Continuous elliptical and exponential power linear dynamic models

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ELLIPTICAL LINEAR DYNAMIC MODELS

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Abstract

This paper shows a practical and easy to compute generalization of the linear dynamic model, made by assuming a continuous elliptical joint distribution for the parameters and errors. Updated distribution and probabilistic characteristics of the current and future vector of state and observations are given. As a particular simple submodel, the one with a multidimensional exponential power initial distribution is developed. An example to show its use is given.

Key words and phrases: continuous elliptical distribution; multidimensional exponential power distribution; global vector; vector of state; prior distribution; updated distribution.

1. Introduction

In this paper we study a generalization of the Harrison and Stevens (1976) linear dynamic model, made by assuming a continuous elliptical joint distribution for the parameters and errors of the model.

The linear dynamic model was developed from the Bayesian viewpoint by Harrison and Stevens (1976) by making use of the normal distribution, but there are many situations where the assumption of normality is not realist, thought the hypothesis of symmetry is adequate. So, several distributions have been employed in this sense with further developments that include mixtures of normal distributions (Girón et al. (1989)) and t distributions (Meinhold and Singpurwalla (1989)). The general elliptical distribution was first used by Chu (1973) and this model was later developed by Girón and Rojano (1994) within the framework of the Bayesian analysis.

Here, we use continuous elliptical distributions and, particularly, the exponential power distribution. Continuous elliptical distributions exhibit a broad range of symmetrical forms and the exponential power family, as a particular case of these, includes all the most important distributions, as the normal, the multivariate double exponential and the multivariate uniform distributions. These families contains distributions with higher or lower tails than the normal one and, besides, allows to test the normality of the errors and to deal even with models in which the normality is not acceptable.

So, we extend the linear dynamic models by using elliptical and, in particular, exponential power distributions, and thus replacing the independence assumption with the weaker assumption of uncorrelation. We take an absolutely continuous elliptical prior and show that the posterior is of the same kind and determine its parameters. We also establish the updated distribution and probabilistic characteristics of the current and future vectors of state and observations. We study in particular the models with a multidimensional exponential power initial distribution as an intermediate case, more general than the normal case but less than the elliptical one.

Throughout this paper properties of the absolutely continuous elliptical distribution and the multidimensional exponential power distribution are used. For a general exposition of the elliptical distributions, including the non-continuous ones, see Fang and Zang (1990) and Cambanis et al. (1981). For the multidimensional exponential power distribution see Gómez et al. (1998). The exponential power distribution is a particular case of the Kotz' type distributions (see Section 3.2 of Fang, Kotz and Ng (1990)).

The elliptical model can be used as an instrument to see the suitability of the normal model. It is possible to fix a number of stages and apply the elliptical model taking as functional parameter the corresponding to the normal model, and finally see the obtained posterior distributions. If these distributions are far from normality we continue applying the elliptical model. By the opposite, if the posterior distributions are close to the normal distribution, the normal model can be systematically used and the elliptical one rejected.

In Section 2 continuous elliptical model is defined. In Section 3 it is shown that the updated distribution remains being continuous elliptical and so the model is permanent over time. In Sections 4 and 5 the updated characteristics of the current and future states and observations are given. In Section 6 the model with a multidimensional exponential power initial distribution is developed. Finally, in Section 7 an application of the model to a pair of economic variables is studied.

2. Definition of continuous elliptical model

The *linear dynamic model* is characterized by the next two relations, called *observation equation* and *state equation*, for $t = 1, ..., n$:

$$\begin{aligned} Y_t &= F_t'\theta_t + v_t, \\ \theta_t &= G_t\theta_{t-1} + w_t, \end{aligned}$$

where, for each t, θ_t is r-dimensional vectors of state and Y_t is s-dimensional vectors of observations; v_t and w_t are vectors of errors with s and r dimensions respectively; F_t is a fixed $r \times s$ matrix of regressors and G_t is a fixed $r \times r$ matrix relating two consecutive states. A detailed interpretation of the linear dynamic model can be found, for example, in West and Harrison (1997).

For each $t = 1, ..., n$ we denote $\mathbf{Y}_t = (Y_1', ..., Y_t')'$, the vector of past observations, whose dimension is st, the scalar $d_t = r+(n-t)(r+s)$ and the *current global vector* $H_t = \left(\theta_t', v_{t+1}', w_{t+1}', ..., v_n', w_n'\right)'$, whose dimension is d_t. We denote, in general, $X \sim E_n(\mu, \Sigma, g)$ if a n-dimensional vector X has the absolutely continuous elliptical distribution with parameters μ, Σ, g as defined in Johnson (1987).

Definition 2.1. (Elliptical Linear Dynamic Model). *The continuous elliptical linear dynamic model is defined as a linear dynamic model where the initial global vector $H_0 = (\theta_0', v_1', w_1', ..., v_n', w_n')'$, whose dimension is $d_0 = r + n(r + s)$, has an absolutely continuous elliptical distribution:*

$$H_0 \sim E_{d_0}\left(\mu_0^H, \Sigma_0^H, g_0^H\right), \tag{2.1}$$

where $\mu_0^H = \left(m_{0,}' 0_{(d_0-r)\times 1}'\right)'$, with $m_0 \in \mathbb{R}^r$; Σ_0^H is a block-diagonal matrix whose diagonal components are symmetrical positive definite matrices C_0, V_1, W_1, $\ldots$, V_n, W_n, of r, s, $r, ...,$ s, r orders respectively; g_0^H is a non negative measurable Lebesgue function such $\int_0^\infty t^{\frac{d_0}{2}-1} g_0^H(t)dt < \infty$.

The elliptical model is clearly a generalization of the usual normal model, which is obtained from the former by taking $g_0^H(z) = \exp\left\{-\frac{1}{2}z\right\}$.

In order to apply the elliptical model to a process with an indefinite number of stages it is possible to use first the model for a large number n of stages and, after that, to use it again for another block of n stages, and so on. In this way we can utilize the information yielded by each block of n stages to adjust the initial distribution of the next block.

3. Permanence of the model over time

In this section we show the permanence over time of the continuous elliptical linear dynamic model. The updated distribution of the current global model H_t conditional to $\mathbf{Y}_t$ remains being continuous elliptical as it was the initial global vector H_0; then, this model behaves as the usual normal model in the sense of the maintenance of the initial distribution. We obtain a theorem about the updated distribution of the current global vector.

We will use the next statement, which is a special case of Lemma 1.3 of Fang, Kotz and Ng (1990): If $a > 0$, $b > 0$, $t \geq 0$ and g is a non-negative measurable function defined in $[0, \infty)$, then

$$\int_0^\infty \int_0^\infty x^{a-1} y^{b-1} g(t+x+y) dx dy = Beta(a,b) \int_0^\infty x^{a+b-1} g(t+x) dx. \tag{3.1}$$

Theorem 3.1. (Conditional distribution of the current global vector). *For each $t = 1, ..., n$ and each possible value $\mathbf{y}_t = (y_1', ..., y_t')'$ of $\mathbf{Y}_t$, the conditional distribution of H_t when $\mathbf{Y}_t = \mathbf{y}_t$ is*

$$(H_t \mid \mathbf{Y}_t = \mathbf{y}_t) \sim E_{d_t}\left(\mu_t^H, \Sigma_t^H, g_t^H\right), \tag{3.2}$$

where $\mu_t^H = \left(m_t', 0'_{(d_t-r)\times 1}\right)'$; Σ_t^H *is a block-diagonal matrix whose diagonal components are the matrices* $C_t, V_{t+1}, W_{t+1}, \ldots, V_n, W_n$; *and* g_t^H *is expressed as*

$$g_t^H(z) = \int_0^\infty w^{\frac{rt}{2}-1} g_0^H\left(z + q_t + w\right) dw, \tag{3.3}$$

where m_t, C_t *and* q_t *are defined as*

$$\begin{aligned} m_t &= G_t m_{t-1} + R_t F_t Q_t^{-1} e_t', \\ C_t &= R_t - R_t F_t Q_t^{-1} F_t' R_t, \\ q_t &= q_{t-1} + e_t' Q_t^{-1} e_t, \end{aligned}$$

with

$$\begin{aligned} R_t &= G_t C_{t-1} G_t' + W_t, \\ Q_t &= F_t' R_t F_t + V_t, \\ e_t &= y_t - F_t' G_t m_{t-1} \end{aligned}$$

and $q_0 = 0$.

Proof. For each $t = 0, 1, ..., n$ let $\overline{\mathbf{Y}}_t = (Y_0', \mathbf{Y}_t')' = (Y_0', Y_1', ..., Y_t')'$, where Y_0 is an arbitrary random vector independent of all the vectors appearing in the linear dynamic model. Then (3.2) is equivalent to

$$\left(H_t \mid \overline{\mathbf{Y}}_t = \overline{\mathbf{y}}_t\right) \sim E_{d_t}\left(\mu_t^H, \Sigma_t^H, g_t^H\right), \tag{3.4}$$

for each $t = 1, ..., n$, where $\overline{\mathbf{y}}_t' = (y_0', \mathbf{y}_t')' = (y_0', y_1', ..., y_t')'$, and y_0 is an arbitrary possible value of Y_0.

For $t = 0$ in (3.4), the expression becomes clearly true. Let it now be true for $t - 1$, that is,

$$\left(H_{t-1} \mid \overline{\mathbf{Y}}_{t-1} = \overline{\mathbf{y}}_{t-1}\right) \sim E_{d_{t-1}}\left(\mu_{t-1}^H, \Sigma_{t-1}^H, g_{t-1}^H\right). \tag{3.5}$$

We have

$$\begin{pmatrix} Y_t \\ H_t \end{pmatrix} = A_t H_{t-1}, \tag{3.6}$$

where A_t is the block matrix defined as

$$A_t = \begin{pmatrix} F_t' G_t & I_s & F_t' & 0 & \cdots & 0 & 0 \\ G_t & 0 & I_r & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & I_s & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & I_s & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & I_r \end{pmatrix}.$$

So $(Y_t', H_t')'$ is an affine transformation of H_{t-1}. Then from (3.5) and (3.6) it follows (see Marín (1998)) that

$$\left(\begin{pmatrix} Y_t \\ H_t \end{pmatrix} \middle| \overline{\mathbf{Y}}_{t-1} = \overline{\mathbf{y}}_{t-1} \right) \sim E_{s+d_t} \left(A_t \mu_{t-1}^H, A_t \Sigma_{t-1}^H A_t', g_t^* \right), \tag{3.7}$$

with

$$g_t^*(z) = \int_0^\infty w^{\frac{r}{2}-1} g_{t-1}^H (z+w) dw.$$

The distribution of H_t conditional to $\overline{\mathbf{Y}}_t = \overline{\mathbf{y}}_t$ is the same as the distribution of H_t in (3.7) conditional to $Y_t = y_t$. This distribution is (see Marín (1998)) $E_{d_t} \left(\mu_t^H, \Sigma_t^H, g_t^H \right)$, with μ_t^H and Σ_t^H as in (3.4) (and (3.2)) and $g_t^H(z) = \int_0^\infty w^{\frac{r}{2}-1} g_{t-1}^H \left(z + e_t' Q_t^{-1} e_t + w \right) dw$ with $Q_t = F_t' R_t F_t + V_t$. The expression (3.3) for g_t^H is easily obtained from the previous one by induction on t, starting from $t = 1$ and making use of (3.1). ■

4. Updated distribution and characteristics of the current vector of state

Theorem 4.1. (Updated distribution of the current vector of state). *With the same hypotheses and notations of theorem 3.1, for each $t = 1, ..., n$*

the conditional distribution of θ_t *when* $\mathbf{Y}_t = \mathbf{y}_t$ *is*

$$(\theta_t \mid \mathbf{Y}_t = \mathbf{y}_t) \sim E_r\left(\mu_t^\theta, \Sigma_t^\theta, g_t^\theta\right), \tag{4.1}$$

where $\mu_t^\theta = m_t$, $\Sigma_t^\theta = C_t$ *and*

$$g_t^\theta(z) = \int_0^\infty w^{\frac{d_t+rt-r}{2}-1} g_0^H\left(z + q_t + w\right) dw. \tag{4.2}$$

Proof. The conditional distribution of θ_t, obtained as the marginal distribution of θ_t in (3.2), results to be the elliptical (4.1) with the above values for μ_t^θ and Σ_t^θ and with $g_t^\theta(z) = \int_0^\infty w^{\frac{d_t-r}{2}-1} g_t^H\left(z+w\right) dw$ (see Marín (1998)). Expression (4.2) is obtained from the previous one for g_t^θ, by taking g_t^H as in (3.3) and making use of (3.1). ■

In the next corollary we show some updated probabilistic characteristics of the current vector of state. In the rest of this article we use the kurtosis measure $\gamma_2[X]$ defined by Mardia et al. (1979) as

$$\gamma_2[X] = E\left[\left((X - E[X])' Var[X]^{-1} (X - E[X])\right)^2\right];$$

we shall also use the notations

$$\begin{aligned} \phi(u) &= \int_0^\infty w^u g_0^H\left(q_t + w\right) dw, \\ \tau(u) &= \int_0^\infty w^u \exp\left\{-\frac{1}{2}\left(q_t + w\right)^\beta\right\} dw. \end{aligned}$$

Corollary 4.2. (Updated characteristics of the current vector of state). *With the same hypotheses and notations as in theorems 3.1 and 4.1, if we additionally suppose that*

$$\int_0^\infty z^{\frac{d_0+3}{2}} g_0^H(z) dz < \infty, \tag{4.3}$$

then, for each $t = 1, ..., n$, the mean vector, covariance matrix and kurtosis measure of vector θ_t conditional to $\mathbf{Y}_t = \mathbf{y}_t$ do exist and they are

$$E\left[\theta_t \mid \mathbf{Y}_t = \mathbf{y}_t\right] = \mu_t^\theta, \tag{4.4}$$

$$Var\left[\theta_t \mid \mathbf{Y}_t = \mathbf{y}_t\right] = \frac{1}{d_t + rt} \frac{\phi\left(\frac{d_t+rt}{2}\right)}{\phi\left(\frac{d_t+rt}{2} - 1\right)} \Sigma_t^\theta, \tag{4.5}$$

$$\gamma_2\left[\theta_t \mid \mathbf{Y}_t = \mathbf{y}_t\right] = \frac{r(r+2)(d_t+rt)}{d_t + rt + 2} \frac{\phi\left(\frac{d_t+rt}{2} - 1\right)\phi\left(\frac{d_t+rt}{2} + 1\right)}{\left(\phi\left(\frac{d_t+rt}{2}\right)\right)^2}. \tag{4.6}$$

Proof. From the stochastic representation $X \overset{d}{=} \mu + A'RU^{(n)}$ of a generic elliptical vector $X \sim E_n(\mu, \Sigma, g)$ (see Cambanis et al. (1981) and Marín (1998)), it can be proved that $E[X] = \mu$, $Var[X] = \frac{1}{n}E\left[R^2\right]\Sigma$ and $\gamma_2[X] = n^2 \frac{E[R^4]}{(E[R^2])^2}$, the moments of the *modular variable* R being $E[R^j] = \frac{\int_0^\infty z^{\frac{n+j}{2}-1} g(z) dz}{\int_0^\infty z^{\frac{n}{2}-1} g(z) dz}$.

Then, the moments $E\left[R^j\right]$, for $j = 1, ..., 4$, of the modular variable R of θ_t conditional to $\mathbf{Y}_t$ are

$$E\left[R^j\right] = \frac{\int_0^\infty v^{\frac{r+j}{2}-1} g_t^\theta(v)\, dv}{\int_0^\infty v^{\frac{r}{2}-1} g_t^\theta(v)\, dv} \tag{4.7}$$

$$= \frac{\int_0^\infty v^{\frac{r+j}{2}-1}\left(\int_0^\infty w^{\frac{d_t+rt-r}{2}-1} g_0^H(z + q_t + w)\, dw\right) dv}{\int_0^\infty v^{\frac{r}{2}-1}\left(\int_0^\infty w^{\frac{d_t+rt-r}{2}-1} g_0^H(z + q_t + w)\, dw\right) dv} \tag{4.8}$$

$$= \frac{Beta\left(\frac{r+j}{2}, \frac{d_t+rt-r}{2}\right)}{Beta\left(\frac{r}{2}, \frac{d_t+rt-r}{2}\right)} \frac{\int_0^\infty w^{\frac{d_t+rt+j}{2}-1} g_0^H(q_t + w)\, dw}{\int_0^\infty w^{\frac{d_t+rt}{2}-1} g_0^H(q_t + w)\, dw}, \tag{4.9}$$

where (4.9) follows from (4.8) by (3.1). The integral in the numerator of (4.9) is finite because of (4.3); therefore the moments do exist. And now expressions (4.4), (4.5) and (4.6) follow immediately. ■

5. Updated distributions and characteristics of future states and observations

Theorem 5.1. (Updated distribution of future states and observations). *With the same hypotheses and notations as in theorems 3.1 and 4.1, for each $t = 1, ..., n-1$ and for each $k = 1, ..., n-t$ the following statements hold.*

(i) The distribution of θ_{t+k} conditional to $\mathbf{Y}_t = \mathbf{y}_t$ is

$$(\theta_{t+k} \mid \mathbf{Y}_t = \mathbf{y}_t) \sim E_r\left(\mu^\theta_{t,k}, \Sigma^\theta_{t,k}, g^\theta_{t,k}\right), \tag{5.1}$$

where $\mu^\theta_{t,k}$ and $\Sigma^\theta_{t,k}$ are given by the recurrence

$$\begin{aligned}\mu^\theta_{t,k} &= G_{t+k}\mu^\theta_{t,k-1}, \\ \Sigma^\theta_{t,k} &= G_{t+k}\Sigma^\theta_{t,k-1}G'_{t+k} + W_{t+k},\end{aligned}$$

with $\mu^\theta_{t,0} = m_t$, $\Sigma^\theta_{t,0} = C_t$, and

$$g^\theta_{t,k}(z) = \int_0^\infty w^{\frac{d_t + rt - r}{2} - 1} g^H_0(z + q_t + w)dw. \tag{5.2}$$

(ii) The distribution of Y_{t+k} conditional to $\mathbf{Y}_t = \mathbf{y}_t$ is

$$(Y_{t+k} \mid \mathbf{Y}_t = \mathbf{y}_t) \sim E_s\left(\mu^Y_{t,k}, \Sigma^Y_{t,k}, g^Y_{t,k}\right), \tag{5.3}$$

where

$$\begin{aligned}\mu^Y_{t,k} &= F'_{t+k}\mu^\theta_{t,k}, \\ \Sigma^Y_{t,k} &= F'_{t+k}\Sigma^\theta_{t,k}F_{t+k} + V_{t+k}, \\ g^Y_{t,k}(z) &= \int_0^\infty w^{\frac{d_t + rt - s}{2} - 1} g^H_0(z + q_t + w)dw.\end{aligned} \tag{5.4}$$

Proof. (i) For each $t = 1, ..., n-1$ and each $k = 1, ..., n-t$ it is

$$H_{t+k} = A_{t+k}H_{t+k-1}, \tag{5.5}$$

where A_{t+k} is the block matrix defined as

$$A_{t+k} = \begin{pmatrix} G_{t+k} & 0 & I_r & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & I_s & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & I_s & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & I_r \end{pmatrix}.$$

Now starting from (3.2) it is easily proved by induction on k that for $k = 1, ..., n-t$ it is

$$(H_{t+k} \mid \mathbf{Y}_t = \mathbf{y}_t) \sim E_{d_{t+k}} \left(\mu^H_{t,k}, \Sigma^H_{t,k}, g^H_{t,k}\right) \tag{5.6}$$

where $\mu^H_{t,k} = \left(\left(\mu^\theta_{t,k}\right)', \left(0_{(d_{t+k}-r)\times 1}\right)'\right)'$; $\Sigma^H_{t,k}$ is the block-diagonal matrix whose diagonal components are $\Sigma^\theta_{t,k}, V_{t+k+1}, W_{t+k+1}, ..., V_n, W_n$; and

$$g^H_{t,k}(z) = \int_0^\infty w^{\frac{r+s}{2}-1} g^H_{t,k-1}(z+w)dw,$$

with $g^H_{t,0}(z) = g^H_t(z)$.

The distribution of $(\theta_{t+k}|\mathbf{Y}_t = \mathbf{y}_t)$ is the corresponding marginal in (5.6); this is elliptical (see Marín (1998)) with $\mu^\theta_{t,k}$ and $\Sigma^\theta_{t,k}$ as location and scale parameters. On the other hand, θ_{t+k} is an affine transformation of H_t, because $\theta_{t+k} = CH_{t+k} = CA_{t+k}A_{t+k-1}...A_{t+1}H_t$, where C is the block matrix $C = (I_r \; 0_{r\times(d_{t+k}-r)})$. Therefore (see Marín (1998)), its functional parameter is $g^\theta_{t,k}(z) = \int_0^\infty w^{\frac{d_t-r}{2}-1} g^H_t(z+w)dw$; by (3.1), this is equivalent to (5.2).

(ii) For each $k = 1, ..., n-t$ we obtain that

$$Y_{t+k} = B_{t+k}H_{t+k-1}, \tag{5.7}$$

where B_{t+k} is the block matrix $B_{t+k} = \begin{pmatrix} F'_{t+k}G_{t+k} & I_s & F'_{t+k} & 0_{s\times(d_{t+k}-r)} \end{pmatrix}$.

Since the conditional distribution of H_{t+k-1} is the elliptical one given by (5.6) (or (3.2), if $k = 1$), the conditional distribution of Y_{t+k} is elliptical, with location and scale parameters $\mu^Y_{t,k}$ and $\Sigma^Y_{t,k}$ given by

$$
\begin{aligned}
\mu^Y_{t,k} &= B_{t+k}\mu^H_{t,k-1} = F'_{t+k}G_{t+k}\mu^\theta_{t,k-1} = F'_{t+k}\mu^\theta_{t,k},\\
\Sigma^Y_{t,k} &= B_{t+k}\Sigma^H_{t,k-1}B'_{t+k} = F'_{t+k}\left(G_{t+k}\Sigma^\theta_{t,k-1}G'_{t+k} + W_{t+k}\right)F_{t+k} + V_{t+k}\\
&= F'_{t+k}\Sigma^\theta_{t,k}F_{t+k} + V_{t+k}.
\end{aligned}
$$

Now, Y_{t+k} is an affine transformation of H_t (see (5.7) and (5.5)). Therefore, its functional parameter is $g^Y_{t,k}(z) = \int_0^\infty w^{\frac{d_t-s}{2}-1}g^H_t(z+w)dw$ and, by (3.1), this is equivalent to (5.4). ■

Corollary 5.2. (Updated characteristics of future states and observations). *With the same hypothesis and notations as in theorem 5.1, if we additionally suppose (4.3), then the following statements hold.*

(i) The mean vector, covariance matrix and kurtosis measure of θ_{t+k} conditional to $\mathbf{Y}_t = \mathbf{y}_t$ do exist and are

$$
\begin{aligned}
E\left[\theta_{t+k} \mid \mathbf{Y}_t = \mathbf{y}_t\right] &= \mu^\theta_{t,k},\\
Var\left[\theta_{t+k} \mid \mathbf{Y}_t = \mathbf{y}_t\right] &= \frac{1}{d_t+rt}\frac{\phi\left(\frac{d_t+rt}{2}\right)}{\phi\left(\frac{d_t+rt}{2}-1\right)}\Sigma^\theta_{t,k},\\
\gamma_2\left[\theta_{t+k} \mid \mathbf{Y}_t = \mathbf{y}_t\right] &= \frac{r\left(r+2\right)\left(d_t+rt\right)}{d_t+rt+2}\frac{\phi\left(\frac{d_t+rt}{2}+1\right)\phi\left(\frac{d_t+rt}{2}-1\right)}{\left(\phi\left(\frac{d_t+rt}{2}\right)\right)^2}.
\end{aligned}
$$

(ii) The mean vector, covariance matrix and kurtosis measure of Y_{t+k} conditional to $\mathbf{Y}_t = \mathbf{y}_t$ do exist and are

$$
\begin{aligned}
E\left[Y_{t+k} \mid \mathbf{Y}_t = \mathbf{y}_t\right] &= \mu^Y_{t,k},\\
Var\left[Y_{t+k} \mid \mathbf{Y}_t = \mathbf{y}_t\right] &= \frac{1}{d_t+rt}\frac{\phi\left(\frac{d_t+rt}{2}\right)}{\phi\left(\frac{d_t+rt}{2}-1\right)}\Sigma^Y_{t,k},\\
\gamma_2\left[Y_{t+k} \mid \mathbf{Y}_t = \mathbf{y}_t\right] &= \frac{s\left(s+2\right)\left(d_t+rt\right)}{d_t+rt+2}\frac{\phi\left(\frac{d_t+rt}{2}+1\right)\phi\left(\frac{d_t+rt}{2}-1\right)}{\left(\phi\left(\frac{d_t+rt}{2}\right)\right)^2}.
\end{aligned}
$$

Proof. The proof is analogous to that of corollary 4.2. ■

6. Models with a multidimensional exponential power initial distribution

In this section we consider the class of elliptical models, in which the distribution of the global initial vector H_0 is multidimensional exponential power (for this distribution see Gómez et al. (1998)), that is, $H_0 \sim PE_{d_0}\left(\mu_0^H, \Sigma_0^H, \beta\right)$ for some $\beta \in (0, \infty)$. This distribution is obtained from (2.1) by taking

$$g_0^H(z) = \exp\left\{-\frac{1}{2}z^\beta\right\}. \tag{6.1}$$

So, this is an intermediate case, more general than the normal one but less than the elliptical one.

In this case the updated distribution of the current global vector H_t is the elliptical (3.2) where the functional parameter g_t^H is

$$g_t^H(z) = \int_0^\infty w^{\frac{rt}{2}-1} \exp\left\{-\frac{1}{2}(z + q_t + w)^\beta\right\} dw. \tag{6.2}$$

Since this distribution is not exponential power, there does not exist a closed "exponential power model". Nevertheless, the form (6.2) of the parameter g_t^H is very related to the form (6.1) of the parameter g_0^H, it maintains the same β as exponent, and, besides, it is relatively simple.

The updated distribution of θ_t, θ_{t+k} and Y_{t+k}, the current and future vectors of state and observations, are the elliptical ones (4.1), (5.1) and (5.3), where the functional parameters g_t^θ, g_{t+k}^θ and g_{t+k}^Y are

$$\begin{aligned} g_t^\theta(z) &= \int_0^\infty w^{\frac{d_t+rt-r}{2}-1} \exp\left\{-\frac{1}{2}(z + q_t + w)^\beta\right\} dw; \\ g_{t,k}^\theta(z) &= \int_0^\infty w^{\frac{d_t+rt-r}{2}-1} \exp\left\{-\frac{1}{2}(z + q_t + w)^\beta\right\} dw; \\ g_{t,k}^Y(z) &= \int_0^\infty w^{\frac{d_t+rt-s}{2}-1} \exp\left\{-\frac{1}{2}(z + q_t + w)^\beta\right\} dw. \end{aligned} \tag{6.3}$$

As for the updated probabilistic characteristics of the above vectors, they result to be simple operations among integrals of the same form as those of (6.2) to (6.3) and can be calculated by numerical integration. This characteristics are as follows. The covariance matrix and the kurtosis measure of θ_t are

$$Var\left[\theta_t \mid \mathbf{Y}_t = \mathbf{y}_t\right] = \frac{1}{d_t + rt} \frac{\tau\left(\frac{d_t+rt}{2}\right)}{\tau\left(\frac{d_t+rt}{2} - 1\right)} \Sigma_t^{\theta},$$

$$\gamma_2\left[\theta_t \mid \mathbf{Y}_t = \mathbf{y}_t\right] = \frac{r(r+2)(d_t + rt)}{d_t + rt + 2} \frac{\tau\left(\frac{d_t+rt}{2} - 1\right)\tau\left(\frac{d_t+rt}{2} + 1\right)}{\left(\tau\left(\frac{d_t+rt}{2}\right)\right)^2}.$$

For the future vectors of state we obtain

$$Var\left[\theta_{t+k} \mid \mathbf{Y}_t = \mathbf{y}_t\right] = \frac{1}{d_t + rt} \frac{\tau\left(\frac{d_t+rt}{2}\right)}{\tau\left(\frac{d_t+rt}{2} - 1\right)} \Sigma_{t,k}^{\theta},$$

$$\gamma_2\left[\theta_{t+k} \mid \mathbf{Y}_t = \mathbf{y}_t\right] = \frac{r(r+2)(d_t + rt)}{d_t + rt + 2} \frac{\tau\left(\frac{d_t+rt}{2} + 1\right)\tau\left(\frac{d_t+rt}{2} - 1\right)}{\left(\tau\left(\frac{d_t+rt}{2}\right)\right)^2}.$$

And for the future observations we have

$$Var\left[Y_{t+k} \mid \mathbf{Y}_t = \mathbf{y}_t\right] = \frac{1}{d_t + rt} \frac{\tau\left(\frac{d_t+rt}{2}\right)}{\tau\left(\frac{d_t+rt}{2} - 1\right)} \Sigma_{t,k}^{Y},$$

$$\gamma_2\left[Y_{t+k} \mid \mathbf{Y}_t = \mathbf{y}_t\right] = \frac{s(s+2)(d_t + rt)}{d_t + rt + 2} \frac{\tau\left(\frac{d_t+rt}{2} + 1\right)\tau\left(\frac{d_t+rt}{2} - 1\right)}{\left(\tau\left(\frac{d_t+rt}{2}\right)\right)^2}.$$

The updated distribution of H_t, θ_t, θ_{t+k} and Y_{t+k} are all of the form $E_d(\mu, \Sigma, g)$, where the functional parameter g is of the form

$$g(z) = \int_0^{\infty} w^{\frac{m}{2}-1} \exp\left\{-\frac{1}{2}(z + q + w)^{\beta}\right\} dw,$$

where m is natural, $q \geq 0$ and $\beta \in (0, \infty)$. This functional parameter is similar, both analytically and graphically, to the parameter $g(z) = \exp\left\{-\frac{1}{2}z^{\beta}\right\}$ of the exponential power distribution. This suggests that an exponential power distribution will be simplifying and fitted enough; and, on the other hand, it will give some "closedness" to the exponential power model. So, we now consider the problem of finding the exponential power distribution $PE_d(\mu^*, \Sigma^*, \beta^*)$ which is in some sense the closest one to the elliptical distribution $E_d(\mu, \Sigma, g)$.

As an easy to implement method to approach this problem we suggest the usual method of the moments. This leads to choose the values of μ^*, Σ^* and β^* which solve the system of equations

$$\left\{ \mu^* = E; \quad \frac{2^{\frac{1}{\beta^*}}\Gamma\left(\frac{d+2}{2\beta^*}\right)}{d\Gamma\left(\frac{d}{2\beta^*}\right)}\Sigma^* = Var; \quad d^2\frac{\Gamma\left(\frac{d+4}{2\beta^*}\right)\Gamma\left(\frac{d}{2\beta^*}\right)}{\left(\Gamma\left(\frac{d+2}{2\beta^*}\right)\right)^2} = \gamma_2 \right\},$$

where E, Var and γ_2 are the mean vector, covariance matrix and kurtosis measure of the distribution $E_d(\mu, \Sigma, g)$. The first member of the third equation is a strictly decreasing function of β^*, therefore there is only a solution for β^*, and this value can be easily calculated numerically: the values of μ^* and Σ^* are straightforward.

7. Example

We show an application of a linear dynamic model, as described in definition 2.1, with a exponential power distribution for vector H_0, to the pair of quantities *activity ratio* (Y_t) and *unemployment ratio* (x_t) in the Community of Valencia (Spain).

We suppose that a linear relation between activity ratio and unemployment ratio exists such that the intercept and the slope can vary along the time (it is equivalent to the variation of θ_t). So, we take $F_t' = (1, x_t)$ in order to include an independent term. We suppose that errors are uncorrelated among them and that they follow a symmetrical distribution, so we modelize them by a exponential power distribution that includes a wide range of symmetrical distributions.

We had 23 pairs (y_t, x_t) of quarterly observations ranging from 1983 to 1988 available. This observations were contributed by J. M. Bernardo and are shown in Girón et al. (1989). We used the seven first pairs to adjust the model and the others to test the model.

First we made $G_t = I$ because there are no reasons to assume a systematic trend in the evolution of θ_t. We adjusted the rest of the model by setting its initial parameters in a non formal way, based on the regression line between Y_t and x_t and on the simulation of a sequence of values of θ_t from consecutive pairs (x_t, y_t), (x_{t+1}, y_{t+1}). This lead us to employ $m_0 = \begin{pmatrix} 50.78 \\ -0.10 \end{pmatrix}$, $C_0 = \begin{pmatrix} 780.73 & -41.59 \\ -41.59 & 2.22 \end{pmatrix}$, $\beta = 2.00$; $V_t = 0.26$ and $W_t = \begin{pmatrix} 0.35 & 0 \\ 0 & 1.19 \end{pmatrix}$.

Then we calculated the updated distribution of θ_t and the updated characteristics of Y_t. Table 1 shows the mean vector, the covariance matrix of θ_t as well as the value β^* of the closest exponential power distribution (in the sense of section 6) to the elliptical distribution obtained for θ_t.

Table 1. Updated Distribution of θ_t

t	Mean	Covariance m.		β^*	t	Mean	Covariance m.		β^*
8	14.85	61.00	-2.89	1.03	16	19.23	81.05	-4.30	1.05
	1.86	-2.89	1.37			1.56	-4.19	0.23	
9	21.53	75.02	-3.51	1.04	17	19.67	82.09	-4.19	1.05
	1.28	-3.51	0.16			1.51	-4.19	0.21	
10	20.52	77.64	-3.81	1.05	18	19.82	83.14	-4.19	1.05
	1.38	-3.81	1.87			1.55	-4.19	0.21	
11	21.77	78.10	-3.62	1.04	19	18.86	82.99	-4.53	1.05
	1.24	-3.62	0.17			1.72	-4.53	0.25	
12	20.32	77.96	-3.94	1.04	20	18.31	84.44	-4.61	1.05
	1.42	-3.94	0.20			1.78	-4.61	0.25	
13	19.81	79.13	-3.99	1.05	21	18.01	85.55	-4.76	1.06
	1.46	-3.99	0.20			1.83	-4.76	0.27	
14	19.50	80.02	-4.11	1.05	22	18.10	86.79	-4.80	1.06
	1.49	-4.11	0.21			1.82	-4.80	0.27	
15	20.21	80.81	-4.02	1.05	23	17.36	86.95	-5.14	1.06
	1.40	-4.02	0.20			1.98	-5.14	0.30	

It can be observed that the independent term is more meaningful that the slope and it is noticed that the covariance between them is negative: when the one increases the other diminishes. The parameter β^* is slightly greater than 1, therefore the distribution is almost normal, though there is a little trend towards more platykurtic distributions.

Table 2 shows, the updated mean and standard deviation of Y_t, as well as the prediction error (the difference between the mean and the real observation) and the standard error (the ratio between the error and the standard deviation).

Table 2. Updated Characteristics of $\mathbf{Y}_t$

t	Mean	S. D.	Error	S. E.	t	Mean	S. D.	Error	S. E.
8	48.75	1.46	-5.39	-3.70	16	46.65	0.61	2.03	3.33
9	54.53	0.22	-5.68	-25.76	17	49.84	0.43	-0.50	-1.15
10	47.59	0.46	0.96	2.10	18	49.67	0.26	0.79	3.01
11	50.17	0.56	-1.72	-3.06	19	48.13	0.73	2.31	3.16
12	46.27	0.76	2.06	2.73	20	50.42	0.24	0.41	1.69
13	48.36	0.23	0.31	1.35	21	50.23	0.30	0.70	2.38
14	48.19	0.28	0.38	1.38	22	51.17	0.26	-0.20	-0.77
15	49.53	0.38	-1.11	-2.94	23	48.83	0.65	1.97	3.02

The errors have no trend and the standard errors are a little large; it may suggest that the value 0.26 chosen for V_t is too small. Nevertheless, the obtained results show that the exponential power model provides good predictions and that the distributions obtained in each time are a little more platykurtic than the normal, so the adjusted model is different to the standard normal one.

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