Ultrametric spaces, valued and semivalued groups arising from the Theory of Shape

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This paper is dedicated to Professor Juan Tarrés on the occasion of his retirement

ABSTRACT

In this paper we construct many generalized ultrametrics in the sets of shape morphisms between topological spaces. We recognize a topology in these sets which is independent on the shape representation of the spaces. We also construct valuations and semivaluations on groups of shape equivalences and on n-th shape groups. With this paper we connect shape theory, for arbitrary topological spaces, with the algebraic theory of generalized ultrametric spaces developed by S.Priess-Crampe and P.Ribenboim among other authors.

Key words: Shape morphisms, shape equivalences, shape groups, ultrametrics, valuations and semivaluations on groups.

2010 Mathematics Subject Classification: 55P55, 54C56, 55Q07, 54H11.

1. Introduction

In [19], [20], [21] and [22] the authors constructed and exploited a non Archimedean metric (or ultrametric) in the set of pointed and unpointed shape morphisms between compact metric spaces. In [6], this construction was extended, in the topological setting, to the arbitrary case.

Independently on this fact, in [12], [13], [23], [24], [25], [26] and [27], it was developed the theory of the so called *"Generalized ultrametric spaces"*, a generalization of the classic definition of ultrametric spaces.

The aim of this paper is, at the same time, to extend our techniques in [19] and [20] and to provide of many topological examples the pure "algebraic" theory developed mainly by S.Priess-Crampe and P.Ribenboim.

The authors have been supported by the DGES

We also obtain, in a natural way, semivaluations and valuations on the groups of shape equivalences and nth-shape groups.

It has been proved that the pure algebraic study of the groups of equivalences or the nth-shape groups, for arbitrary topological spaces, is not enough to obtain topological information on the spaces, but if we consider the more rich structure of topological groups, that we give on them, then it could be possible to reach better topological information on the spaces, as we point out in [20], [21] and [22]. Since in the case of good spaces as polyhedra, manifolds (more generally ANR's), these topological groups are discrete, the new information given by the conjunction of Algebra and Topology reduces, in this case, to algebraic information on that groups and the history of Algebraic Topology has proved that it is adequate to obtain topological or geometrical information on these special spaces. In summary, we think that "if you want to obtain topological information on general spaces on the line suggested by homotopy theory, you must change homotopy by shape and you should consider the natural structure of topological groups not only the algebraic ones".

Information on shape theory can be found in [4], [8] and [16]. In this case we recommend [16] for definitions and notations used herein.

2. Generalized ultrametric spaces of shape morphisms associated to HPol expansions

Let X, Y be topological spaces. Assume

$$\underline{\mathbf{Y}} = (Y_\mu, q_{\mu,\mu'}, M)$$

to be an inverse system in HPol and let

$$\underline{\mathbf{q}} = \{q_\mu\}_{\mu \in M} : Y \longrightarrow \underline{\mathbf{Y}}$$

be an HPol-expansion of Y (see [16] for the basic definitions). We can suppose that $(M, \leq)$ is a directed set. Denote by $\mathcal{L}(M)$ the set of all lower classes in M ordered by inclusion. Let us recall that $\Delta \subset M$ is called a *lower class* if for every $\delta \in \Delta$ and $\mu \in M$ with $\mu \leq \delta$, then $\mu \in \Delta$. From now on we consider the empty set $\varnothing$ as a lower class. Moreover, given two lower classes $\Delta, \Delta' \in \mathcal{L}(M)$ we say that $\Delta \leq \Delta'$ if and only if $\Delta \supset \Delta'$.

Proposition 2.1. *$(\mathcal{L}(M), \leq)$ is a partially ordered set with a least element (we will denote it by 0). Futhermore $\mathcal{L}(M)^* = \mathcal{L}(M) - \{0\}$ is downward directed.*

Proof. The least element is just the lower class M.

Suppose now that $\Delta, \Delta' \in \mathcal{L}(M)^*$ and define $\Delta'' = \Delta \cup \Delta'$. First of all let us see that Δ'' is a lower class. For a $\alpha'' \in \Delta''$, we imply that $\alpha'' \in \Delta$ or $\alpha'' \in \Delta'$. In any case, if $\alpha \leq \alpha''$ it is $\alpha \in \Delta$ or $\alpha \in \Delta'$, and consequently Δ'' is a lower class. Let us prove now that $\Delta'' \in \mathcal{L}(M)^*$; that is, $\Delta'' \neq M$. On the contrary, assume that $M = \Delta'' = \Delta \cup \Delta'$. Since $\Delta, \Delta' \in \mathcal{L}(M)^*$, it follows that $\Delta \neq \Delta \cap \Delta' \neq \Delta'$. Take $\alpha \in \Delta - \Delta'$, $\alpha' \in \Delta' - \Delta$. Since M is a directed set, there exists an element $\alpha'' \in M$ with $\alpha'' \geq \alpha, \alpha'$. As $\alpha'' \in \Delta$ or $\alpha'' \in \Delta'$, it follows that $\alpha' \in \Delta$ or $\alpha \in \Delta'$ which is a contradiction. So, $\Delta'' \in \mathcal{L}(M)^*$. $\square$

The basic definition of (generalized) ultrametric we are going to use here is:

Definition 2.2. (see [12], page 25). Let X be a set and $(\Gamma, \leq)$ be a partial ordered set with a least element 0. An *ultrametric* on X is a map $d : X \times X \longrightarrow \Gamma$ such that for $x, y \in X$, $\gamma \in \Gamma$, it satisfies:

1) $d(x, y) = 0$ iff $x = y$.

2) $d(x, y) = d(y, x)$.

3) if $d(x, y) \leq \gamma$ and $d(y, z) \leq \gamma$, then $d(x, z) \leq \gamma$.

Now we can prove:

Theorem 2.3. *Let X, Y be topological spaces. Assume that $\underline{\mathbf{Y}} = (Y_\mu, q_{\mu,\mu'}, M)$ is an inverse system in $HPol$ and let $\underline{\mathbf{q}} = \{q_\mu\} : Y \longrightarrow \underline{\mathbf{Y}}$ be an $HPol$-expansion of Y. Let $Sh(X, Y)$ be the set of all shape morphisms from X to Y. For $\alpha, \beta \in Sh(X, Y)$ the formula*

$$d(\alpha, \beta) = \{\mu \in M \mid q_\mu \circ \alpha = q_\mu \circ \beta \text{ (in the shape category)}\}$$

defines an ultrametric

$$d : Sh(X, Y) \times Sh(X, Y) \longrightarrow (\mathcal{L}(M), \leq).$$

Proof. Consider $\alpha = (\alpha_\mu)_{\mu \in M}$, $\beta = (\beta_\mu)_{\mu \in M}$ as representations of the corresponding shape morphisms (see [16]). If $\mu, \mu' \in M$, $\mu \leq \mu'$, we have the following equalities in homotopy:

$$\alpha_\mu = q_{\mu,\mu'} \circ \alpha_{\mu'} \text{ and } \beta_\mu = q_{\mu,\mu'} \circ \beta_{\mu'}\,.$$

Moreover $\alpha_\mu = q_\mu \circ \alpha$, $\beta_\mu = q_\mu \circ \beta$, where the compositions are within the shape category. Now, it is very easy to check that $d(\alpha, \beta) \in \mathcal{L}(M)$ and 1), 2) and 3) in Definition 2.2 (using the transitive property of the homotopy relation for the proof of 3)). □

In order to connect this construction with our first paper ([19]) on ultrametrics and shape, we have the following result whose proof is straightforward:

Proposition 2.4. *Let $(\mathcal{Q}, \rho)$ be the Hilbert cube with a fixed metric ρ. Suppose $Y \subset \mathcal{Q}$ is a closed subset and, for each real number $\varepsilon > 0$ take*

$$Y_\varepsilon = B(Y, \varepsilon) = \{q \in \mathcal{Q} \mid \rho(Y, q) < \varepsilon\}\,.$$

Given $\varepsilon' > \varepsilon > 0$, define $q_{\varepsilon,\varepsilon'} : Y_\varepsilon \longrightarrow Y_{\varepsilon'}$ and $q_\varepsilon : Y \longrightarrow Y_\varepsilon$ as the corresponding inclusions. Assume the reverse usual order in $\mathbb{R}^+$ and denote by $(\mathbb{R}^+)^{-1}$ the corresponding ordered set. Finally consider the inverse system

$$\underline{\mathbf{Y}} = (Y_\varepsilon, q_{\varepsilon,\varepsilon'}, (\mathbb{R}^+)^{-1})$$

and the $HPol$-expansion

$$\underline{\mathbf{q}} = \{q_\varepsilon\}_{\varepsilon \in \mathbb{R}^+} : Y \longrightarrow \underline{\mathbf{Y}}\,.$$

Then the generalized ultrametric d constructed in Theorem 2.3 (for this $HPol$-expansion) is just the complete non-Archimedean metric constructed in [19].

3. The canonical and the intrinsic topologies

Using the same idea as in [12] (page 34) we have:

Proposition 3.1. *Let X, Y be topological spaces. Assume that $\underline{\mathbf{Y}} = (Y_\mu, q_{\mu,\mu'}, M)$ is an inverse system in $HPol$ and that $\underline{\mathbf{q}} : Y \longrightarrow \underline{\mathbf{Y}}$ is an $HPol$-expansion. For every $\Delta \in \mathcal{L}(M)^*$ and $\alpha \in Sh(X,Y)$ consider*

$$B_\Delta(\alpha) = \{\beta \in Sh(X,Y) \mid d(\alpha,\beta) \leq \Delta\}.$$

Then the family

$$\{B_\Delta(\alpha) \mid \alpha \in Sh(X,Y), \Delta \in \mathcal{L}(M)^*\}$$

is a base for a topology in $Sh(X,Y)$ which is completely regular Haussdorf and zero-dimensional. We call (as in [12]) this topology the canonical topology induced by the ultrametric d.

One of the main trouble to use this topology to obtain information related to shape theory is the fact that it depends, as we will prove in the next proposition, on the particular HPol-expansion used and not on the shape of the space involved:

Proposition 3.2. *Let Y be a topological space and $\underline{\mathbf{Y}} = (Y_\mu, q_{\mu,\mu'}, M)$ an inverse system on $HPol$. Suppose that $\underline{\mathbf{q}} = \{q_\mu\} : Y \longrightarrow \underline{\mathbf{Y}}$ is an $HPol$-expansion. Consider now the usual product order in $M \times M$ and define the new inverse system*

$$\underline{\mathbf{Y}}' = (Y_{(\mu,\gamma)}, q_{(\mu,\gamma),(\mu',\gamma')}, M \times M)$$

where $Y_{(\mu,\gamma)} = Y_\mu$, $q_{(\mu,\gamma),(\mu',\gamma')} = q_{\mu,\mu'}$. Then

$$\underline{\mathbf{q}}' = \{q'_{(\mu,\gamma)}\} : Y \longrightarrow \underline{\mathbf{Y}}'$$

is an $HPol$-expansion, where $q'_{(\mu,\gamma)} = q_\mu$. Furthermore for every topological space X, the canonical topology on $Sh(X,Y)$ induced by the ultrametric d (associated to the $HPol$-expansion $\underline{\mathbf{q}}'$ as in Theorem 2.3) is just the discrete one.

Proof. Fix a non maximal element $\gamma_0 \in M$ and take

$$D = \{(\mu,\gamma) \in M \times M \mid \gamma = \gamma_0\}.$$

Let $\Delta_D \in \mathcal{L}(M \times M)^*$ be the minimal lower class in $\mathcal{L}(M \times M)$ containing D, it is

$$\Delta_D = \{(\mu,\gamma) \in M \times M \mid \gamma \leq \gamma_0\}.$$

Now, if $\alpha \in Sh(X,Y)$ then $B_{\Delta_D}(\alpha) = \{\alpha\}$. $\square$

Note that Propositions 2.4 and 3.2 in this paper and the fact (proved in [19]) that $Sh(X,Y)$ is, in general, non-discrete for compact metric spaces X, Y allow us to deduce that the canonical topologies depend on the particular HPol-expansion used.

The counterpart of the last proposition will be given in the next one, but we need first some words and definitions to motivate it. In [6] we defined a topology on the sets of shape morphisms which depends only on the shape of the involved spaces. It allowed us to construct many new shape invariants and to obtain some relationships between shape theory and $\mathbb{N}$-compactness (see [13] for some relations between the canonical topologies and $\mathbb{N}$-compactness). Later, we saw in [12] (page 52, Corollary 9.6) the way to obtain the results in [6] as a by-product of the present paper. Let now $(M, \leq)$ be a direct set and consider $(\mathcal{L}(M), \leq)$ the corresponding ordered set of lower classes in M. For every $\alpha \in M$ denote by $[\alpha]$ the lower class generated by α, that is,

$$[\alpha] = \{\alpha' \in M \mid \alpha \geq \alpha'\}.$$

Define

$$\begin{array}{rccc} \phi: & (M, \leq) & \longrightarrow & (\mathcal{L}(M), \leq) \\ & \mu & \mapsto & [\mu] \end{array}$$

ϕ needs not to be injective (for example, think on $(M, \leq)$ as the Čech system associated to a topological space -see [14]-), but we have that if $\mu \geq \mu'$ then $[\mu] \leq [\mu']$ and $(\phi(M), \leq)$ is a partial ordered set while $(M, \leq)$ maybe not. Note also that $(\phi(M), \leq)$ is downward directed in $\mathcal{L}(M)$ because M is a directed set. Suppose now that $\underline{\mathbf{Y}} = (Y_\mu, q_{\mu,\mu'}, M)$ is an inverse system and that $\underline{\mathbf{q}} = \{q_\mu\} : Y \longrightarrow \underline{\mathbf{Y}}$ is an HPol-expansion. Let X be an arbitrary topological space and consider $Sh(X, Y)$ the set of shape morphisms from X to Y. Then the family

$$\left\{B_{[\mu]}(\alpha) \mid \alpha \in Sh(X, Y), \mu \in M\right\}$$

is a base for a topology on $Sh(X, Y)$ (see [12]). We call it the *intrinsic topology*. The main reason is the following result:

Proposition 3.3. *The intrinsic topology on $Sh(X, Y)$ is independent on the fixed HPol-expansion of Y and it coincides with the topology defined and studied in [6].*

From [19], [6] and [13] we can obtain the next result which, in particular, says that, outside from the compact metric case, the intrinsic topology may not be ultrametrizable.

Proposition 3.4. *a) If Y is a compact metric space and X is an arbitrary topological space, the intrinsic topology on $Sh(X, Y)$ is induced by an ultrametric.*

b) If $*$ is a one point space, the intrinsic topology on $Sh(*, \{0, 1\}^{\omega_1})$ ($\{0, 1\}$ discrete) is not induced by an ultrametric.

Proof. a) It is enough to use the HPol-expansion $\underline{\mathbf{q}}$ described in Proposition 2.4.

b) From [6] it follows that the intrinsic topology on $Sh(*, \{0, 1\}^{\omega_1})$ is just, up one identification, the product topology on $\{0, 1\}^{\omega_1}$ and it is not ultrametrizable ([13]). □

4. Semivalued and valued groups of shape equivalences

First of all, given a topological space Y, let us denote by $E(Y)$ the group of shape equivalences. Take now an inverse system $\underline{\mathbf{Y}} = (Y_\mu, q_{\mu,\mu'}, M)$ and an HPol-expansion $\underline{\mathbf{q}} = \{q_\mu\} : Y \longrightarrow \underline{\mathbf{Y}}$. If d is the ultrametric on $Sh(Y, Y)$ defined in Theorem 2.3, we can state the next

Theorem 4.1. *The map*

$$\begin{array}{rccc} \nu: & E(Y) & \longrightarrow & (\mathcal{L}(M), \leq) \\ & f & \mapsto & \nu(f) = d(f, id_Y) \end{array}$$

is a semivaluation in the sense of [12] (page 37. See also [27]).

Proof. We must prove the following:

a) $\nu(f) = 0$ iff $f = id_Y$.

b) $\nu(f^{-1}) = \nu(f)$ for every $f \in E(Y)$.

c) If $\nu(f) \leq \Delta$, $\nu(g) \leq \Delta$, then $\nu(g \circ f) \leq \Delta$.

a) is obvious because d is an ultrametric. Take now $f, g \in E(Y)$ with $d(f,g) \leq \Delta$. This means that

$$q_\mu \circ f = q_\mu \circ g$$

for every $\mu \in \Delta$. For each $h \in E(Y)$

$$q_\mu(g \circ h) = (q_\mu \circ g) \circ h = (q_\mu \circ f) \circ h = q_\mu(f \circ h)\,,$$

hence

$$d(f \circ h, g \circ h) \leq d(f,g)\,.$$

On the other hand

$$d(f,g) = d((f \circ h) \circ h^{-1}, (g \circ h) \circ h^{-1}) \leq d(f \circ h, g \circ h)\,.$$

This means that d is a right invariant ultrametric on $E(Y)$, that is,

$$d(f \circ h, g \circ h) = d(f,g)$$

for every $f, g, h \in E(Y)$. In particular,

$$\nu(f) = d(f, id_Y) = d(f \circ f^{-1}, id_Y \circ f^{-1}) = d(id_Y, f^{-1}) = \nu(f^{-1})$$

and b) is proved.

Now suppose that $\nu(f) \leq \Delta$ and $\nu(g) \leq \Delta$. Thus,

$$d(id_Y, f^{-1}) = d(f, id_Y) = \nu(f) \leq \Delta$$

and

$$d(g, id_Y) = \nu(g) \leq \Delta\,.$$

Since d is an ultrametric,

$$d(g, f^{-1}) \leq \Delta\,.$$

Consequently,

$$\nu(g \circ f) = d(g \circ f, id_Y) = d(g, f^{-1}) \leq \Delta$$

and the proof is finished. □

With the same notation as in Theorem 4.1, we obtain

Corollary 4.2. *(See [28] for related results) For every $\Delta \in \mathcal{L}(M)$, $B_\Delta(id_Y)$ is a subgroup of $E(Y)$.*

For the theory of shape, it could be convenient the study of E(Y) with the intrinsic topology. For simplicity and clearness, we are going to give an outline of such study within the compact metric case. Note that, in this case, ν is a valuation.

Proposition 4.3. *Let Y be a compact metric space. $E(Y)$, with the intrinsic topology, is a completely ultrametrizable (a real metric) separable topological group.*

Proof. Let us recall (see [19] or Theorem 1 in [6]) that the composition is continuous and that, in this case (compact metric), the intrinsic topology in $Sh(Y, Y)$ is induced by the non Archimedean metric d constructed in [19]. The separability of $E(Y)$ follows from the fact that $(Sh(Y, Y), d)$ is homeomorphic to a closed subspace of the irrationals (see [19] Theorem 1.9).

Now, in order to prove that $(E(Y), d\mid_{E(Y)})$ is a topological group, it is enough to state that the assignment

$$\begin{array}{ccc} E(Y) & \longrightarrow & E(Y) \\ f & \mapsto & f^{-1} \end{array}$$

is continuous. Let $\{f_n\}_{n\in\mathbb{N}} \longrightarrow f$ in $E(Y)$, that is, $\{d(f_n, f)\}_{n\in\mathbb{N}} \longrightarrow 0$. Since d is a right invariant metric we deduce that

$$\{d(id_Y, f \circ f_n^{-1})\}_{n\in\mathbb{N}} = \{d(f_n \circ f_n^{-1}, f \circ f_n^{-1})\}_{n\in\mathbb{N}} \longrightarrow 0\,.$$

As the composition is continuous we have that

$$\{d(f^{-1}, f_n^{-1})\}_{n\in\mathbb{N}} = \{d(f^{-1}, f^{-1} \circ (f \circ f_n^{-1}))\}_{n\in\mathbb{N}} \longrightarrow 0$$

and therefore $(E(Y), d\mid_{E(Y)})$ is a topological group.

Finally, it is not difficult to prove that the formula

$$d'(f, g) = \max\{d(f, g), d(f^{-1}, g^{-1})\}$$

defines an equivalent complete non Archimedean metric on $E(Y)$. □

As a consequence we obtain

Corollary 4.4. *$E(Y)$ is countable iff $(E(Y), d)$ is uniformly discrete, that is, iff there is an $\varepsilon > 0$ such that if $d(f, g) < \varepsilon$ then $f = g$ ($f, g \in E(Y)$).*

Proof. Suppose that $E(Y)$ is countable. From the Baire theorem we have that $E(Y)$ is discrete (because of homogeneity). Let $\varepsilon > 0$ be such that $B(id_Y, \varepsilon) = \{id_Y\}$. Take $f, g \in E(Y)$ with $d(f, g) < \varepsilon$. From the right invariance of d,

$$d(id_Y, g \circ f^{-1}) = d(f, g) < \varepsilon\,,$$

hence $id_Y = g \circ f^{-1}$, consequently

$$0 = d(id_Y, g \circ f^{-1}) = d(f, g)$$

and $f = g$.

Assume now that $E(Y)$ is uniformly discrete. So, from separability, $E(Y)$ is countable. □

Remark 4.5. a) Note that if Y is an FANR-space (more generally, a calm compactum [5]) then $E(Y)$ is countable (because it is discrete).

b) If Y is an ANR then $E(Y)$ is just the group of homotopy equivalences (not maps but classes) and it is also countable.

From Theorem 3.1 in [19] we have that the natural projection

$$s : \mathcal{C}(X, Y) \longrightarrow Sh(X, Y)$$

is uniformly continuous, where $\mathcal{C}(X, Y)$ is the space of continuous functions from X to Y with the uniform convergence metric. In particular we obtain:

Proposition 4.6. *Let Y be a compact metric space and $H(Y)$ the topological group of autohomeomorphisms (with the uniform convergence topology), then the projection*

$$s : H(Y) \longrightarrow E(Y)$$

is a continuous (uniformly continuous) group homomorphism.

Remark 4.7. Many consequences can be drawn out from Proposition 4.6. In particular, we obtain many clopen subsets of $H(Y)$ depending on the shape classification of the elements of $H(Y)$. In the case of a compact ANR Y, the subgroup $H_0(Y)$ of $H(Y)$ formed by all homeomorphisms which are homotopic to the identity is a clopen normal subgroup of $H(Y)$ and the cardinal of the homotopy classes in $H(Y)$ is just the index of $H_0(Y)$ in $H(Y)$. We hope, in the future, to go further into this type of facts. Other aspects can be studied considering the projection $s : \mathcal{C}(X, Y) \longrightarrow Sh(X, Y)$. In many cases, for example if Y is calm, the limit of homeomorphisms is a shape equivalence and, in some cases, it is a homotopy equivalence. Another fact we reach is that the shape classes are closed in $\mathcal{C}(X, Y)$ while homotopy classes are, in general, not.

In order to show that our construction is not vacuous in meaning we have the following

Proposition 4.8. *For a compact zero-dimensional space Y the projection $s : H(Y) \longrightarrow E(Y)$ is a uniformly continuous homeomorphism (with uniformly continuous inverse).*

We could give a direct demonstration of the above proposition (we left it to the reader) but note that it is also a consequence of the well-known Banach's open mapping theorem for separable and completely metrizable topological groups.

The last proposition provides us of many examples of spaces Y such that $(E(Y), d)$ is noncomplete. The completeness of $(E(Y), d)$ is an important fact, the reason is the next result:

Proposition 4.9. *$(E(Y), d)$ is complete iff the limit of any sequence of shape equivalences in $Sh(X, Y)$ is a shape equivalence.*

The first result along this line (from [19]) is the following:

Proposition 4.10. *Let Y be a calm compact metric space (for example, a space shape dominated by a polyhedron). Then $(E(Y), d)$ is uniformly discrete, so complete.*

As a direct consequence we have:

Corollary 4.11. *Let Y be a calm compact metric space (for example, a solenoid, a maniflod, a polyhedrom...) and $\mathcal{C}(Y, Y)$ be the space of continuous functions with the uniform convergence topology. If the sequence $\{f_n\}_{n \in \mathbb{N}} \subset \mathcal{C}(Y, Y)$ converges to f and every f_n generate a shape equivalence (in particular, if each f_n is a homeomorphism) then f generates a shape equivalence.*

Proof. In fact the shape morphisms generated by f_n and f are equals for almost all $n \in \mathbb{N}$. □

Now from Corollary 4.4 we imply:

Proposition 4.12. *If the group $E(Y)$ is finitely (even countably) generated, then $(E(Y), d)$ is complete.*

Another result on completeness is:

Proposition 4.13. *When $E(Y)$ is abelian, $(E(Y), d)$ is complete.*

Proof. Since d is right invariant and $E(Y)$ abelian, then d is in fact invariant (i.e., left and right invariant). From [15] and the topological completeness of $E(Y)$ it follows that d is complete. □

Remark 4.14. a) Note that the above procedure permits us to prove that the center $(Z(Y), d)$ of $(E(Y), d)$ is always complete.

b) We could think that, in general, the center $Z(Y)$ is "small", at least topologically, but in [18] there is an example (in the pointed case) where $Z(Y)$ is not compact.

Let us establish now:

Proposition 4.15. *If $E(Y)$ is locally compact, then $(E(Y), d)$ is complete.*

Proof. Since $E(Y)$ is locally compact, there exists an $\varepsilon > 0$ such that $B(e, \varepsilon)$ is a compact subgroup of $E(Y)$, where e is the neutral element of $E(Y)$. Moreover, for every $g \in E(Y)$ it is $B(g, \varepsilon) = B(e, \varepsilon) \circ g$, thus $\{B(g, \varepsilon) \mid g \in E(Y)\}$ is a clopen partition of $E(Y)$. For a Cauchy sequence $\{f_n\}_{n\in\mathbb{N}}$ in $E(Y)$ we can assume the existence of $g \in E(Y)$ such that $\{f_n\}_{n\in\mathbb{N}} \subset B(g, \varepsilon)$. Take now $\{h_n\}_{n\in\mathbb{N}} \subset B(e, \varepsilon)$ with $f_n = h_n \circ g$. As $\{f_n\}_{n\in\mathbb{N}} = \{h_n \circ g\}_{n\in\mathbb{N}}$ and d is right invariant, $\{h_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. If $h = \lim_{n\to\infty} h_n$, $h \in B(e, \varepsilon)$ and, therefore, $\lim_{n\to\infty} h_n^{-1} = h^{-1}$. In this way,

$$\{f_n\}_{n\in\mathbb{N}} \longrightarrow h \circ g$$

and

$$\{f_n^{-1}\}_{n\in\mathbb{N}} = \{g^{-1} \circ h_n^{-1}\}_{n\in\mathbb{N}} \longrightarrow g^{-1} \circ h^{-1}\,,$$

so $(E(Y), d)$ is complete. □

Using analogous arguments as in the precedent proof we can obtain

Proposition 4.16. *a) If $(Sh(Y, Y), d)$ (see [19]) is locally compact then $E(Y)$ is locally compact (and so complete).*

b) If $Sh(Y, Y)$ is compact then $E(Y)$ is compact.

Remark 4.17. a) There are many examples showing that $E(Y)$ is compact but $Sh(Y, Y)$ is not (take $Y = S^n$, $n \in \mathbb{N}$).

b) From [7] and Proposition 4.8 we can obtain many examples of zero-dimensional compact metric spaces Y with $(E(Y), d)$ non complete.

c) Problem: Is $(Z(Y), d)$ always locally compact?

5. Semivaluation and valuation on shape groups

The n-th shape groups play the same role in shape theory as the n-th homotopy groups do in homotopy theory and they coincide in the class of ANR's (for example, manifolds, polyhedra, ...).

The intrinsic topology (in the pointed case) allows us, in particular, to give a natural structure of topological group on the n-th shape group. As in the third section, we confine ourselves into the compact metric case. The way to go further is that pointed out in Sections 1 and 2 of this paper (but in the pointed case).

Some of the authors (see [20]) have described a way to construct invariant ultrametrics (in the sense of [28]) on the shape groups. Then, they constructed valued groups in the sense of [27] (or normed groups [10]).

Let us recall the following

Proposition 5.1. *(see [20], Proposition 3) For any pointed compact metric space Y and any $n \in \mathbb{N}$ there exists a norm $\|\cdot\|$ on the shape group $\check{\Pi}_n(Y)$ such that*

i) $\|\cdot\|$ *is canonical, i.e., independent on any concrete presentation of the group.*

ii) $\|\alpha\beta\alpha^{-1}\| = \|\beta\|$ *for each* $\alpha, \beta \in \check{\Pi}_n(Y)$.

iii) $\|\alpha\| = \|\alpha^{-1}\|$ *for every* $\alpha \in \check{\Pi}_n(Y)$.

iv) $\|\cdot\|$ *leads to a left and right invariant complete ultrametric on* $\check{\Pi}_n(Y)$ *given by* $d(\alpha, \beta) = \|\alpha\beta^{-1}\|$.

Remark 5.2. a) As we said before, the norm $\|\cdot\|$ in the last proposition satisfies the properties i), ii), iii) of [10] (page 386) in the Farkas's definition of norms on groups.

b) One of the first authors who considered special kind of norms (non Archimedean) on groups related to topology was Alexander in homology theory ([1] and [2]). See also Markov [17] for some general contructions.

c) The kind of topology that we obtain on the shape groups by means of the norms $\|\cdot\|$ is just that defined in [11] under the name of subgroup topology, redefined (in some sense) in [3] and used in [9] and [14] to get Whitehead type theorems in shape theory.

The unique results we want to point out now for the shape groups with the topology defined by $\|\cdot\|$ are the following consequences of Theorem 3.5 in [11]:

Proposition 5.3. *For a pointed metric compact space (Y, y_0), the normed shape group $(\check{\Pi}_n(Y, y_0), \|\cdot\|)$ is compact iff the normal subgroups $B(e, \varepsilon) = \{\alpha \in \check{\Pi}_n(Y, y_0) : \|\alpha\| < \varepsilon\}$ are of finite index in $\check{\Pi}_n(Y, y_0)$.*

Corollary 5.4. *Let (Y, y_0) be a pointed metric compact space such that*

$$(Y, y_0) = \varprojlim \left\{(P_n, \{p_0\}_n), \phi_n\right\},$$

where $(P_n, \{p_0\}_n)$ are pointed polyhedra with finite m-homotopy group (for all $n \in \mathbb{N}$). Then, the mth-shape group $\Pi_m(Y, y_0)$ is compact.

Remark 5.5. Note that the analogous result for the groups of (unpointed) shape equivalences is not true. For example, the Cantor discontinuum C is the inverse limit of finite spaces -so with finite groups of shape equivalences- but the topological group of shape equivalences on C is not compact.

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