\pi; 0, \xi)$ which, owing to the assumption of global existence and uniqueness of solutions, is a homeomorphism of $\mathbb{R}^3$. We are interested in the case when $P$ is volume contracting. This happens, for instance, when $X$ is of class $\mathcal{C}^1$ and its divergence with respect to the $x$ variable is negative everywhere: $\operatorname{div}_x X = \sum \frac{\partial X_i}{\partial x_i}(x, t) < 0$ for every $x$ and every $t$.

Equation (2) can be turned into an autonomous equation on $\mathbb{R} \times \mathbb{R}^3$ by the usual device of introducing the variable $y = (t, x)$ and the vectorfield $Y(t, x) = (1, X(t, x))$, so that $\dot{y}(t) = Y(y)$. Since $X$ is $2\pi$–periodic, $Y$ descends to a continuous vectorfield $\overline{Y}$ on the quotient $\mathbb{S}^1 \times \mathbb{R}^3$, where $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$. Thus equation (2) can be viewed as an autonomous equation on $\mathbb{S}^1 \times \mathbb{R}^3$, and we will adopt this point of view from now on.

**Theorem 18.** *Suppose $P$ contracts volume and assume that equation* (2) *has a connected attractor $K \subseteq \mathbb{S}^1 \times \mathbb{R}^3$. Then at least one of the following holds:*

i) *$K$ contains periodic orbits.*

ii)

$$\check{H}^d(K; \mathbb{Q}) = \begin{cases} \mathbb{Q} \bigoplus \mathbb{Q} & \textit{for } d = 1, \\ \mathbb{Q} & \textit{for } d = 2, \\ 0 & \textit{for } d > 2. \end{cases}$$

To prove Theorem 18 we need the following lemma, whose proof is deferred to an appendix. Given a space $L$ and a map $g : L \longrightarrow L$, recall that the *mapping torus* of $g$ is defined as the result of quotienting the space $L \times [0, 1]$ with the relation $(p, 0) \sim (g(p), 1)$.

**Lemma 19.** *Let $L$ be a space and $g : L \longrightarrow L$ a homeomorphism. Denote $L_g$ the mapping torus of $g$. Then there is an exact sequence*

$$(3) \quad \ldots \longleftarrow \check{H}^d(L_g) \xleftarrow{\Delta} \check{H}^{d-1}(L) \xleftarrow{\mathrm{id}-g^*} \check{H}^{d-1}(L) \longleftarrow \check{H}^{d-1}(L_g) \xleftarrow{\Delta} \ldots$$

*Proof of Theorem 18.* It is easy to see that the Poincaré map $P$ has an attractor $L \subseteq \mathbb{R}^3$ such that the mapping torus of $P|_L$ is (homeomorphic to) $K$. Suppose that $K$ does not contain periodic orbits. Then $P$ does not have fixed or periodic points in $L$, so by Theorem 17 each component $L_i$ of $L$ has cohomology

$$\check{H}^d(L_i; \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{for } d = 1, \\ 0 & \text{for } d \geq 2. \end{cases}$$

We want to use the exact sequence of Lemma 19 to compute the cohomology of $K$ and show that (ii) holds. In order to do this we need to analyze the maps $\psi_0$ and $\psi_1$ defined by

$$\psi_d := \mathrm{id} - (P|_L)^* : \check{H}^d(L; \mathbb{Q}) \longrightarrow \check{H}^d(L; \mathbb{Q}) \ \ (d = 0, 1).$$

Specifically, we are going to prove that dim ker $\psi_0 =$ dim ker $\psi_1 = 1$. The equality concerning $\psi_0$ is easy to obtain: using Lemma 19 and the hypothesis that $K$ is connected there is an exact sequence

$$\ldots \longleftarrow \check{H}^0(L) \xleftarrow{\psi_0} \check{H}^0(L) \longleftarrow \check{H}^0(K) = \mathbb{Q} \longleftarrow 0$$

which inmediately implies dim ker $\psi_0 = 1$. The equality concerning $\psi_1$ is not so straightforward, and we deal with it now.

Let $L_1, \ldots, L_s$ be the components of $L$ (they are finite in number because $\check{H}^0(L;\mathbb{Q})$ is finitely generated by Theorem 1). Since $P|_L$ is a homeomorphism, it takes each $L_i$ homeomorphically onto some $L_{j(i)}$, so we may write $P(L_i) = L_{j(i)}$ for some permutation $j$ of $\{1, \ldots, s\}$. Now suppose that $j(I) = I$ for some set $I$ of indices. Denoting $I'$ the set of remaning indices, clearly both $\bigcup_{i \in I} L_i$ and $\bigcup_{i' \in I'} L_{i'}$ are invariant by $P$. If both were nonempty then the mapping torus of $P|_L$ would have at least two connected components, contrary to our assumption that $K$ is connected. Thus either $I = \emptyset$ or $I = \{1, \ldots, s\}$ (actually this is a restatement of the fact that dim ker $\psi_0 = 1$).

We claim that, for any one $i$, the sequence

$$i, j(i), j^2(i), \ldots, j^{s-1}(i)$$

has no repetitions. For, suppose it had. Then there would exist $0 \leq k < \ell < s$ such that $j^k(i) = j^\ell(i)$ and letting $I := \{j^k(i), j^{k+1}(i), \ldots, j^\ell(i)\}$ it is clear that $j(I) = I$ which contradicts the previous paragraph. A similar reasoning shows that $j^s(i) = i$.

Pick a generator $\ell_1$ for $\check{H}^1(L_1;\mathbb{Q})$. Applying $P^*$ repeatedly we obtain generators $\ell_{j(1)} = P^*(\ell_1)$ for $\check{H}^1(L_{j(1)};\mathbb{Q})$, then $\ell_{j^2(1)} = P^*(\ell_{j(1)})$ for $\check{H}^1(L_{j^2(1)};\mathbb{Q})$, and so on. The previous paragraph guarantees that $\{\ell_1, \ell_{j(1)}, \ell_{j^2(1)}, \ldots, \ell_{j^{s-1}(1)}\}$ contains precisely one generator for each $\check{H}^1(L_i;\mathbb{Q})$; thus it is a basis for $\check{H}^1(L;\mathbb{Q})$. Also, since $j^s(1) = 1$, it follows from the addendum of Theorem 17 (applied to $P^s$) that $P^*(\ell_{j^{s-1}(1)}) = (P^*)^s(\ell_1) = \ell_1$. Now it is very easy to see that $\ell_1 + \ell_{j(1)} + \ldots + \ell_{j^{s-1}(1)}$ belongs to ker $\psi_1$ and, actually, generates it. Thus dim ker $\psi_1 = 1$.

The proof can be quickly finished now. From Lemma 19 we have

$$\check{H}^2(L) = 0 \longleftarrow \check{H}^2(K) \xleftarrow{\Delta} \check{H}^1(L) = \mathbb{Q}^s \xleftarrow{\psi_1} \check{H}^1(L) = \mathbb{Q}^s \longleftarrow \ldots$$

which implies dim $\check{H}^2(K) =$ dim im $\Delta = s -$ dim im $\psi_1 = 1$. Finally, from

$$\ldots \xleftarrow{\psi_1} \check{H}^1(L) = \mathbb{Q}^s \longleftarrow \check{H}^1(K) \xleftarrow{\Delta} \check{H}^0(L) = \mathbb{Q}^s \xleftarrow{\psi_0} \ldots$$

we see that dim $\check{H}^1(K) =$ dim ker $\psi_1 +$ dim im $\Delta =$ dim ker $\psi_1 + s -$ dim im $\psi_0 = 2$. $\square$

Let $X(x)$ be a $\mathcal{C}^1$ vectorfield on $\mathbb{R}^3$. Assume that the autonomous system

$$(S) : \dot{x} = X(x), \quad x \in \mathbb{R}^3$$

has a connected attractor $L \subseteq \mathbb{R}^3$ such that $\check{H}^1(L;\mathbb{Q}) \neq \mathbb{Q}$, and suppose also that $\mathrm{div}_x\ X < 0$ on $L$. As an interesting application of Theorem 18 we will now show that periodic points appear in $L$ when a sufficiently small periodic perturbation $\varepsilon(t, x)$ is added to $X(x)$. A piece of terminology is needed: given a real number

$\epsilon > 0$, we say that a $2\pi$–periodic function $\varepsilon(t,x) : \mathbb{R} \times \mathbb{R}^n$ of class $\mathcal{C}^1$ is *$\epsilon$–small on $L$* if

$$\|\varepsilon(t,x)\| \leq \epsilon \quad \text{and} \quad |\mathrm{div}_x\, \varepsilon(t,x)| \leq \epsilon$$

for every $(t,x) \in [0,2\pi] \times L$.

**Corollary 20.** *In the situation just described, there exists $\epsilon > 0$ such that if $\varepsilon(t,x)$ is an $\epsilon$–small $2\pi$–periodic function, the perturbed nonautonomous system*

$$(S_\varepsilon) : \dot{x} = X(x) + \varepsilon(t,x), \quad x \in \mathbb{R}^3$$

*has an attractor $K \subseteq \mathbb{S}^1 \times \mathbb{R}^3$ which contains periodic orbits.*

Before proving the corollary we need to recall the following result originally due to Hastings [7]: if $L$ is an attractor for a *flow* and $N$ is a compact, positively invariant neighbourhood of $L$ contained in its basin of attraction $\mathcal{A}(L)$, then the inclusion $L \subseteq N$ induces isomorphisms in Čech cohomology. The reader might recognize this as a close relative of the fact —mentioned in the introduction— that the inclusion $L \subseteq \mathcal{A}(L)$ also induces isomorphisms in Čech cohomology.

*Proof of Corollary 20.* Let $\mathcal{L}$ be any differentiable Lyapunov function for $L$. For any sufficiently small regular value $c$ of $\mathcal{L}$ the set $N := \mathcal{L}^{-1}[0,c]$ is a compact 3–manifold neighbourhood of $L$ such that $\mathrm{div}_x\ X < 0$ for each $x \in N$. Also, the vectorfield $X$ points transversally towards int $N$ at each point of $\partial N$ and therefore $N$ is positively invariant for $(S)$, which implies that the inclusion $L \subseteq N$ induces isomorphisms in Čech cohomology by the theorem of Hastings stated above. Thus $N$ is connected and $\check{H}^1(N;\mathbb{Q}) \neq \mathbb{Q}$.

Denote $X_\varepsilon := X(x) + \varepsilon(t,x)$ the vectorfield that governs $(S_\varepsilon)$. Since $N$ and $\partial N$ are compact, there exists $\epsilon > 0$ such that if the $2\pi$–periodic perturbation $\varepsilon(t,x)$ is $\epsilon$–small then $X_\varepsilon(t,x)$ still points transversally towards int $N$ at each point of $\partial N$ (for every $t \in [0,2\pi]$) and also $\mathrm{div}_x\ X_\varepsilon < 0$ for every $x \in N$. In particular, $\mathbb{R} \times N \subseteq \mathbb{R} \times \mathbb{R}^3$ descends to a compact, positively invariant set $\overline{N}$ in $\mathbb{S}^1 \times \mathbb{R}^3$, which therefore contains an attractor $K$ for $(S_\varepsilon)$. Once again, the inclusion $K \subseteq \overline{N}$ induces isomorphisms in Čech cohomology.

It is very easy to compute the cohomology of $\overline{N}$ in terms of the cohomology of $N$ (one may use Lemma 19 with $L = N$ and $g = \mathrm{id}$), and it turns out that

$$\check{H}^1(\overline{N};\mathbb{Q}) = \check{H}^1(N;\mathbb{Q}) \oplus \check{H}^0(N;\mathbb{Q})$$

so

$$\check{H}^1(K;\mathbb{Q}) = \check{H}^1(L;\mathbb{Q}) \oplus \check{H}^0(L;\mathbb{Q}) \neq \mathbb{Q} \oplus \mathbb{Q}$$

since $\check{H}^0(L;\mathbb{Q}) = \mathbb{Q}$ because $L$ is connected but $\check{H}^1(L;\mathbb{Q}) \neq \mathbb{Q}$ by assumption. Thus it follows from the alternative of Theorem 18 that $K$ contains periodic orbits. □

Models of population dynamics type are natural contexts where the existence of periodic behaviour is an important question. Many authors have studied them using two dimensional fixed point index techniques or reducing the original problem to a two dimensional one via the carrying simplex method (see [9], [11], [10], [17], [18] or [20] for some recent references). With the aid of Corollary 20 we can adopt a different point of view and obtain sufficient conditions for the existence of periodic behaviour provided there is an attractor and certain information about its cohomology is known. Let us illustrate this by briefly discussing a standard model for three interacting species.

Suppose $u_1(t)$, $u_2(t)$ and $u_3(t)$ denote the population, at time $t$, of three species (of course, $u_i \geq 0$). Taking into account the interaction of each species with the other ones, a frequently used model for the evolution in time of $u_i$ is the system

$$(*) \quad \dot{u_i} = \Big(a_i(t) - \sum_{j \in \{1,2,3\}} b_{ij}(t) u_j\Big) u_j$$

where $a_i$, $b_{ij}$ are $\mathcal{C}^1$ and $2\pi$–periodic, this last condition accounting for seasonal effects on population. It is to be assumed that $b_{ii} > 0$ for $i = 1, 2, 3$, which means that each species, if the others are not present, has a logistic behaviour. However, no condition is required on the sign of $b_{ij}$ with $i \neq j$ so we can consider simultaneously different interactions between the three species (cooperation, competition, etc.)

Consider the change of variables $u_i = \exp(x_i)$, which turns $(*)$ into

$$(**) \quad \dot{x_i} = a_i(t) - \sum_{j \in \{1,2,3\}} b_{ij}(t) \exp(x_j).$$

Denoting $X(t, x)$ the right hand side of $(**)$ one has

$$\mathrm{div}_x X(t, x) = - \sum_{i \in \{1,2,3\}} b_{ii}(t) \exp(x_i) < 0$$

so the Poincaré map $P$ associated with $(**)$ is volume contracting. Therefore an application of Corollary 20 shows that if $(*)$ has an attractor $L$ with $\check{H}^1(L; \mathbb{Q}) \neq \mathbb{Q}$ when the seasonal effects are discarded (so $a_i$, $b_{ij}$ do not depend on time) then it has an attractor with periodic behaviour when the seasonal effects are taken into account again, provided they are small enough.

## 6. Appendix: proof of Lemma 19

Let $\pi : L \times [0, 1] \longrightarrow L_g$ be the canonical projection. Identify $L$ with $\pi(L \times 0)$ and let $L' := \pi(L \times 1/2)$, which is another copy of $L$ placed halfway around the mapping torus. There is a homeomorphism $h : L \longrightarrow L'$ given by $h(\pi(p, 0)) := \pi(p, 1/2)$. Denote $i : L \subseteq L_g$ the inclusion. The unlabeled arrows in sequence (3) are induced by $i$. Our goal is to show that (3) is exact.

Consider the open sets $U := L_g - \pi(L \times [5/8, 7/8])$ and $V := L_g - \pi(L \times [1/8, 3/8])$. Refer to Figure 1.

We compute $\check{H}^*(L_g)$ using the Mayer–Vietoris sequence that corresponds to the decomposition $L_g = U \cup V$; namely

(4)

$$\ldots \longleftarrow \check{H}^d(L_g) \xleftarrow{\delta} \check{H}^{d-1}(U \cap V) \xleftarrow{\psi} \check{H}^{d-1}(U) \oplus \check{H}^{d-1}(V) \xleftarrow{\varphi} \check{H}^{d-1}(L_g) \longleftarrow \ldots$$

where $\delta$ denotes the connecting homomorphism, $\varphi(z) = (z|_U, z|_V)$ and $\psi(u, v) = u|_{U \cap V} - v|_{U \cap V}$. Here we have used the common notation for images of cocycles under inclusion induced homomorphisms; for instance $z|_U$ means $j^*(z)$, where $j : U \subseteq L_g$ is the inclusion, and so on.

Notice that the inclusion $L \cup L' \subseteq U \cap V$ is a homotopy equivalence, and so we may identify $\check{H}^*(U \cap V)$ with $\check{H}^*(L) \oplus \check{H}^*(L')$. Moreover, $\check{H}^*(L')$ can be further identified with $\check{H}^*(L)$ via $h^*$. Then we have that the map $\psi : \check{H}^*(U) \oplus \check{H}^*(V) \longrightarrow \check{H}^*(L) \oplus \check{H}^*(L)$ reads $\psi(u, v) = (u|_L - v|_L, h^*(u|_{L'}) - h^*(v|_{L'}))$.

The inclusions $L \subseteq U$ and $L \subseteq V$ are homotopy equivalences too, and again lead to identifications of $\check{H}^*(U)$ and $\check{H}^*(V)$ with $\check{H}^*(L)$. In particular, any cocycles $u$

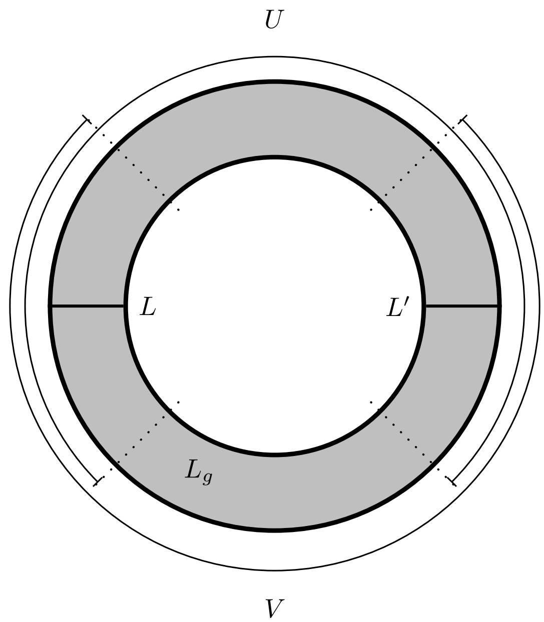


FIGURE 1. Computing the cohomology of $L_g$

and $v$ in $\check{H}^{d-1}(U)$ and $\check{H}^{d-1}(V)$ may be represented as extensions $u = \hat{z}$ and $v = \hat{w}$ of cocyles $z, w \in \check{H}^{d-1}(L)$. We want to express $\psi(u, v)$ in terms of $z$ and $w$ alone.

Trivially we have $u|_L = \hat{z}|_L = z$ and $v|_L = \hat{w}|_L = w$. We claim that $h^*(u|_{L'}) = z$ and $h^*(v|_{L'}) = g^*(w)$. To check this we may reason as follows. Denote $i'_U$ the inclusion of $L'$ in $U$ and $i'_V$ the inclusion of $L'$ in $V$, and similarly denote $i_U$ the inclusion of $L$ in $U$ and $i_V$ the inclusion of $L$ in $V$. It is easy to see that $i'_U \circ h \simeq i_U$ and $i'_V \circ h \simeq i_V \circ g$. From these relations it follows inmediately that $h^*(u|_{L'}) = u|_L$ and $h^*(v|_{L'}) = g^*(v|_L)$, which proves our claim.

From the above arguments it follows that $\psi$ can be expressed as a mapping $\psi : \check{H}^*(L) \oplus \check{H}^*(L) \longrightarrow \check{H}^*(L) \oplus \check{H}^*(L)$, with $\psi(z, w) = (z - w, z - g^*(w))$; a similar reasoning shows that $\varphi$ can be viewed as a mapping $\varphi : \check{H}^*(L_g) \longrightarrow \check{H}^*(L) \oplus \check{H}^*(L)$ with $\varphi(z) = (z|_L, z|_L)$. Recall that $z|_L$ is an alternative expression for $i^*(z)$, where $i : L \subseteq L_g$ denotes the inclusion. Thus (4) becomes

$$(5) \quad \ldots \longleftarrow \check{H}^d(L_g) \xleftarrow{\delta} \check{H}^{d-1}(L) \oplus \check{H}^{d-1}(L) \xleftarrow{\psi}$$
$$\xleftarrow{\psi} \check{H}^{d-1}(L) \oplus \check{H}^{d-1}(L) \xleftarrow{\varphi} \check{H}^{d-1}(L_g) \longleftarrow \ldots$$

with $\psi(z, w) = (z - w, z - g_*(w))$ and $\varphi(z) = (z|_L, z|_L)$.

Now let $\Delta : \check{H}^{d-1}(L) \longrightarrow \check{H}^d(L_g)$ be defined by $\Delta(z) := \delta(0, z)$. We are going to show that, with this definition, (3) is exact.

• <u>$\text{im } i^* = \ker\, (\text{id} - g^*)$</u>. Let $z \in \check{H}^{d-1}(L)$. Then $(\text{id} - g^*)(z) = 0 \Leftrightarrow z = g^*(z) \Leftrightarrow \psi(z, z) = 0 \Leftrightarrow (z, z) \in \text{im } \varphi \Leftrightarrow z \in \text{im } i^*$.

• <u>$\text{im } (\text{id} - g^*) = \ker \Delta$</u>. Let $z \in \check{H}^{d-1}(L)$. Then $\Delta(z) = 0 \Leftrightarrow \delta(0, z) = 0 \Leftrightarrow (0, z) \in \text{im } \psi \Leftrightarrow (0, z) = \psi(z', w')$ for some $(z', w')$, but since $\psi(z', w') = (z' - w', z' - g^*(w'))$, the latter is equivalent to $z = z' - g^*(z') \Leftrightarrow z \in \text{im } (\text{id} - g^*)$.

• im $\Delta$ = ker $i^*$. Before proving this observe the following: for any $(z', w')$, the equality $\delta(z', w') = \delta(0, w' - z')$ holds. Indeed: we have $(-z', -z') = \psi(-z', 0) \in$ im $\psi$ = ker $\delta$, so $\delta(z', w') = \delta((z', w') + (-z', -z')) = \delta(0, w' - z')$. Now, $i^*(z) = 0 \Leftrightarrow \varphi(z) = (0, 0) \Leftrightarrow z = \delta(z', w') = \delta(0, w' - z') = \Delta(w' - z')$.

## References


[1] K. Borsuk, *Concerning homotopy properties of compacta*, Fund. Math. 62 (1968) 223-254.

[2] K. Borsuk, *Theory of shape*, Monografie Matematyczne 59, Polish Scientific Publishers, Warszawa 1975.

[3] J. Dydak, J. Segal, *Shape theory: an introduction*, Lecture Notes in Math. 688. Springer-Verlag, Berlin 1978.

[4] A. Giraldo, M.A. Morón, F.R. Ruiz del Portal, J.M.R. Sanjurjo, *Shape of global attractors in topological spaces*, Nonlinear Analysis T.M.A. 60 (2005) 837847.

[5] M. Gobbino, *Topological properties of attractors for dynamical systems*, Topology 40 (2000) 279-298.

[6] J.B. Günther, J. Segal, *Every attractor of a flow on a manifold has the shape of a finite polyhedron*, Proc. Amer. Math. Soc. 119 (1993) 321-329.

[7] H.M. Hastings, *A higher dimensional Poincaré-Bendixson theorem*, Glasnik Mat. 34 (1979) 263-268.

[8] A. Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002.

[9] J. Jezierski, W. Marzantowicz, *Homotopy Methods in Topological Fixed and Periodic Points Theory*, Springer, 2005.

[10] J. Jiang, Y. Wang, *Uniqueness and attractivity of the carrying simplex for discrete-time competitive dynamical systems*, J. Differential Equations 186 (2002), 611-632.

[11] J. Jiang, J. Mierczyński, Y. Wang, *Smoothness of the carrying simplex for discrete-time competitive dynamical systems: a characterization of neat embedding*, J. Differential Equations 246 (2009), 1623-1672.

[12] L. Kapitanski, I. Rodnianski, *Shape and Morse theory of attractors*, Communications in Pure and Applied Math. 53, (2000) 218-242.

[13] P. Le Calvez, F.R. Ruiz del Portal, J.M. Salazar, *Fixed point index of the iterates of $R^3$-homeomorphisms at fixed points which are isolated invariant sets*, J. London Math. Soc. 83 (2010), pp. 683-696.

[14] J. Milnor, *On axiomatic homology theory*, Pacific J. Math. 12 (1962) 337-341.

[15] S. Mardešić, J. Segal, *Shape theory*, North-Holland, Amsterdam, 1982.

[16] M.A. Morón, F.R. Ruiz del Portal, *A note about the shape of attractors of discrete semidynamical systems*, Proc. Amer. Math. Soc. 134 (2006), no. 7, 2165-2167.

[17] R. Ortega, F.R. Ruiz del Portal, *Attractors with vanishing rotation number*, J. Eur. Math. Soc. (JEMS) 13 (2011) 1569-1590.

[18] A. Ruiz-Herrera, *Topological criteria of global attraction with applications in population dynamics*, Nonlinearity 25 (2012) 2823-2841.

[19] J.J. Sánchez-Gabites, *On the Borsuk shape of attractors of discrete dynamical systems*, preprint.

[20] W. Shen, Y. Wang, *Carrying simplices in nonautonomous and random competitive Kolmogorov systems*, J. Differential Equations 245 (2008), 1-29.