Attractors with vanishing rotation number

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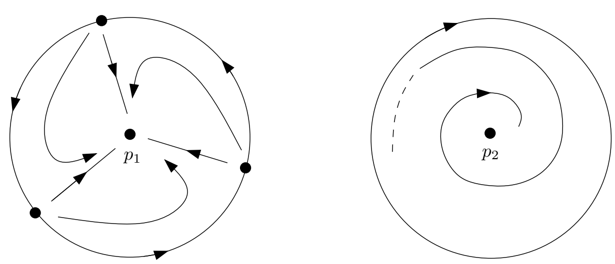
1 Introduction

Rotation numbers can be assigned to attractors in two dimensions. This is illustrated by the figures below



At an intuitive level we can say that the rotation number vanishes for p_1 and must be a non-zero number (linked to the period of the closed orbit) for p_2. In this paper we analyze some properties of attractors with zero rotation number.

The idea of associating a rotation number to an attractor has its origin in the work by Birkhoff [5] and was fully developed by Cartwright and Littlewood in [10]. More recently Alligood and Yorke [2] have used these ideas to explore fractal boundaries. Our approach has many points in common with [2] but our goal is somehow different. We place the emphasis on results about global attraction and their applicability to differential equations. To be more precise, we consider an orientation-preserving homeomorphism of the plane, denoted by h, and a fixed point $p = h(p)$ that is asymptotically stable. The region of attraction

$$U = \{q \in \mathbb{R}^2 : \lim_{n\to+\infty} h^n(q) = p\}$$

is an open and simply connected subset of $\mathbb{R}^2$. While this implies that U is homeomorphic to the open unit disc the boundary of U can have a complicated structure. Assuming that $U \neq \mathbb{R}^2$, Carathéodory's theory of prime ends associates a copy of $\mathbb{S}^1$ to the boundary of U and the map h induces an orientation-preserving homeomorphism of $\mathbb{S}^1$. The corresponding rotation number will be denoted by $\rho = \rho(h, U)$. Our main result says that if $\rho = 0$, h is dissipative and U is unbounded then there exists a fixed point lying in $\mathbb{R}^2 \setminus U$. As a corollary one obtains a criterion for global attraction when p is the unique fixed point. Dissipativity means that ∞ is a repeller for h. The assumption on the unboundedness of the region of attraction is satisfied as soon as h is area-contracting. These are typical assumptions motivated by the theory of nonlinear oscillations.

For maps h coming from differential equations it is not easy to determine the rotation number. This fact was pointed out by Cartwright and Littlewood when they were dealing with the forced Van der Pol equation (see in particular section 7.1 of [10]). In order to make our result applicable we obtain some criteria for the computation of the rotation number. From here we derive consequences for orientation-reversing maps, extinction in population dynamics or global attraction in nonlinear oscillators.

The paper is organized as follows. The main result is stated and proved in Section 4, after two sections on preliminaries. Criteria for $\rho = 0$ are obtained in Section 5. The last two sections, 6 and 7, are devoted to applications.

2 Asymptotic stability, prime ends and rotation numbers

We work on the plane $\mathbb{R}^2$ and sometimes on the Riemann sphere $\mathbb{S}^2 = \mathbb{R}^2 \cup \{\infty\}$. The topological operations of closure, boundary and interior will be denoted by $\mathrm{cl}(A)$, ∂A and $\mathrm{int}(A)$ and understood relative to the plane. If these operations are taken with respect to the sphere, it will be explicitly indicated in the notation. A set D contained in the plane or the sphere is a (topological) disc if it is homeomorphic to

$$\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}.$$

The class of homeomorphisms of the plane is denoted by $\mathcal{H}$. The notation $\mathcal{H}_+$ will be employed for the subclass of orientation-preserving homeomorphisms. Similarly $\mathcal{H}_-$ is employed for orientation-reversing maps in $\mathcal{H}$.

Assume now that p is a fixed point of a map $h \in \mathcal{H}$. We say that p is *stable* if there exists a basis of positively invariant neighborhoods. This means that there exists a sequence of open sets $\{\mathcal{U}_n\}$ satisfying

$$\mathcal{U}_{n+1} \subset \mathcal{U}_n, \quad \bigcap_{n\geq 0} \mathcal{U}_n = \{p\}, \quad h(\mathcal{U}_n) \subset \mathcal{U}_n.$$

The region of attraction of the fixed point p is defined as

$$U = \{x \in \mathbb{R}^2 : \lim_{n\to\infty} h^n(x) = p\}$$

and it is an invariant set. The point p is called an *attractor* when it is contained in the interior of U. A fixed point is *asymptotically stable* whenever it is a stable attractor.

Given an asymptotically stable fixed point, the region of attraction is an open and simply connected subset of the plane (see [4]). A classical result due to Kerékjarto [16, 6] says that the restriction of h to U is topologically conjugate to one of the following maps in $\mathbb{C}$,

$$z \mapsto \frac{1}{2}z \quad \text{or} \quad z \mapsto \frac{1}{2}\overline{z}.$$

Notice that this alternative depends on whether h belongs to $\mathcal{H}_+$ or to $\mathcal{H}_-$. Extensions to higher dimensions of this result can be found in [13] and [14].

The above result shows that we must go to the boundary of the region of attraction, ∂U or $\partial_{\mathbb{S}^2} U$, to establish topological differences among asymptotically stable fixed points. The simplest instance occurs when $U = \mathbb{R}^2$ and the point is called *globally asymptotically stable*. When $U \neq \mathbb{R}^2$ and h is orientation-preserving, it is possible to assign a rotation number to the fixed point. To this end we must enter into Carathéodory's theory of prime ends applied to $\partial_{\mathbb{S}^2} U$. We follow [23] and discuss briefly this theory. A crosscut C of U is a Jordan arc in $\mathbb{S}^2$ that lies on U excepting for the two endpoints. Notice that if U is unbounded one of the endpoints can be ∞. The complement $U \setminus C$ is split in two connected components. The next step is to define a null-chain and for this purpose we fix a point x_0 in U and consider a sequence of crosscuts $\{C_n\}$ with the following properties,

$$C_n \cap C_m = \emptyset, \;\; \text{diam } C_n \to 0 \text{ as } n \to \infty \;\; \text{and } V_{n+1} \subset V_n,$$

where V_n is the component of $U \setminus C_n$ not containing x_0.

In the previous conditions the diameter of C_n must be understood on the Riemann sphere.

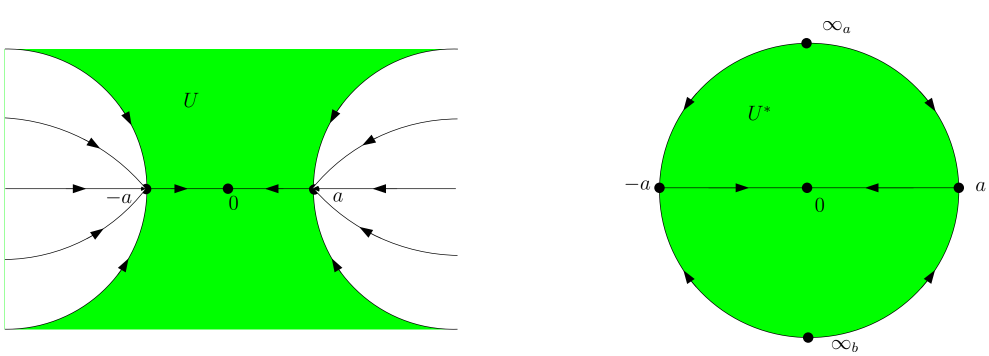
Two null-chains (C_n) and (C'_n) are equivalent if, given m,

$$V_n \subset V'_m, \quad V'_n \subset V_m$$

for n large enough.

The space of prime ends $\mathbb{P} = \mathbb{P}(U)$ is composed by the equivalence classes of null-chains. The disjoint union $U^\star = U \cup \mathbb{P}(U)$ becomes a topological space in the natural way. The class of open sets contains the subsets of $U^\star$ determined by a crosscut and also the open subsets of U. Indeed $U^\star$ is homeomorphic to $\mathbb{D}$ and its boundary is precisely $\mathbb{P}$. In Carathéodory's approach this homeomorphism was obtained as an extension of the conformal map between $\mathrm{int}(\mathbb{D})$ and U given by Riemann's theorem on conformal mappings. After these definitions it is easy to extend $h : U \cong U$ to a homeomorphism of $U^\star$. This homeomorphism will preserve the boundary and the restriction to $\mathbb{P}$ will be denoted by $h^\star : \mathbb{P} \to \mathbb{P}$. Since $\mathbb{P}$ is homeomorphic to $\mathbb{S}^1$, $h^\star$ is conjugate to a homeomorphism of the unit circle. This homeomorphism is orientation preserving if and only if $h \in \mathcal{H}_+$. This is a consequence of well known results in the theory of manifolds with boundary. The rotation number is defined for orientation preserving homeomorphisms of $\mathbb{S}^1$. If it interpreted as an angle rather than a number it becomes an invariant under conjugacy. We will assume that rotation numbers are defined in $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and denote by $\rho(h, U)$ the rotation number of $h^\star$. To simplify the notation we will identify $\rho(h, U)$ with a number in the interval $[0, 1[$.

To illustrate the previous discussions we consider a flow in the plane as indicated in the next figure.



The origin is a local attractor while the equilibria a and $-a$ are unstable. We assume that this flow has been parameterized so that all solutions are defined in the whole real line and consider the homeomorphism h_1 determined by the time-1 map associated to this flow. The region of attraction U is the shaded region and the space $\mathbb{P}$ can be identified to ∂U together with two prime ends representing infinity from above and below, say ∞_a and ∞_b. To describe the dynamics of $h_1^\star$ we observe that it has four fixed points, the attractors a and $-a$ and the repellers ∞_a and ∞_b. This implies that $\rho(h_1, U) = \overline{0}$. Consider now a homeomorphism in $\mathcal{H}_-$ obtained as the composition $h_2 = S_y \circ h_1$, where S_y is the symmetry with respect to the y-axis. The region of attraction U does not change and so the space $\mathbb{P}$ remains the same. The dynamics of $h_2^\star$ is as follows, the fixed points ∞_a and ∞_b are repellers and $\{a, -a\}$ is an attracting 2-cycle. For $h_3 = S_x \circ h_1$ there is a repelling 2-cycle $\{\infty_a, \infty_b\}$ and two attracting fixed points a and $-a$.

In the previous discussions the fixed point p could be replaced by an invariant continuum $K \subset \mathbb{R}^2$ having trivial shape. We recall that a planar continuum has trivial shape if and only if the complement does not decompose the plane (see [8]). Indeed this more general situation can be reduced to the case of fixed points. To do this it is sufficient to observe that the quotient space $\mathbb{R}^2/K$ is homeomorphic to $\mathbb{R}^2$ (see page 313 in [8]) and so the induced map $\overline{h} : \mathbb{R}^2/K \cong \mathbb{R}^2/K$ has a fixed point $|K|$. Moreover this fixed point is asymptotically stable whenever K is a stable attractor.

Finally we observe that the construction of $h^\star$ does not requires an asymptotically stable fixed point. It is sufficient to start with an open and simply connected subset of the plane which is proper and invariant under h. For this reason we have chosen the notation $\rho(h, U)$ instead of making reference to the fixed point p.

3 Dissipative and area-contracting homeomorphisms

A map $h \in \mathcal{H}$ is called *dissipative* if there exists a compact set $W \subset \mathbb{R}^2$ that is positively invariant and attracts uniformly all compact sets. This means that $h(W) \subset W$ and for each $x \in \mathbb{R}^2$

$$\operatorname{dist}(h^n(x), W) \to 0 \;\text{ as } n \to \infty$$

uniformly on balls $||x|| \leq r$, $r > 0$.

This notion can be presented in several equivalent formulations and we refer to [18, 21, 24] for more information on this class of mappings. Dissipativity can be interpreted in terms of stability theory. To do this we first extend h to $\mathbb{S}^2$ with $h(\infty) = \infty$. Then ∞ becomes a fixed point under h and the notions introduced in the previous section can be adapted. With some work it can be proven that the dissipativity of h is equivalent to the asymptotic stability of ∞ with respect to h^{-1}. As a consequence of this observation we can deduce a result that will be useful later.

Lemma 1 *Assume that $h \in \mathcal{H}$ is dissipative. Then there exists a sequence of topological discs $\{L_n\}$ in $\mathbb{S}^2$ satisfying*

$$\infty \in \mathrm{int}_{\mathbb{S}^2}(L_n), \quad \bigcap_n L_n = \{\infty\}, \quad L_{n+1} \subset \mathrm{int}_{\mathbb{S}^2}(L_n)$$

and

$$L_n \subset \mathrm{int}_{\mathbb{S}^2}(h(L_n)).$$

Proof. The region of attraction of ∞ with respect to h^{-1} will be denoted by R. This is an open set in $\mathbb{S}^2$ that is invariant under h. Moreover on this set h^{-1} is topologically conjugate to $z \mapsto \frac{1}{2}z$ or $z \mapsto \frac{1}{2}\overline{z}$. In consequence the restriction of h to R is conjugate to $z \mapsto 2z$ or to $z \mapsto 2\overline{z}$. For these maps the discs $|z| \geq n$ satisfy the required conditions.$\square$

A map $h \in \mathcal{H}$ is called *area-contracting* if the Lebesgue measure μ is contracted under the action of h. This means that

$$\mu(h(B)) < \mu(B)$$

for each Borel set B of $\mathbb{R}^2$.

These maps have an important property: the measure of any invariant Borel set is either zero or infinity. If U is the region of attraction of an asymptotically stable fixed point, the measure must be infinite since we are dealing with an open set. In particular the region U is always unbounded.

4 A fixed point theorem

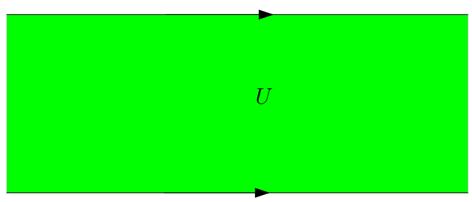
In this section we prove the existence of a fixed point outside an invariant region with zero rotation number.

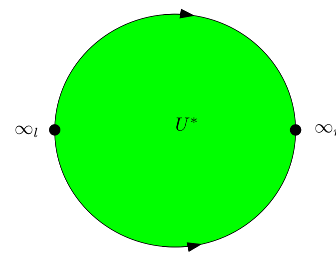
Proposition 2 *Assume that $h \in \mathcal{H}_+$ is dissipative and U is a simply connected open subset of the plane that is unbounded and proper, $\emptyset \neq U \subsetneq \mathbb{R}^2$. In addition,*

$$h(U) = U, \quad \rho(h, U) = 0.$$

Then h has a fixed point in $\mathbb{R}^2 \setminus U$.

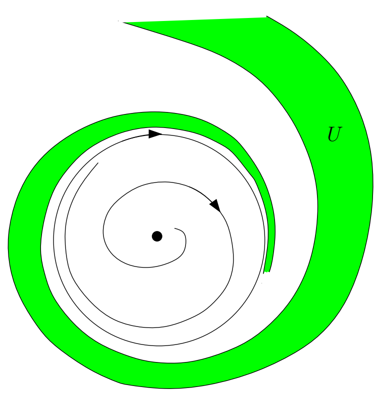
The dissipativity of h is essential. This can be shown by considering the translation $h(x_1, x_2) = (x_1 + 1, x_2)$ and the set $U = \mathbb{R}\times]0,1[$. This map has no fixed points and the rotation number on U vanishes. To justify that $\rho(h, U) = 0$ it is enough to describe the dynamics of $h^\star$ and observe that it has two fixed points.





The result by Barge and Gillette in [3] leads to a result similar to Proposition 2 but for bounded domains. The typical strategy of Complex Analysis of extending results from bounded to unbounded domains by working on the Riemann sphere is not applicable in this case. In the assumptions of Proposition 2 the point of infinity is always a fixed point lying on $\partial_{\mathbb{S}^2} U$.

It can happen that the fixed point found by Proposition 2 does not lie on the boundary of U. As an example we can consider the time 1 map associated to the van der Pol flow. The region U is determined by two orbits emanating from infinity and attracted by the closed orbit. Assuming that 1 is not a period of the closed orbit it is easy to verify that all the assumptions of Proposition 2 hold and the only fixed point is the unstable equilibrium.



Before the proof we need some additional information on prime ends and several preliminary results. Given a prime end p in $\mathbb{P}$, a point $x \in \mathbb{S}^2$ is a principal point of p if the prime end can be represented by a null-chain (C_n) such that the sequence of sets C_n converges to x. The set of principal points of p is denoted by $\Pi(p)$. It is a non-empty continuum. From the definitions it is clear that if $h^\star(p) = p_1$ then $h(\Pi(p)) = \Pi(p_1)$. In particular $\Pi(p)$ is an invariant continuum if p is fixed under $h^\star$. Assume now that γ is a crosscut of U with end points a, b lying in $\partial_{\mathbb{S}^2} U$. We employ the notation $\dot{\gamma} = \gamma \setminus \{a, b\}$. Also we denote by V one of the two components of $U \setminus \gamma$. The set of all prime ends that can be represented by null-chains (C_n) with $\dot{C}_n \subset V$ determines an arc in $\mathbb{P}$. To justify this assertion we employ Proposition 2.14 and Theorem 2.15 in [22]. The closure of this arc will be denoted by $\alpha_V = \widehat{pq}$ and the end points p, q are such that $\Pi(p) = \{a\}$ and $\Pi(q) = \{b\}$. The simple structure of the principal sets in this case is a consequence of Corollary 2.17 in [22]. This is also implied by Lemma 5.1 in [2].

The Riemann conformal map extends to a homeomorphism between $\mathbb{D}$ and $U^\star$ and the pre-image of $\dot{\gamma}$ is an open arc in $\mathrm{int}(\mathbb{D})$ ending at two different points of the boundary. Translating this observation from the disc to $U^\star$ we observe that $\dot{\gamma} \cup \{p, q\}$ is an arc in $U^\star$.

We also make some remarks on accessible points. A point $b \in \partial_{\mathbb{S}^2} U$ is accessible if there exists an arc β such that b is an end point and $\beta \setminus \{b\}$ is contained in U. In this case there exists a prime end $p_\beta \in \mathbb{P}$ such that $\beta^\star = (\beta \setminus \{b\}) \cup \{p_\beta\}$ is an arc in $U^\star$ and $\Pi(p_\beta) = \{b\}$. This is again a consequence of the above mentioned results in [22]. Finally we recall that accessible points are dense on the boundary of U.

Lemma 3 *Assume that $p \in \mathbb{P}$ is fixed under $h^\star$ and $\infty \notin \Pi(p)$. Then h has a fixed point lying in $\mathbb{R}^2 \setminus U$.*

Proof. The set $\Pi(p)$ is a non-empty continuum contained in $\mathbb{R}^2$. If this set has trivial shape we can apply the Cartwright-Littlewood Fixed Point Theorem [10, 9] to deduce the existence of a fixed point lying in $\Pi(p)$. Since this set is contained in ∂U we have found the searched point. When $\Pi(p)$ has no trivial shape one decomposes $\mathbb{S}^2 \setminus \Pi(p)$ in a family of connected components $\{\Omega_\lambda\}_{\lambda \in \Lambda}$ with

$$\bigcup_{\lambda \in \Lambda} \Omega_\lambda = \mathbb{S}^2 \setminus \Pi(p).$$

The component containing ∞ is denoted by Ω_{λ_0}. U is a connected subset of $\mathbb{S}^2 \setminus \Pi(p)$ and so it must be contained in some component, being unbounded this component is precisely Ω_{λ_0}. The set

$$K = \mathbb{S}^2 \setminus \Omega_{\lambda_0}$$

is non-empty and compact. Moreover it is invariant under h. This is so because Ω_{λ_0} is invariant. To prove that K is connected we observe that this set can be expressed as

$$K = \bigcup_{\lambda \neq \lambda_0} \Omega_\lambda \cup \Pi(p) = \bigcup_{\lambda \neq \lambda_0} \mathrm{cl}(\Omega_\lambda) \cup \Pi(p).$$

Thus K is a union of continua contained in a common disc and all of them have non-empty intersection with $\Pi(p)$. This implies that K is a continuum. We know from its definition that K has trivial shape and so the Cartwright-Littlewood Theorem is applicable. The fixed point lying on K is not in U because $K \cap U = \emptyset$.

Remark. The fixed point belongs to ∂U when h is area-contracting. Notice that $\Pi(p)$ must have trivial shape, for otherwise the components Ω_λ with $\lambda \neq \lambda_0$ would be invariant open sets with positive (but finite) measure.

Lemma 4 *Assume that γ is a crosscut of U satisfying*

$$\gamma \cap h(\gamma) = \emptyset$$

and let V be a component of $U \setminus \gamma$ such that $U \setminus V$ contains at least one fixed point. Then one of the following alternatives holds

$$(i) \qquad h(V) \subset V \quad \textit{and} \quad h^\star(\alpha_V) \subset \dot{\alpha}_V$$

$$(ii) \qquad V \subset h(V) \quad \textit{and} \quad \alpha_V \subset h^\star(\dot{\alpha}_V)$$

$$(iii) \qquad V \cap h(V) = \emptyset \quad \textit{and} \quad \alpha_V \cap h^\star(\dot{\alpha}_V) = \emptyset.$$

Proof. We first notice that the arcs α_V and $h^\star(\alpha_V)$ cannot have common end points. That is,

$$\{p, q\} \cap \{p_1, q_1\} = \emptyset \tag{1}$$

where $\alpha_V = \widehat{pq}$ and $h^\star(\alpha_V) = \widehat{p_1 q_1}$. This will be proved by contradiction, assuming that $p = q_1$ (the other cases are analogous). Since $q_1 = h^\star(q)$ we deduce that

$$\{a\} = \Pi(p) = \Pi(q_1) = h(\Pi(q)) = \{h(b)\}.$$

But $a = h(b)$ would imply that $\gamma \cap h(\gamma) \neq \emptyset$. Once we know that (1) holds we observe the following alternative for the positions of V and $h(V)$,

$$h(V) \subset V, \qquad V \subset h(V), \qquad V \cap h(V) = \emptyset.$$

The proof of this trichotomy will require some work. Let W denote the other component of $U \setminus \gamma$. We observe that $W \cap h(W)$ is non-empty since it contains a fixed point. The set $h(\dot{\gamma})$ is connected and so it must lie in one of the components of $U \setminus \gamma$. We distinguish two cases:

- **Case 1:** $h(\dot{\gamma}) \subset V$.

  This implies that $h(\gamma) \cap W = \emptyset$. The components of $U \setminus h(\gamma)$ are $h(V)$ and $h(W)$ and so the connected set W must lie in one of them. Since we know that $W \cap h(W) \neq \emptyset$ we conclude that $W \subset h(W)$. Taking complements with respect to U,

  $$h(V) \cup h(\dot{\gamma}) \subset V \cup \dot{\gamma}.$$

  Since $h(V)$ is open we deduce that $h(V)$ is contained in V and the first alternative holds.

- **Case 2:** $h(\dot{\gamma}) \subset W$

  Now $h(\gamma) \cap V = \emptyset$ and so either $V \subset h(V)$ or $V \subset h(W)$. The first inclusion is precisely the second alternative. From the inclusion $V \subset h(W)$ we deduce that

  $$V \cap h(V) \subset h(W) \cap h(V) = \emptyset$$

  and the third alternative holds.

Now the rest of the proof follows from the definition of α_V. Assuming for instance that $h(V) \subset V$ then $h^\star(\alpha_V) \subset \alpha_V$, implying by (1) that $h^\star(\alpha_V) \subset \dot{\alpha}_V$. The other two cases are treated similarly.

Remark. The previous proof also applies to generalized crosscuts with the same end points. By this we mean a set γ that is homeomorphic to $\mathbb{S}^1$ and has a point $a \in \gamma \cap \partial_{\mathbb{S}^2} U$ with $\gamma \setminus \{a\} \subset U$. Notice that in this case α_V can be a singleton, a proper arc or the whole space $\mathbb{P}$.

Proof of Proposition 2. The homeomorphism h is dissipative and so it has at least one fixed point. From now on we assume that

$$\mathrm{Fix}(h) \subset U,$$

for otherwise the result is already proved. The dissipativity also implies that $\mathrm{Fix}(h)$ is compact and so it is possible to find a topological disc D such that

$$\mathrm{Fix}(h) \subset \mathrm{int}(D) \subset D \subset U. \tag{2}$$

We also fix an arc Γ with the following properties:

$$\Gamma = \widehat{\xi\eta}, \quad \xi \in \mathrm{Fix}(h), \quad \eta \in \partial U, \quad \Gamma \setminus \{\eta\} \subset U. \tag{3}$$

Since $\rho(h, U) = 0$ we know that $h^\star$ has at least one fixed point. We distinguish two cases: a) ∞ is not a principal point of some fixed prime end, b) ∞ is a principal point of each fixed prime end. In view of lemma 3 it is clear that we can find a fixed point of h outside U in case a). The rest of the proof will be devoted to showing that case b) cannot occur. We proceed by absurd and assume that

$$\infty \in \Pi(p), \quad \text{for each } p \in \mathrm{Fix}(h^\star). \tag{4}$$

Our goal is to arrive at a contradiction.

After an application of Lemma 1 we find a disc L in $\mathbb{S}^2$ with

$$D \cap L = \emptyset, \quad \Gamma \cap L = \emptyset, \quad \infty \in \mathrm{int}_{\mathbb{S}^2}(L), \quad L \subset \mathrm{int}_{\mathbb{S}^2} h(L).$$

The set U intersects both components of $\mathbb{S}^2 \setminus \partial L$ and so

$$U \cap \partial L \neq \emptyset.$$

We can apply Proposition 2.13 in [22] and deduce the existence of a family of crosscuts $\{\gamma_\lambda\}_{\lambda \in \Lambda}$, $\emptyset \neq \Lambda \subset \mathbb{N} \setminus \{0\}$, such that $\gamma_\lambda \subset \partial L$ and U can be expressed as a disjoint union

$$U = U_0 \cup \bigcup_{\lambda \in \Lambda} U_\lambda \cup \bigcup_{\lambda \in \Lambda} \dot{\gamma}_\lambda,$$

where U_0 is the component of $U \setminus \partial L$ containing ξ and U_λ is a domain with

$$\dot{\gamma}_\lambda = U \cap \partial U_\lambda \subset U \cap \partial U_0.$$

Each crosscut γ_λ splits U in two components and the proof of Proposition 2.13 in [22] shows that U_λ is precisely the component of $U \setminus \gamma_\lambda$ not containing ξ. Then U_λ determines an arc α_{U_λ} in $\mathbb{P}$. From now on this arc will be simply denoted by α_λ.

The crosscuts in [22] are understood in a generalized sense and it can happen that the two end points of γ_λ coincide. This is a rather exceptional situation and when it occurs the set of indexes Λ must be a singleton. As a hypothetical illustration of this situation the reader can consider $U = \mathbb{R}^2 \setminus ([0, \infty[\times \{0\})$, $\partial L = \mathbb{S}^1$. It is important to observe that α_λ is always a proper arc in $\mathbb{P}$, that is, $\alpha_\lambda = \widehat{pq}$ with $p \neq q$. This is clear for a standard crosscut because p and q have different principal points. Next we prove that it also holds for generalized crosscuts. In principle we know that α_λ is a connected set in $\mathbb{P}$ and so it is enough to prove that it contains more than one point but not the whole circle. The point η is accessible through the arc Γ defined in (3) and so we can find $p_\Gamma \in \mathbb{P}$ with $\Pi(p_\Gamma) = \{\eta\}$ and such that $\Gamma^\star = (\Gamma \setminus \{\eta\}) \cup \{p_\Gamma\}$ is an arc in $U^\star$. We claim that p_Γ is not in α_λ. Indeed given any chain of crosscuts (C_n) defining p_Γ we observe that $C_n \to \{\eta\}$. Therefore $C_n \cap L = \emptyset$ for n large enough. From here we deduce that $\dot{C}_n$ is contained either in U_0 or in another component U_λ. The definition of p_Γ implies that $\dot{\Gamma}$ and $\dot{C}_n$ must intersect. Since $\Gamma \setminus \{\eta\}$ is contained in U_0 we deduce that $\dot{C}_n \subset U_0$. Taking into account the definition of α_λ we observe that some open arc around p_Γ does not intersect α_λ. Now we know that $\alpha_\lambda \neq \mathbb{P}$. To prove that α_λ is non-empty we observe that accessible points are dense on the boundary of U. Taking accessible points in $\partial_{\mathbb{S}^2} U \cap \mathrm{int}_{\mathbb{S}^2}(L)$ one can construct infinitely many points in α_λ.

We point out two additional properties of these arcs:
(i) the end points of α_λ are not fixed under $h^\star$,
(ii) $\dot{\alpha}_\lambda \cap \dot{\alpha}_\mu = \emptyset$ if $\lambda \neq \mu$.
The first property is a consequence of (4) since the principal points associated to the end points of α_λ belong to ∂L. The second property holds because U_λ and U_μ are disjoint.

Next we observe that

$$h(\gamma_\lambda) \cap \gamma_\lambda \subset h(\partial L) \cap \partial L = \emptyset$$

and so Lemma 4 and the remark after its proof are applicable. Thus one of the alternatives $h(U_\lambda) \subset U_\lambda$, $U_\lambda \subset h(U_\lambda)$ or $U_\lambda \cap h(U_\lambda) = \emptyset$ holds. Also, $h^\star(\alpha_\lambda) \subset \dot{\alpha}_\lambda$, $\alpha_\lambda \subset h^\star(\dot{\alpha}_\lambda)$ or $\alpha_\lambda \cap h^\star(\alpha_\lambda) = \emptyset$.

The next step is to prove that $\{\dot{\alpha}_\lambda\}_{\lambda \in \Lambda}$ is an open covering of $\mathrm{Fix}(h^\star)$. To this end we take $p \in \mathrm{Fix}(h^\star)$ and select a chain of crosscuts (C_n) determining p and such that $C_n \to \infty$. Here we are using that ∞ is a principal point thanks to (4). For n large enough, C_n will be contained in $\mathrm{int}_{\mathbb{S}^2}(L)$. The connected set $\dot{C}_n$ must be contained in some component of $U \setminus \partial L$. This component cannot be $U_0 \subset \mathbb{S}^2 \setminus L$. Let $\lambda \in \Lambda$ be the index such that $\dot{C}_n$ is contained in U_λ. As noticed in [22] after Proposition 2.13, this index is unique. We conclude from the definition of α_λ that p belongs to this arc. Moreover p cannot be an end point of α_λ. Indeed $\infty \in \Pi(p)$ and the end points of α_λ have as unique principal points the end points of γ_λ.

Once we know that $\{\dot{\alpha}_\lambda\}$ is an open covering of the compact set $\mathrm{Fix}(h^\star)$, we can extract a finite sub-covering $\{\dot{\alpha}_\lambda\}_{\lambda \in F}$. This sub-covering can be chosen so that every open arc contains at least one fixed point. This excludes the possibility $\alpha_\lambda \cap h^\star(\alpha_\lambda) = \emptyset$ and so we must have either $h^\star(\alpha_\lambda) \subset \dot{\alpha}_\lambda$ or $\alpha_\lambda \subset h^\star(\dot{\alpha}_\lambda)$ for each $\lambda \in F$.

The rest of the proof is based on the theory of fixed point index on polyhedra (see [12] or [15]). First we construct the double space $DU^\star$ from the disc $U^\star$. This space is obtained from two copies of $U^\star$ by identifying the boundaries. It is homeomorphic to $\mathbb{S}^2$ and can be split as

$$DU^\star = U^+ \cup U^- \cup \mathbb{P},$$

where $\mathbb{P}$ is the equator and the two hemispheres are composed by points $(x, +)$ and $(x, -)$ with $x \in U$. Next we construct the homeomorphism $H : DU^\star \to DU^\star$ defined as

$$H(x, \pm) = (h(x), \pm) \;\text{ if } x \in U, \quad H(p) = h^\star(p) \;\text{ if } p \in \mathbb{P}.$$

Since h is orientation preserving the same property holds for H. In consequence the Lefschetz number of H is precisely 2, the Euler characteristic of the sphere. The set of fixed points of H can be decomposed in three parts, those lying on $D^+ = D \times \{+\}$ or $D^- = D \times \{-\}$ and those on the equator. We are going to compute the corresponding fixed points indexes. First we observe that, since h is dissipative, the index of h on large balls of the plane is 1. In view of (2) also the index on D is 1. This implies that

$$i(H, D^+) = i(H, D^-) = 1. \tag{5}$$

The fixed points on the equator are precisely the prime ends fixed under $h^\star$ and they are covered by the family of discs $\{\Delta_\lambda\}_{\lambda \in F}$, where each Δ_λ is the closure in $DU^\star$ of $U_\lambda^+ \cup U_\lambda^-$ with $U_\lambda^\pm = U_\lambda \times \{\pm\}$. We observe that these discs contain the arc α_λ and their boundary is composed by $\gamma_\lambda^\pm$ and the end points of α_λ. Thus H does not have fixed points in the boundary of Δ_λ and one of the following conditions holds, either $H(\Delta_\lambda) \subset \Delta_\lambda$ or $\Delta_\lambda \subset H(\Delta_\lambda)$. In any of the two cases one has,

$$i(H, \Delta_\lambda) = 1. \tag{6}$$

This is a consequence of Lemma 2.2.25 in [15] for the attracting case. For the repelling case it is enough to observe that $i(H, \Delta_\lambda) = i(H^{-1}, \Delta_\lambda)$. As a consequence of the properties (i) and (ii) of the family of arcs we deduce that, for $\lambda \neq \mu$, Δ_λ and Δ_μ can only intersect at their boundaries. Since $\xi \in D \cap U_0$ and D is a connected subset of $U \setminus \partial L$ we conclude that $D \subset U_0$.

In consequence $D^{\pm}$ is disjoint with Δ_λ. This allows us to combine the excision property with Hopf index theorem to conclude that

$$2 = \chi(\mathbb{S}^2) = i(H, DU^\star) = i(H, D^+) + i(H, D^-) + \sum_{\lambda \in F} i(H, \Delta_\lambda).$$

Since the set of indexes F is non-empty we arrive at a contradiction with (5) and (6).

In view of the remark after the proof of Lemma 3 the previous proof can be modified to obtain a refinement for area-contracting maps.

Corollary 5 *Assume that all the conditions of Proposition 2 holds. In addition h is area-contracting and there exists a topological disc $D \subset U$ such that*

$$\mathrm{Fix}(h) \cap U \subset \mathrm{int}(D) \quad \textit{and} \quad i(h, D) = 1.$$

Then h has a fixed point lying on ∂U.

5 Computing the rotation number

To make Proposition 2 useful for applications we need some conditions on h and U implying that the rotation number vanishes. We list some of these conditions in the next result.

Proposition 6 *Assume that $h \in \mathcal{H}_+$ and $U = h(U)$ is a simply connected open subset of the plane that is unbounded and proper, $\emptyset \neq U \subsetneqq \mathbb{R}^2$. Then*

$$\rho(h, U) = 0$$

if any of the conditions below holds

(i) ∂U is connected and ∞ is accessible from U

(ii) $h = r \circ r$ with $r \in \mathcal{H}_-$ and $r(U) = U$

(iii) There exists an arc $\gamma \subset \mathbb{S}^2$ having ∞ as one of the end points with $\gamma \setminus \{\infty\} \subset U$ and $h(\gamma) \subset \gamma$

(iv) There exists a sector $K = \{\rho e^{i\theta} : \rho \geq 0,\ \theta \in [\Theta_-, \Theta_+]\}$, $\Theta_- < \Theta_+$, and a disc $D = \{|z| \leq R\}$ such that $h(K \setminus D) \subset K$ and $K \setminus D \subset U$.

The condition (i) requires the accessibility of the point of infinity and this is essential. Later we will construct an example where the boundary ∂U is connected but the rotation number is $\frac{1}{2}$. We recall that ∞ is accessible from U when there is an arc γ in $\mathbb{S}^2$ having ∞ as one end point and such that $\gamma \setminus \{\infty\}$ is contained in U. The condition (ii) can be illustrated with the linear case. If $h(x) = Ax$ then $h = r^2$ with $r(x) = Bx$, $\det B < 0$, as soon as the matrix A has two positive eigenvalues with $\lambda_1 \neq \lambda_2$. The condition (iii) is inspired by the work of Alarcón, Guíñez and Gutiérrez [1]. They introduced a similar assumption to get results on global asymptotic stability. Positively invariant cones appear often in the theory of differential equations and for this reason we have also stated condition (iv).

Before going to the proof of the proposition we present the example linked to (i). We start with the system of differential equations in the plane

$$\dot{x} = \phi(x), \;\; \dot{y} = \psi(x, y)$$

where ϕ and ψ are C^∞ bounded functions satisfying

$$\phi(x) = 0 \;\text{ if } x \in A := \{\frac{1}{n} : n \in \mathbb{Z},\; n \neq 0\} \cup \{0\}, \;\; x\phi(x) > 0 \text{ otherwise}$$

$$\psi(x, y) = 0 \;\text{ if } (x, y) \in (\mathbb{R} \times \{0\}) \cup (A \times \{1, -1\}),\; y\psi(x, y) > 0 \text{ otherwise.}$$

The study of the phase portrait shows that $K = (A \times [-1, 1]) \cup ([-1, 1] \times \{0\})$ is invariant under the flow $\{\phi_t\}_{t \in \mathbb{R}}$. Next we compose the time one map with the rotation of 180 degrees $R : (x, y) \mapsto (-x, -y)$. The map $h_1 = R \circ \phi_1$ is orientation-preserving and can be extended to $\mathbb{S}^2$. The region $V = \mathbb{S}^2 \setminus K$ is invariant under h_1 and simply connected and so the rotation number $\rho(h_1, V)$ is defined. We claim that $\rho(h_1, V) = \frac{1}{2}$. To justify it we observe that the points $(1, 1)$ and $(-1, -1)$ are a 2-cycle of h_1 and there are unique prime ends $p_\pm \in \mathbb{P}(V)$ with $\Pi(p_+) = \{(1, 1)\}$ and $\Pi(p_-) = \{(-1, -1)\}$. Hence $p_\pm$ is a 2-cycle for $h_1^\star$. The domain V contains the point of infinity and so it is not in the class considered by Proposition 6. This is easily solved with the change of variables $z \mapsto \frac{1}{z}$, performed in the Riemann sphere. Now the region $U = \{\frac{1}{z} : \;\; z \in V\}$ is invariant under $h(z) = 1/h_1(\frac{1}{z})$ and we observe that ∂U is homeomorphic to $K \setminus \{0\}$. The origin is not accessible from V and the same property holds for ∞ and U.

We prepare the proof of Proposition 6 with some results on prime ends.

Lemma 7 *Let $U \neq \mathbb{R}^2$ be open and simply connected. In addition it satisfies the conditions*

(a) ∞ is accessible from U

(b) $\mathbb{R}^2 \setminus U$ is connected.

Then there exists a unique prime end $p \in \mathbb{P}$ with $\Pi(p) = \{\infty\}$.

The simplest example of a domain satisfying the conditions of the lemma is $U =]0, \infty[\times]0, 1[$. In this case $\Pi(p)$ is a singleton for every $p \in \mathbb{P}$ and the correspondence between prime ends and principal points is a bijection from $\mathbb{P}$ to $\partial_{\mathbb{S}^2} U$. To construct a more sophisticated example we consider the curve

$$C: \quad y = e^{1/x} \sin(\frac{1}{x}), \quad x > 0.$$

After inflating this curve without creating self-intersections we obtain a domain U. Notice that U must become thinner as one approaches the line $x = 0$. The arc $C \cap \{x \geq 1\}$ connects U with ∞ and so (a) holds. The condition (b) is also valid although $\mathbb{R}^2 \setminus U$ is not arc-wise connected. Finally we observe that there are two prime ends p_0 and p_∞ having ∞ as a principal point, indeed $\Pi(p_0) = \{x = 0\} \cup \{\infty\}$ and $\Pi(p_\infty) = \{\infty\}$.

Proof of Lemma 7. First of all we need some additional information on the behavior of the Riemann conformal map from the open disc $\mathrm{int}(\mathbb{D})$ onto U. We fix such a map and denote it by $\mathcal{R}$. Given a continuous function $\gamma : [0, 1[\to U$ with $\gamma(t) \to a \in \partial_{\mathbb{S}^2} U$ as $t \uparrow 1$, then $\mathcal{R}^{-1}(\gamma(t)) \to \xi$ for some $\xi \in \partial\mathbb{D}$. This is a consequence of Proposition 2.14 in [22]. In the same chapter of the book [22] there is a characterization of the prime ends for which $\Pi(p)$ is a singleton. Following Corollary 2.17 in [22] we assume that $\xi \in \partial\mathbb{D}$, $p = \hat{\mathcal{R}}(\xi)$ and $a \in \partial_{\mathbb{S}^2} U$. Here $\hat{\mathcal{R}}$ stands for the extension of the Riemann map to $\mathbb{D} \cong U^\star$. The following statements are equivalent:

- $\Pi(p) = \{a\}$
- $\mathcal{R}(t\xi) \to a$ as $t \uparrow 1$
- $\mathcal{R}(\Gamma(t)) \to a$ as $t \uparrow 1$ for some continuous function $\Gamma : [0, 1[\to \mathrm{int}(\mathbb{D})$ with $\Gamma(t) \to \xi$ as $t \uparrow 1$.

With this background we are ready to prove the Lemma. First we prove the *existence* of such a prime end. From the condition (a) we find a continuous function $\gamma : [0, 1[\to U$ with $\gamma(t) \to \infty$ as $t \uparrow 1$. The limit of $\mathcal{R}^{-1}(\gamma(t))$ is denoted by $\xi \in \partial\mathbb{D}$ and the prime end $p = \hat{\mathcal{R}}(\xi)$ satisfies $\Pi(p) = \{\infty\}$.
The *uniqueness* is proven by a contradiction argument. Assume that $p_1, p_2 \in \mathbb{P}$, $p_1 \neq p_2$ are such that $\Pi(p_1) = \Pi(p_2) = \{\infty\}$. The pre-images $\xi_1 = \hat{\mathcal{R}}^{-1}(p_1)$ and $\xi_2 = \hat{\mathcal{R}}^{-1}(p_2)$ are well defined and $\mathcal{R}(t\xi_i) \to \infty$ as $t \uparrow 1$. Let r_1 and r_2 be the arcs in $\mathbb{S}^2$ parameterized by $r_i(t) = \mathcal{R}(t\xi_i)$. They share the end points, $\mathcal{R}(0)$ and ∞, and $\dot{r}_1 \cap \dot{r}_2 = \emptyset$. In consequence $J = r_1 \cup r_2 \cup \{\infty\}$ is a Jordan curve. Let V and W be the connected components of $\mathbb{S}^2 \setminus J$. Since $J \setminus \{\infty\}$ is contained in U, the set $\mathbb{R}^2 \setminus U$ is disjoint with J. Hence it must

lie in one of the components, say $\mathbb{R}^2 \setminus U \subset V$. It is at this point that we have applied the condition (b). Our interest will be in the other component W that must be contained in U. The Jordan-Schonflies Theorem implies that $W \cup J$ is a topological disc and so we can draw an arc σ in $W \cup J$ having end points at $\mathcal{R}(0)$ and ∞ and touching r_1 and r_2 infinitely many times. More precisely, we assume that $\sigma(t)$ is a parameterization, $\sigma(0) = \mathcal{R}(0)$, $\sigma(1) = \infty$ and $\sigma(t_n) \in r_1$, $\sigma(s_n) \in r_2$ for sequences t_n and s_n converging to 1^-. Let $\xi_3 \in \partial\mathbb{D}$ be the limit of $\mathcal{R}^{-1} \circ \sigma$. We know that such a limit exists since $\sigma(t) \to \infty$ as $t \uparrow 1$ and $\sigma(t) \in W \cup (J \setminus \{\infty\}) \subset U$ if $t \in [0, 1[$. The prime end $p_3 = \hat{R}(\xi_3)$ is such that $\sigma(t) \to p_3$ in $U^\star$. At the same time the sequence $\sigma(t_n)$ is contained in r_1 and converges to p_1. This implies that $p_1 = p_3$. The same reasoning applied to the sequence s_n leads to $p_2 = p_1$, the searched contradiction.

Remark. The condition (b) in the previous Lemma can be replaced by

$(b^\star)$ ∂U *is connected.*

First we fix a point ξ in U and, given any x in $\mathbb{R}^2 \setminus U$, we denote by $x^\star$ the first point in the segment $[x, \xi]$ lying in ∂U. To prove the connectedness of $\mathbb{R}^2 \setminus U$ we show that any two points x_1 and x_2 in $\mathbb{R}^2 \setminus U$ can be joined by a connected subset of $\mathbb{R}^2 \setminus U$. This set can be $C = [x_1, x_1^\star] \cup [x_2, x_2^\star] \cup \partial U$.

Proof of Proposition 6. As it is well known, the class of orientation-preserving homeomorphisms of $\mathbb{S}^1$ having zero as rotation number coincides with those having at least one fixed point. Next we prove that under every assumption from (i) to (iv) there exists a fixed point of $h^\star$.

(i) In view of the above Remark and Lemma 7 we know that there is a unique prime end with $\Pi(p) = \{\infty\}$. The prime end $h^\star(p)$ has the same principal set and so by uniqueness we conclude that p is fixed.

(ii) Since U is invariant under h and r, these maps induce homeomorphisms of $\mathbb{P}$ and they satisfy $h^\star = r^\star \circ r^\star$. The extension to $U^\star$ must be orientation-reversing for r and orientation-preserving for h. The same properties are valid for $r^\star$ and $h^\star$ as maps of $\mathbb{P}$. It is well known that an orientation-reversing homeomorphism of the unit circle has exactly two fixed points. This result applies to $r^\star$ and so $h^\star$ will have at least two fixed points.

(iii) The arc γ is such that $\mathcal{R}^{-1} \circ \gamma$ is a path in $\mathbb{D}$ with $\lim_{t\uparrow 1} \mathcal{R}^{-1} \circ \gamma(t) = \xi$ for some $\xi \in \mathbb{D}$. Since $h(\gamma)$ is a sub-arc of γ ending at ∞ we deduce that $\lim_{t\uparrow 1} \mathcal{R}^{-1}(h(\gamma(t))) = \xi$. The prime end $p = \hat{\mathcal{R}}(\xi)$ will be fixed under $h^\star$.

(iv) First we recall some additional facts from the theory of prime ends. They are extracted from the book [19]. Given $a \in \partial_{\mathbb{S}^2} U$ we consider the class $\mathcal{A}(a, U)$ of arcs γ having a as one end point and such that $\gamma \setminus \{a\} \subset U$. Every arc in $\mathcal{A}(a, U)$ defines a prime end p in $\mathbb{P}$. Indeed if $\gamma(t)$ is a parameterization with $\gamma(1) = a$, then $\lim_{t \uparrow 1} \gamma(t) = p$ in $U^\star$. Two arcs γ_1 and γ_2 in $\mathcal{A}(a, U)$ define the same prime end if there exists a third arc γ_3 in $\mathcal{A}(a, U)$ and sequences (x_n) and (y_n) of points in U satisfying

$$x_n \to a, \ \ y_n \to a, \ \ x_n \in \gamma_1 \cap \gamma_3, \ \ y_n \in \gamma_2 \cap \gamma_3.$$

These results can be proved following along the lines of the proof of Lemma 7. We are ready to apply these remarks to prove (iv). First we fix an angle $\Theta \in]\Theta_-, \Theta_+[$. The continuity of h at $\infty = h(\infty)$ implies the existence of $R_1 > R$ with $h(\rho e^{i\Theta}) \notin D$ if $\rho \geq R_1$. The ray $\gamma(t) = \frac{R_1}{1-t} e^{i\Theta}$ is in the class $\mathcal{A}(\infty, U)$ and, from the assumption, we deduce that also $h \circ \gamma$ is in this class. Indeed both arcs lie inside $K \setminus D \subset U$. This allows us to construct an arc γ_3 in $\mathcal{A}(\infty, U)$ to show the equivalence of γ and $h \circ \gamma$. For example we can take an arc in $K \setminus D$ bouncing infinitely many times between $\theta = \Theta_-$ and $\theta = \Theta_+$, Assume that p is the prime end defined by γ (and also by $h \circ \gamma$). Since $\gamma(t) \to p$ in $U^\star$, the continuity of the extension of h to $U^\star$ implies that $h(\gamma(t)) \to h^\star(p)$ in $U^\star$. Since both arcs define the same prime end we conclude that p is fixed.

6 Orientation-reversing homeomorphisms

The rest of the paper will be devoted to find applications of Propositions 2 and 6 in different instances. This will lead to some novel results in stability theory. We start with the condition (ii) of Proposition 6 that is very suitable for the orientation reversing case. Throughout this section h is a map in $\mathcal{H}_-$. The first result is a characterization of asymptotic stability in the global sense.

Theorem 8 *Assume that $h \in \mathcal{H}_-$ is area-contracting and dissipative. In addition $p = h(p)$ is an asymptotically stable fixed point with region of attraction $U \subset \mathbb{R}^2$. The following conditions are equivalent*

(i) $\quad U = \mathbb{R}^2$

(ii) $\quad \mathrm{Fix}(h^2) = \{p\}$.

Proof. We notice that when (i) holds all points are attracted by the point p and so there are no other fixed points or two cycles. This shows the implication $(i) \Rightarrow (ii)$. To prove $(ii) \Rightarrow (i)$ we employ an indirect argument and assume that $U \neq \mathbb{R}^2$. Since U is simply connected and unbounded it is possible to apply Proposition 6 in the case (ii) to the map $h^2 \in \mathcal{H}_+$. At this point it must be observed that the regions of attraction of p for h and h^2 coincide. The conclusion is that $\rho(h^2, U)$ vanishes. A classical result due to Browder and Krasnoselskii [17] says that the index around an asymptotically stable fixed point is 1. In particular, $i(h^2, D) = 1$ for any small disc centered at p. The region of attraction does not contain any recurrent point different from p, in particular $\mathrm{Fix}(h^2) \cap U = \{p\}$ and so Corollary 5 is applicable. We deduce that h^2 must have a fixed point lying on ∂U and this is a conclusion that is not compatible with the assumption (ii).

Remark. The previous proof leads to a stronger conclusion. If one assumes that the conditions of the previous theorem hold and $U \neq \mathbb{R}^2$, then

$$\mathrm{Fix}(h^2) \cap \partial U \neq \emptyset.$$

The study of orientation-reversing homeomorphisms of the plane has interest in itself and recently has received a lot of attention, see [7]. Sometimes the question arises as to whether this theory can be useful in the study of differential equations. Some discussions in this direction for autonomous systems can be found in [25]. For periodic equations the Poincaré map plays a crucial role in the understanding of the dynamics, but this map is always orientation-preserving (see [20] for more details). Next we show how orientation-reversing maps do appear in the context of periodic differential equations with symmetries. Let S denote the symmetry with respect to the horizontal axis $S(x_1, x_2) = (x_1, -x_2)$ and consider the differential system

$$\dot{x} = X(t, x), \quad x \in \mathbb{R}^2 \tag{7}$$

where $X : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ satisfies

$$X(t + \pi, Sx) = SX(t, x), \text{ for each } (t, x) \in \mathbb{R} \times \mathbb{R}^2.$$

We also assume that the vector field X is continuous and such that there is global existence and uniqueness for the initial value problem. The solution satisfying the initial condition $x(t_0) = \xi$ will be denoted by $x(t; t_0, \xi)$ and it is well defined for all $t \in \mathbb{R}$. We define the maps

$$P_1(\xi) = x(\pi; 0, \xi) \quad \text{and} \quad P_2(\xi) = x(2\pi; \pi, \xi).$$

They satisfy

$$S \circ P_1 = P_2 \circ S.$$

Indeed we notice that if $x(t)$ is a solution then $Sx(t+\pi)$ is also a solution. Hence

$$x(t;0,S\xi) = Sx(t+\pi;\pi,\xi).$$

This implies that $S \circ P_2 = P_1 \circ S$ and from here it is easy to arrive at the above identity. The symmetry of the vector field implies that system (7) is 2π-periodic with respect to time and so the Poincaré map $P(\xi) = x(2\pi;0,\xi)$ is the key to understand the dynamics of the equation. It satisfies

$$P = P_2 \circ P_1 = P_2 \circ S \circ S \circ P_1 = (S \circ P_1)^2.$$

It is well known that the maps P_1, P_2 and P belong to $\mathcal{H}_+$. The previous discussions show that P can be expressed as $P = r \circ r$ with $r = S \circ P_1 \in \mathcal{H}_-$.

We present a consequence of Theorem 8. It employs the notion of dissipativity for periodic differential equations. This notion is analyzed in [21] and also in [27] with a different terminology. In particular it implies the dissipativity of the Poincaré map.

Corollary 9 *Consider the system (7) in the above conditions and assume in addition that X is C^1, $X(t,0) = 0$ for all t, and the three conditions below hold,*

- *System (7) is dissipative*
- $\mathrm{div}_x X(t,x) := \frac{\partial X_1}{\partial x_1}(t,x) + \frac{\partial X_2}{\partial x_2}(t,x) < 0$ *everywhere*
- *The linearized system*

  $$\dot{y} = \frac{\partial X}{\partial x}(t,0)y$$

  is asymptotically stable.

Then the trivial solution $x = 0$ is globally asymptotically stable if and only if there are no more 2π-periodic solutions.

Proof. We apply the previous theorem to $r = S \circ P_1$. In view of the first condition the map P is dissipative and so the same property must hold for r. The Jacobi-Liouville formula together with the second condition imply that P_1 is area-contracting. Hence the same property is enjoyed by r. The linearization principle and the third condition imply that the solution $x = 0$ is asymptotically stable for (7). This is equivalent to saying that $p = 0$ is asymptotically stable as a fixed point of P. In consequence it has the same property with respect to r and we can invoke the theorem.

To illustrate the previous result consider the system

$$\dot{x}_1 = -x_1 + \psi(x_2), \quad \dot{x}_2 = -x_2 + \lambda(\sin t)x_1,$$

where $\psi \in C^1(\mathbb{R})$ is even and bounded, $\psi(0) = \psi'(0) = 0$ and $\lambda \in \mathbb{R}$ is a parameter. The general conditions imposed to system (7), including the symmetry, are satisfied in this case. To check the dissipativity one can employ the Lyapunov function

$$V(x_1, x_2) = \alpha x_1^2 + \beta x_2^2$$

with α and β positive numbers satisfying $\alpha > \frac{\lambda^2}{4}\beta$. It satisfies

$$\dot{V}(x) \leq -\gamma V(x) \quad \text{whenever } ||x|| \geq R.$$

Here γ and R are positive numbers that can be determined. This is sufficient to guarantee dissipativity. The divergence of the system is constant and equal to -2 so that the second condition holds. It remains to study the stability of the 2π-periodic linear system

$$\dot{y}_1 = -y_1, \quad \dot{y}_2 = -y_2 + \lambda(\sin t)y_1.$$

It has the Floquet solution $y(t) = \text{col}(0, e^{-t})$ and so the corresponding multiplier is $\mu_1 = e^{-2\pi}$. The Jacobi-Liouville formula implies that the product of the multipliers satisfies $\mu_1\mu_2 = e^{-4\pi}$ and so also $\mu_2 = e^{-2\pi}$. This proves the asymptotic stability of the linearized system.

The previous example has been prepared in order to fulfill the required conditions but at least it shows that concrete results can be achieved with this methodology.

7 Miscellaneous results in stability theory

In this Section we present applications of Proposition 6 in the cases (i), (iii) and (iv).

7.1 Existence of periodic points

The next result deals with the region of attraction U of an asymptotically stable fixed point. The conclusion is that we must expect periodic points outside U if the boundary ∂U is not complicated. An interesting feature of the proof is the use of Proposition 2 in a situation where the invariant set is not a region of attraction.

Theorem 10 *Let h be a dissipative map in $\mathcal{H}$ having an asymptotically stable fixed point p with unbounded region of attraction $U \neq \mathbb{R}^2$. In addition assume that ∞ is accessible from U and ∂U has a finite number of connected components. Then h has a periodic orbit lying in $\mathbb{R}^2 \setminus U$.*

Proof. Let $C_1, \ldots, C_p$ be the components of ∂U. Since $\partial_{\mathbb{S}^2} U$ is connected each component C_i must be unbounded. The homeomorphism h induces a permutation on the finite set $\{C_1, \ldots, C_p\}$ and so there exists some integer $1 \leq N \leq p!$ with $h^N(C_i) = C_i$ for each i. It is not restrictive to assume that N is even. The set $\mathbb{R}^2 \setminus C_1$ is open and we denote by V the connected component containing U. We observe that V is simply connected. Otherwise there should exist a Jordan curve $\gamma \subset V$ not contractible to a point and this would imply that C_1 should be contained in the bounded component of $\mathbb{R}^2 \setminus \gamma$. This is absurd since C_1 is unbounded. The sets C_1 and U are invariant under h^N and the same must happen to V. We intend to apply Proposition 6 in the case (i) to the map $h^N \in \mathcal{H}_+$ and the set V. To check the remaining assumptions we notice that the boundary of V is $\partial V = C_1$ and so it is connected. Moreover, since ∞ is accessible from U the same must hold with respect to the larger set V. We can now conclude that $\rho(h^N, V) = 0$. Finally we apply Proposition 2 in the same setting and conclude that h^N has a fixed point outside V. This is the searched periodic point and it is interesting to observe that we have some control on the size of the period N in terms of the number of connected components.

7.2 Invariant rays and population dynamics

An interesting characterization of global asymptotic stability was obtained recently in [1].

Theorem 11 *(Alarcón, Guíñez, Gutiérrez). Assume that $h \in \mathcal{H}_+$ is dissipative and p is an asymptotically stable fixed point of h. The following conditions are equivalent,*

(a) *p is globally asymptotically stable*

(b) $\mathrm{Fix}(h) = \{p\}$ *and there exists an arc $\gamma \subset \mathbb{S}^2$ with end points at p and ∞ such that $h(\gamma) = \gamma$.*

The proof in [1] is based in Brouwer's theory of fixed point free homeomorphisms of the plane. Next we present an alternative proof based on the theory of prime ends.

Proof. $(a) \Rightarrow (b)$ We observe that h is conjugate to a linear contraction. This is a consequence of the results by Kerékjarto mentioned in Section 2. For a linear contraction we can take the ray $\gamma = ([0, \infty[\times\{0\}) \cup \{\infty\}$.
$(b) \Rightarrow (a)$ By a contradiction argument we assume that the region of attraction U of p is not the whole plane. The restriction of h to γ is a homeomorphism of the arc fixing the end points. Since there are no fixed points in the interior of the arc, every orbit must be attracted by an end point. As the map is dissipative we conclude that p attracts all orbits in $\dot{\gamma}$. In other words, $\gamma \setminus \{\infty\} \subset U$. In particular U is unbounded and all the conditions required by the case (iii) of Proposition 6 are satisfied. Once we know the rotation number vanishes we apply Proposition 2 and conclude that there exists a second fixed point. This is not compatible with (b).

The previous result is applicable to systems with two populations. In these systems the coordinate axes are invariant and they produce the invariant ray. More precisely we consider the system

$$\dot{u} = uF(t, u, v), \quad \dot{v} = vG(t, u, v), \qquad u, v \geq 0, \tag{8}$$

where $F, G : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ are of class C^1 and 1-periodic in time. The periodicity in time reflects the seasonal effects. For more ecological insight we refer to [11]. We also assume that F and G are such that there is global existence for the associated initial value problem on the first quadrant. We think of $u(t)$ and $v(t)$ as the sizes of two species and say that there is extinction for (8) if

$$u(t) \to 0, \quad v(t) \to 0 \quad \text{as } t \to +\infty$$

for each solution $(u(t), v(t))$.

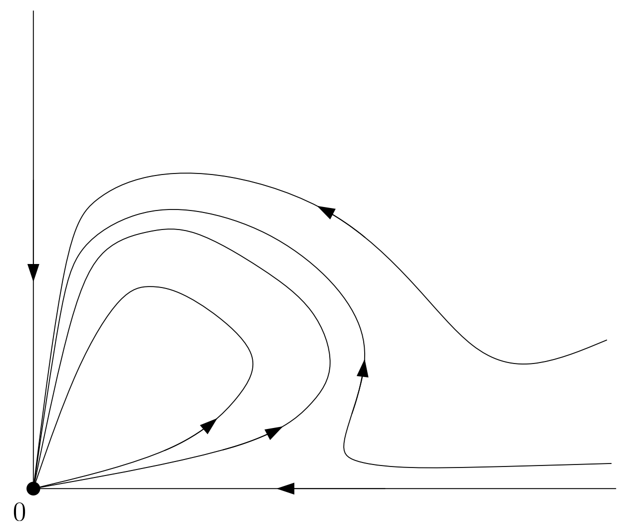
We shall assume that the system is dissipative and this is a very natural assumption in population dynamics, particularly when logistic effects are involved. We also employ the condition

$$\int_0^1 F(t, 0, 0)dt < 0, \quad \int_0^1 G(t, 0, 0)dt < 0. \tag{9}$$

This condition says that the averaged balance between birth and death is negative when the populations are very small.

Theorem 12 *Assume that the system (8) is dissipative and the condition (9) holds. Then there is extinction if and only if $u = v = 0$ is the unique 1-periodic solution.*

Remark. The condition (9) is necessary to exclude cases where the origin is an unstable attractor. We illustrate this situation in the following phase portrait



Proof. System (8) has period $T = 1$ and the associated Poincaré map P is an orientation-preserving homeomorphism of the first quadrant

$$\mathbb{R}^2_+ = \{(u,v) \in \mathbb{R}^2 : \ u \geq 0, v \geq 0\}.$$

The coordinate axes $u = 0$ and $v = 0$ are invariant under (8) and also under the map P. This allows us to extend P to a homeomorphism $h \in \mathcal{H}_+$ by successive reflections. The map h leaves invariant each quadrant and the dynamics of P is reproduced. The global asymptotic stability for $p = 0$ implies the extinction for (8). We intend to apply Theorem 11 and to this end we must verify that $p = 0$ is asymptotically stable. Going back to the differential equations we observe that the linearized system at $u = v = 0$ is

$$\dot{\xi} = F(t,0,0)\xi, \quad \dot{\eta} = G(t,0,0)\eta.$$

This system is uncoupled and can be easily integrated. The condition (9) says that there is asymptotic stability. In consequence $u = v = 0$ is asymptotically stable with respect to the nonlinear system and so the fixed point $p = 0$ is asymptotically stable for h. To check the condition (b) of the Theorem it is enough to find an invariant ray. We can take the half line $\gamma : \ u \geq 0, v = 0$.

7.3 Sectorial attraction in forced oscillators

Consider the equation

$$\ddot{x} + c\dot{x} + g(x) = p(t) \tag{10}$$

where $c > 0$, $g : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz-continuous and p is continuous and 2π-periodic. In the terminology of Pliss [21] this equation is *convergent* if it has a 2π-periodic solution that is globally asymptotically stable. We say that the equation has the property of Σ-*uniqueness* it it has a unique 2π-periodic solution and this solution is asymptotically stable. This property is weaker than convergence but we want to show that they are equivalent as soon as the solutions starting at some angular sector are attracted.

Let us assume that g has finite limits at $\pm\infty$ and

$$g(-\infty) < \overline{p} < g(+\infty), \tag{11}$$

where $\overline{p} = \frac{1}{2\pi}\int_0^{2\pi} p(t)dt$ is the mean value of p. This condition has a mechanical meaning, it says that the averaged force $-g(x) + \overline{p}$ points towards the origin, at least in a neighborhood of infinity. In agreement with this intuition it is known that (11) is sufficient to guarantee the dissipativity of the first order system associated to (10). See [27] for more details, in particular pages 70 and 71. From now on $\varphi(t)$ will denote the unique 2π-periodic solution of (10). Given another solution $x(t)$ we say that it is attracted by φ if

$$|x(t) - \varphi(t)| + |\dot{x}(t) - \dot{\varphi}(t)| \to 0 \quad \text{as } t \to +\infty.$$

Proposition 13 *Assume that (11) holds and there is Σ-uniqueness for (10). In addition there are positive numbers ρ and ϵ such that all the solutions satisfying*

$$x(0) \geq \rho, \quad |\dot{x}(0)| \leq \epsilon x(0)$$

are attracted by φ. Then (10) is convergent.

Before proving this result we state a lemma on linear equations.

Lemma 14 *Assume that $h \in C[0, 2\pi]$ and $x(t)$ is a solution of*

$$\ddot{x} + c\dot{x} = h(t).$$

Given $r > 0$ there exists $R > 0$ such that

$$|x(0)| + |\dot{x}(0)| \geq R \quad \Rightarrow \quad |x(t)| + |\dot{x}(t)| \geq r, \ t \in [0, 2\pi].$$

The number R only depends upon r, c and $||h||_\infty$.

The proof can be obtained using the formula of variation of constants.

Proof of Proposition 13. Since g is bounded all the solutions of (10) are globally defined. Consider the planar set

$$K = \{(x_1, x_2) \in \mathbb{R}^2 : \ |x_2| \le \epsilon x_1\}.$$

We observe that if $x(t)$ is a solution then

$$\frac{d}{dt}(\dot{x}(t) \pm \epsilon x(t)) = -(c \mp \epsilon)\dot{x}(t) - g(x(t)) + p(t).$$

Assuming $\epsilon < c$ we can find $r > \rho$ such that if $|x(t)| + |\dot{x}(t)| \ge r$ and $(x(t), \dot{x}(t)) \in \partial K$ then $(x(s), \dot{x}(s))$ enters into K when $s > t$ is close enough to t. Next we apply the above Lemma with $h = p - g \circ x$ and find the corresponding number R associated to r and ϵ. Notice that h depends on the chosen solution $x(t)$ but R only depends on $||g||_\infty + ||p||_\infty$. Now it is standard to prove that the set K has the following invariance property: if $x(t)$ is a solution of (10) with $|x(0)| + |\dot{x}(0)| \ge R$ and $(x(0), \dot{x}(0)) \in K$ then $(x(t), \dot{x}(t)) \in K$ for each $t \in [0, 2\pi]$. With this information we consider the system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -g(x_1) - cx_2 + p(t)$$

and the associated Poincaré map for $T = 2\pi$, that will be denoted by P. This map belongs to $\mathcal{H}_+$ and, since the divergence of the vector field is $-c$, it is area-contracting. At this point it is convenient to observe that the vector field is not necessarily smooth but the divergence is always well defined. More details on this point can be found in [26]. Also we observe that the condition (11) implies that P is dissipative. Now we can apply Proposition 6 in the case (iv) and Proposition 2 to complete the proof.

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