ANOTHER PROOF FOR THE PRESENTATION OF THE QUANTUM COHOMOLOGY OF THE MODULI OF BUNDLES OVER A RIEMANN SURFACE

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Abstract. The presentation of the quantum cohomology of the moduli space of stable vector bundles of rank two and odd degree with fixed determinant over a Riemann surface of genus $g > 2$ is obtained. The argument avoids the use of gauge theory, providing an alternative proof to the one given in [1].

Let $\Sigma = \Sigma_g$ be a closed Riemann surface of genus $g > 2$ and let M_Σ be the moduli space of rank 2 holomorphic stable vector bundles E with $\det E = \Lambda$, where Λ is a fixed line bundle of odd degree over Σ. In [1] a presentation of the quantum cohomology ring $QH^*(M_\Sigma)$ of M_Σ was given by using the relationship of the Gromov-Witten invariants of M_Σ with some Donaldson invariants of the 4-manifold $\Sigma \times \mathbb{S}^2$. In this way, the knowledge of the instanton Floer homology of the 3-manifold $\Sigma \times \mathbb{S}^1$ gathered in [2] allowed us to find the presentation of $QH^*(M_\Sigma)$.

It is the purpose of this note to give a purely algebro-geometrical argument to get the presentation of $QH^*(M_\Sigma)$, avoiding any reference to the material in [2]. This provides a method which does not use gauge theory and does not require any knowledge of Donaldson invariants. It originated in a question asked to the author by Bernd Siebert (see also [3, section 3]). We shall follow those arguments in [1] which are algebro-geometrical and give alternative proofs wherever any reference to [2] is given.

By [1] the quantum cohomology $QH^*(M_\Sigma)$ is generated by elements α, β and $\psi_1, \dots, \psi_{2g}$, canonically associated to elements in the homology of Σ. Here $\deg \alpha = 2$, $\deg \beta = 4$ and $\deg \psi_i = 3$. There is an action of the symplectic group $\mathrm{Sp}\,(2g, \mathbb{Z})$ on $QH^*(M_\Sigma)$ acting on $\{\psi_i\}_{1 \le i \le 2g}$ in the standard way (and trivially on α and β). The element $\gamma = -2 \sum_{i=1}^g \psi_i \psi_{g+i}$ is invariant under $\mathrm{Sp}\,(2g, \mathbb{Z})$. The starting point is the following result

Proposition 1. ([1, proposition 10]) *The $Sp\,(2g, \mathbb{Z})$-decomposition of $QH^*(M_\Sigma)$ is*

$$QH^*(M_\Sigma) = \bigoplus_{k=0}^{g-1} \Lambda_0^k H^3 \otimes \frac{\mathbb{Q}[\alpha, \beta, \gamma]}{(Q^1_{g-k}, Q^2_{g-k}, Q^3_{g-k})}$$

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where Q^i_r are defined recursively by setting $Q^1_0=1$, $Q^2_0=0$, $Q^3_0=0$ and, for $0\le r\le g-1$, by

$$\left\{\begin{array}{l} Q^1_{r+1}=\alpha Q^1_r+r^2Q^2_r \\ Q^2_{r+1}=(\beta+c_{r+1})Q^1_r+\frac{2r}{r+1}Q^3_r \\ Q^3_{r+1}=\gamma Q^1_r \end{array}\right.$$

for some numbers $c_r\in\mathbb{Q}$, $1\le r\le g$, dependent on g and r. □

Here $H^3=<\psi_1,\dots,\psi_{2g}>$ and $\Lambda^k_0H^3$ is the irreducible sub-Sp$(2g,\mathbb{Z})$-representation of Λ^kH^3 with dominant vector $\psi_1\cdots\psi_k$. The recurrence in [1, proposition 18] is stated with $Q^3_{r+1}=\gamma Q^1_r+d_{r+1}Q^2_r$, for some $d_{r+1}\in\mathbb{Q}$. But this number $d_{r+1}=0$, since by [1, lemma 17] we have $\gamma J_r\subset J_{r+1}$, where $J_r=(Q^1_r,Q^2_r,Q^3_r)$. So $\gamma Q^1_r\in J_{r+1}$ and this forces $Q^3_{r+1}=\gamma Q^1_r$.

The purpose of this paper is to give an alternative proof (avoiding the use of gauge theory) of the following result

Theorem 2. ([1, proposition 19]) *We have $c_r=(-1)^{g+r}8$, for $1\le r\le g$.*

In order to get c_r we need to use the Gromov-Witten invariants of degree 1 computed in [1] by purely algebro-geometrical methods. More concretely, let A be the positive generator of $\pi_2(M_\Sigma)=\mathbb{Z}$ and let $\mathbb{A}(\Sigma)=\mathbb{Q}[\alpha,\beta]\otimes\Lambda^*(\psi_1,\cdots,\psi_{2g})$ be the free graded algebra on the elements α, β and ψ_i, $1\le i\le 2g$. Then we have

Lemma 3. ([1, lemma 10]) *Let $\alpha^a\beta^b\psi_{i_1}\cdots\psi_{i_r}\in\mathbb{A}(\Sigma)$ have degree $6g-2$ where $g\ge3$. Then the Gromov-Witten invariant of M_Σ of degree one is*

$$\Psi^{M_\Sigma}_A(\alpha,\overset{(a)}{\dots},\alpha,\beta,\overset{(b)}{\dots},\beta,\psi_{i_1},\dots,\psi_{i_r})=\langle(4\omega+X)^a(X^2)^b\phi_{i_1}\cdots\phi_{i_r}X^r,[J]\rangle, \quad (1)$$

evaluated on the Jacobian J of Σ, where $X^{2g-1+i}=\frac{(-8)^i}{i!}\omega^i\in H^*(J)$. □

Here $\{\phi_i\}$ is a standard symplectic basis of $H^1(J)$ canonically associated to $\{\psi_i\}$ and $\omega=\sum_{i=1}^g\phi_i\wedge\phi_{g+i}$ is the volume form for the Jacobian.

We write for any $z=\alpha^a\beta^b\psi_{i_1}\cdots\psi_{i_r}\in\mathbb{A}(\Sigma)$ of degree $6g-6+4d$, $d\ge0$,

$$\langle\alpha^a\beta^b\psi_{i_1}\cdots\psi_{i_r}\rangle_{g,d}=\Psi^{M_\Sigma}_{dA}(\alpha,\overset{(a)}{\dots},\alpha,\beta,\overset{(b)}{\dots},\beta,\psi_{i_1},\dots,\psi_{i_r}),$$

and extend the definition to any homogeneous element of $\mathbb{A}(\Sigma)$ by linearity. Note that $\langle\rangle_{g,0}$ is evaluation of an element in $H^*(M_\Sigma)$ of degree $6g-6$ against the fundamental class. As a corollary of lemma 3 we get that $\langle\psi_g\psi_{2g}z\rangle_{g,1}=\langle z\rangle_{g-1,1}$, for any $z\in\mathbb{A}(\Sigma_{g-1})$ of degree $6g-8$, where $\mathbb{A}(\Sigma_{g-1})=\mathbb{Q}[\alpha,\beta]\otimes\Lambda^*(\psi_1,\dots\psi_{g-1},\psi_{g+1},\dots,\psi_{2g})\subset\mathbb{A}(\Sigma)$. This is true for any $g\ge4$. To have a similar statement for any genus, we define $\langle\alpha^a\beta^b\psi_{i_1}\cdots\psi_{i_r}\rangle_{g,1}$ as the right hand side of (1) for any $g\ge1$. Therefore $\langle\psi_g\psi_{2g}z\rangle_{g,1}=\langle z\rangle_{g-1,1}$, for any $g\ge2$ and any z of degree $6g-8$. Note that $\langle\psi_g\psi_{2g}z\rangle_{g,0}=-\langle z\rangle_{g-1,0}$, for $z\in\mathbb{A}(\Sigma_{g-1})$ of degree $6g-12$ and any $g\ge2$ (see [5]).

From proposition 1, we may write an iterative formula for $Q_r=Q^1_r$ as follows:

$$Q_{r+1}=\alpha Q_r+r^2(\beta+c_r)Q_{r-1}+2r(r-1)\gamma Q_{r-2},$$

for $0 \leq r \leq g-1$. Write $Q_r = q_r + C_r + \cdots$, where q_r is the leading term and C_r is the first quantum correction. Therefore q_r satisfies the recursive relation

$$q_{r+1} = \alpha q_r + r^2 \beta q_{r-1} + 2r(r-1)\gamma q_{r-2} \tag{2}$$

with $q_0 = 1$, and C_r satisfies the recursive relation

$$C_{r+1} = \alpha C_r + r^2 \beta C_{r-1} + 2r(r-1)\gamma C_{r-2} + r^2 c_r q_{r-1} \tag{3}$$

with $C_0 = 0$. The fact that Q_r is a relation for a suitable piece of the $\mathrm{Sp}\,(2g, \mathbb{Z})$-decomposition of $QH^*(M_\Sigma)$ given in proposition 1 implies that

$$\psi_{g-k+1} \cdots \psi_g Q_{g-k+1} = 0$$

as a quantum product in $QH^*(M_\Sigma)$. Multiplying by β^{g-k-1} and $\psi_{2g-k+1} \cdots \psi_{2g}$ we have an element of degree $6g-2$,

$$\psi_{g-k+1}\psi_{2g-k+1} \cdots \psi_g \psi_{2g} Q_{g-k+1} \beta^{g-k-1} = 0.$$

This implies

$$(-1)^k \langle C_{g-k+1} \beta^{g-k-1} \rangle_{g-k,0} + \langle q_{g-k+1} \beta^{g-k-1} \rangle_{g-k,1} = 0, \tag{4}$$

for any $0 \leq k \leq g-1$.

Recall that $\beta^g = 0$ and $\beta^{g-1}\gamma = 0$ in $H^*(M_\Sigma)$, by [5, page 148], so in (4) we only need C_{g-k+1} modulo β and γ. By the recursive relation (2), $q_r = \alpha^r \pmod{\beta, \gamma}$ and by (3),

$$C_{g-k+1} = \sum_{r=1}^{g-k} r^2 c_r \alpha^{g-k-1} \qquad \pmod{\beta, \gamma}, \tag{5}$$

for $g-k+1 \geq 2$. Now recall that

$$\langle \alpha^{g-k-1} \beta^{g-k-1} \rangle_{g-k,0} = (-1)^{g-k-1} 4^{g-k-1} (g-k-1)! \tag{6}$$

again by [5]. Finally we need

Proposition 4. *We have $\langle q_{g+1} \beta^{g-1} \rangle_{g,1} = (-1)^g 4^g (g+1)!$, for any $g \geq 1$.*

Proof. By lemma 3 we need to evaluate $q_{g+1}(4\omega + X, X^2, -2\omega X^2) X^{2g-2}$, where $X^{2g-1+i} = \frac{(-8\omega)^i}{i!}$, on the Jacobian J (this holds for $g \geq 3$, and for $g = 1, 2$ by the convention above). Since this expression is homogeneous, it equals

$$\omega^g \frac{1}{X} q_{g+1}(4 + X, X^2, -2X^2)$$

under the substitution $X^i \mapsto \frac{(-8)^i}{i!}$. Make the change of variable $X = -8Y$, so we want to compute

$$g! \frac{1}{-8Y} q_{g+1}(4 - 8Y, 64Y^2, -128Y^2) = \left(g! \frac{1}{-8} 4^{g+1} \right) \frac{1}{Y} q_{g+1}(1 - 2Y, 4Y^2, -2Y^2),$$

where $Y^i \mapsto \frac{1}{i!}$. This is the residue

$$Res_{z=0} \left(-2^{2g-1} g! \frac{q_{g+1}(1 - 2z, 4z^2, -2z^2)}{z^2} e^{1/z} \right). \tag{7}$$

Define the generating function for q_g as (see [4])

$$F(t)=\sum_{g=0}^{\infty}\frac{q_g}{g!}t^g.$$

The recursion relation (2) gives

$$F'=\alpha F+\beta t^2F'+\beta tF+2\gamma t^2 F,$$

which yields (see [4, proposition 2.5])

$$\frac{F'}{F}=\frac{\alpha+\beta t+2\gamma t^2}{1-\beta t^2}.$$

Substituting $\alpha=1-2z$, $\beta=4z^2$ and $\gamma=-2z^2$ we get

$$\frac{F'}{F}=\frac{1-2z+4z^2t-4z^2t^2}{1-4z^2t^2}=1-\frac{2z}{1+2zt},$$

from where we deduce (recalling that $F(0)=1$)

$$F(t)=\frac{e^t}{1+2zt}.$$

Now we compute

$$Res_{z=0}(\frac{F(t)}{z^2}e^{1/z})=-Res_{z=-1/2t}(\frac{F(t)}{z^2}e^{1/z})=-\frac{1}{2t}\cdot\frac{e^t}{1/4t^2}e^{-2t}=-2te^{-t}.$$

In this expression, the coefficient of t^{g+1} is $-2(-1)^g/g!$, so (7) equates to

$$2^{2g-1}g!(-1)^g2(g+1)=(-1)^g4^g(g+1)!$$

□

With this proposition together with (5) and (6), we get that (4) reduces to

$$(-1)^k\sum_{r=1}^{g-k}r^2c_r(-1)^{g-k-1}4^{g-k-1}(g-k-1)!+(-1)^{g-k}4^{g-k}(g-k+1)!=0.$$

for $g-k\geq 1$. This gives

$$\sum_{r=1}^{g-k}r^2c_r=(-1)^k4(g-k+1)(g-k)=(-1)^k8\binom{g-k+1}{2},$$

which solves to $c_r=(-1)^{g+r}8$, $1\leq r\leq g$.

References

[1] V. Muñoz, 'Quantum cohomology of the moduli space of stable bundles over a Riemann surface', *Duke Math. J.* 98 (1999) 525–540.

[2] V. Muñoz, 'Ring structure of the Floer cohomology of $\Sigma \times \mathbb{S}^1$', *Topology* 38 (1999) 517–528.

[3] B. Siebert, 'An update on (small) quantum cohomology', *AMS/IP Stud. Adv. Math.* 10 (1999) 279–312.

[4] B. Siebert and G. Tian, 'Recursive relations for the cohomology ring of moduli spaces of stable bundles', *Turkish J. Math.* 19 (1995) 131–144.

[5] M. Thaddeus, 'Conformal field theory and the cohomology of the moduli space of stable bundles', *J. Differential Geom.* 35 (1992) 131–150.

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