Moduli spaces of bundles and of pairs

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Let Λ be a fixed line bundle of degree d, and let $r \geq 2$.

For a bundle E, the slope is $\mu(E) = \frac{d}{r}$.
E is stable (resp. semistable) if for any proper $E' \subset E$, we have $\mu(E') < \mu(E)$ (resp. $\leq$).

$M_X(r, \Lambda) = \{$ moduli space of stable vector bundles $E \to X$ of rank r and det $E \cong \Lambda\}$
$\overline{M}_X(r, \Lambda) = \{$ moduli space of semistable vector bundles $\}$

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Physical interpretation

Fix a normalized metric of constant negative curvature on X.
Let E be a C^∞ hermitian vector bundle (of degree d and rank r).
Let $\mathcal{A}_E$ be the space of hermitian connections.
Let $\mathcal{G}_E$ be the group of unitary automorphisms of E.

Then $\overline{M}_X(r,\Lambda) \cong \{A \in \mathcal{A}_E \mid *F_A = \frac{d}{r} Id\}/\mathcal{G}_E$

Fix a point $p \in X$, and a small loop γ around p.
$\overline{M}_X(r,\Lambda) \cong \{\rho : \pi_1(X-p) \to SU(r) \mid \rho(\gamma) = e^{2\pi i d/r} Id\}$

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A pair is (E, ϕ), where $E \to X$ is a vector bundle and $\phi \in H^0(E)$.

Let L_0 be the trivial bundle.
Then $L_0 \xrightarrow{\phi} E$.
Let $\sigma \in \mathbb{R}$. We define the σ-slope:

- For $T = (L_0 \to E)$, $\mu_\sigma(T) = \frac{d}{r+1} + \sigma \frac{1}{r+1}$
- For $T = (0 \to E)$, $\mu_\sigma(T) = \frac{d}{r}$

T is σ-stable (resp. σ-semistable) if for any proper $T' \subset T$ we have $\mu_\sigma(T') < \mu_\sigma(T)$ (resp. $\leq$).

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This is a vortex type equation.

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The study of the moduli of pairs helps to study the moduli of bundles.

Important facts:

1. σ lies in an interval $\sigma \in (\sigma_m, \sigma_M)$.
2. There are finitely many *critical values*

$\sigma_m < \sigma_1 < \sigma_2 < \ldots < \sigma_L < \sigma_M$ such that:

- $\mathcal{N}_\sigma$ is constant in $\sigma \in (\sigma_{i-1}, \sigma_i)$
- $\mathcal{N}_{\sigma_i^-}$ and $\mathcal{N}_{\sigma_i^+}$ are birational, for $\sigma_i^\pm := \sigma_i \pm \epsilon$.

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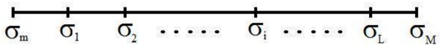

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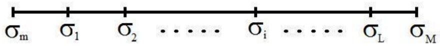

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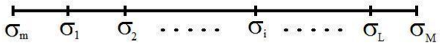

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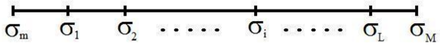

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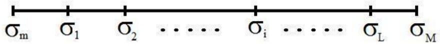

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Therefore $\mathcal{N}_{\sigma_i^+} - \mathcal{S}_{\sigma_i^+} = \mathcal{N}_{\sigma_i^-} - \mathcal{S}_{\sigma_i^-}$.

Obviously:

$$\mathcal{S}_{\sigma_i^\pm} = \{(E,\phi) \mid \sigma_i^\pm\text{-stable but } \sigma_i^\mp\text{-unstable}\}$$

In particular, such (E,ϕ) are properly σ_i-semistable.

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3. Let $\sigma = \sigma_m^+ := \sigma_m + \epsilon$. Then

$$\begin{array}{ccc} \mathcal{N}_{\sigma_m^+}(r,\Lambda) & \longrightarrow & \overline{M}_X(r,\Lambda) \\ (E,\phi) & \mapsto & E \end{array}$$

For all $E \in M_X(r,\Lambda)$, and any $\phi \neq 0$, we have that (E,ϕ) is σ_m^+-stable. Hence there is a fibration

$$\mathbb{P}H^0(E) \to \mathcal{U}_m \to M_X(r,\Lambda)\,,$$

and $\mathcal{U}_m \subset \mathcal{N}_{\sigma_m^+}$ is open.

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Let $\mathcal{S}_m := \mathcal{N}_{\sigma_m^+} - \mathcal{U}_m$.

4. Let $\sigma = \sigma_M^-$. Any σ-stable (E, ϕ) satisfies that

$$L_0 \to E \to F$$

is exact and F is semistable. Therefore there is a morphism

$$\begin{array}{rcl} \pi : \mathcal{N}_{\sigma_M^-}(r, \Lambda) & \longrightarrow & \overline{M}_X(r-1, \Lambda) \\ (E, \phi) & \mapsto & F \end{array}$$

If F is stable, then any nontrivial extension $L_0 \to E \to F$ gives a σ_M^--stable (E, ϕ). So

$$\mathbb{P}H^1(F^*) \to \mathcal{U}_M \to M_X(r-1, \Lambda)$$

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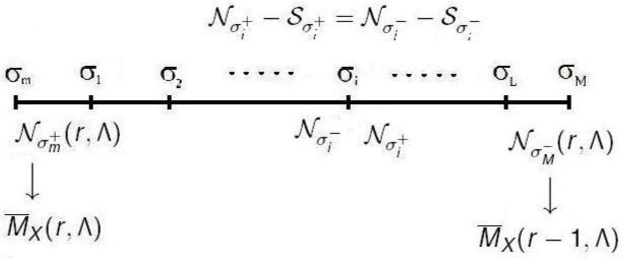

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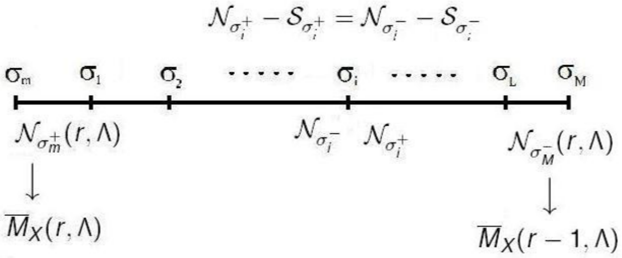

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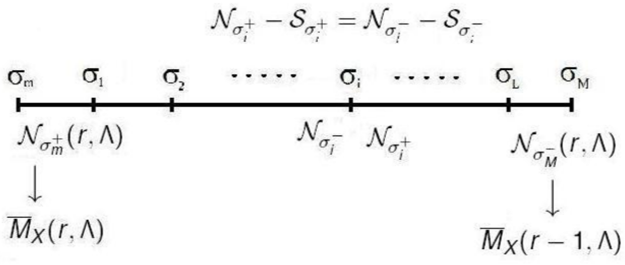

Irreducibility

Theorem 1

$M_X(r,\Lambda)$ and $\mathcal{N}_\sigma(r,\Lambda)$ are irreducible.

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Torelli theorem

Theorem 3 [Torelli theorem for the moduli spaces of pairs Math. Proc. Cambridge Phil. Soc., 146, 2009, 675-693]

Let $r, g \geq 2$ such that $r + g \geq 6$. Then

- $M_X(r, \Lambda) \cong M_{X'}(r', \Lambda') \implies X \cong X'$,
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Proof. We compute one Hodge structure:

- $H^3(M_X(r,\Lambda)) = H^1(X)$,
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Starting point: $H^3(\mathcal{N}_{\sigma_L^-}) = H^1(\mathcal{S}_{\sigma_L^-}) \oplus H^3(\mathbb{P}^N) = H^1(X)$ (by describing $\mathcal{S}_{\sigma_L^-}$ explicitly).

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The Brauer group of a smooth variety parametrizes the projective bundles modulo those who come from vector bundles.

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$$\begin{array}{ccccccc}
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\mathbb{Z}\cdot cl(\mathbb{P}H^1) & \to & Br(M_X(r,\Lambda)) & \to & Br(\mathcal{N}_{\sigma_M^-}(r+1,\Lambda)) & \to & 0
\end{array}$$

So $Br(\mathcal{N}_{\sigma_M^-}(r+1,\Lambda)) = Br(\mathcal{N}_{\sigma_m^+}(r,\Lambda))$.

Now use that $Br(\mathcal{N}_\sigma(r,\Lambda))$ is independent of σ (by birationality), and $\mathcal{N}_{\sigma_M^-}(2,\Lambda) = \mathbb{P}^N$.

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Stable rationality

Z is stably rational if $Z \times \mathbb{P}^N$ is rational for some N, i.e. $Z \times \mathbb{P}^N \simeq \mathbb{P}^{n+N}$.

Theorem 5 [I. Biswas, M. Logares and V.M., Brauer group of moduli spaces of pairs, arxiv-1009.5204]

$\mathcal{N}_\sigma(r, \Lambda)$ is stably rational.

Proof. Recall the fibrations:

$$\mathbb{P}^n \longrightarrow \mathcal{N}_{\sigma_M^-}(r+1, \Lambda) \longrightarrow M_X(r, \Lambda)$$
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As the Brauer group is trivial,
$Z \simeq \mathcal{N}_{\sigma_M^-}(r+1,\Lambda) \times \mathbb{P}^m \simeq \mathcal{N}_{\sigma_m^+}(r,\Lambda) \times \mathbb{P}^n$

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Theorem 7 [Hodge structures of the moduli space of pairs, Inter. J. Math. To appear]

$H^k(M_X(r,\Lambda)) \subset \bigotimes^* H^1(X)$
$H^k(\mathcal{N}_\sigma(r,\Lambda)) \subset \bigotimes^* H^1(X)$

Proof.
It follows from a detailed study of the flip locus.

Starting point: $H^*(JacX) \subset \bigotimes^* H^1(X)$

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A σ_i-semistable triple is an extension

$$\begin{array}{ccc} L_0 & \to & E' \\ \downarrow & & \downarrow \\ L_0 & \to & E \\ \downarrow & & \downarrow \\ 0 & \to & E'' \end{array}$$

(or upside-down). So the flip locus is a fibration over $\mathcal{N}_{\sigma_i}(r', \Lambda') \times M(r'', d'') \approx \mathcal{N}_{\sigma_i}(r', \Lambda') \times M(r'', \Lambda'') \times \mathrm{Jac} X$, and work by induction on r.

Project [J. Sánchez]

Motives of $M_X(r, \Lambda)$ and $\mathcal{N}_\sigma(r, \Lambda)$ are generated by the motive of $X \rightsquigarrow M_X(r, \Lambda)$ and $\mathcal{N}_\sigma(r, \Lambda)$ satisfy the Hodge conjecture for X generic.

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Hodge polynomials

For Z compact smooth algebraic variety, $e(Z) = \sum h^{pq}(Z) u^p v^q$

For Z non-smooth or non-compact,
$e(Z) = \sum \left(\sum_k h^{pq}(H^k_c(Z)) \right) u^p v^q$
Then for $X = Y \sqcup U$, we have $e(X) = e(Y) + e(U)$.

Theorem 8 [V.M., D. Ortega and M-J. Vázquez, Hodge polynomials of the moduli spaces of pairs, Inter. J. Math., 18, 2007, 695-721]

For $r = 2$ and $gcd(r, d) = 1$, we have that $e(M_X(2, \Lambda)) =$

$$= \frac{(1+u^2v)^g(1+uv^2)^g - (uv)^g(1-u)^g(1-v)^g}{(1-uv)(1-(uv)^2)}$$

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$e(\mathcal{N}_{\sigma_i^-}) = e(\mathcal{N}_{\sigma_i^+}) - e(\mathcal{S}_{\sigma_i^+}) + e(\mathcal{S}_{\sigma_i^-})$

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Detailed study of flip locus $\rightsquigarrow e(\mathcal{S}_{\sigma_i^\pm})$

Then use $e(\mathcal{N}_{\sigma_m^+}) = e(M_X(2,\Lambda))e(\mathbb{P}H^0(E))$.

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Theorem 9 [Hodge polynomials of the moduli spaces of rank 3 pairs Geom. Dedicata, 136, 2008, 17-46]

For $r = 3$, $gcd(r, d) = 1$, we have $e(M_X(3, \Lambda)) =$

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Open questions

Some other problems:

- Hodge polynomials of the moduli spaces of pairs for any rank r.
- Analyse moduli spaces of bundles for $gcd(r, d) \neq 1$.
- Other moduli spaces: coherent systems (E, V), triples $\phi : E_0 \to E_1$, Higgs bundles $\Phi : E \to E \otimes K$, etc.
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