# Symplectically aspherical manifolds with nontrivial $\pi_2$ and with no Kähler metrics

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**Abstract**

In a previous paper, the authors show some examples of compact symplectic solvmanifolds, of dimension six, which are cohomologically Kähler and they do not admit Kähler metrics because their fundamental groups cannot be the fundamental group of any compact Kähler manifold. Here we generalize such manifolds to higher dimension and, by using Auroux symplectic submanifolds [3], we construct four-dimensional symplectically aspherical manifolds with nontrivial $\pi_2$ and with no Kähler metrics.

## 1 Introduction

During the last years, the study of *symplectic manifolds* has been of much interest. These manifolds appeared first in mathematical physics, but they are now of independent interest due to their relationship to differential and algebraic geometry.

A *symplectic manifold* is a pair $(M, \omega)$ where $M$ is a $2n$–dimensional differentiable manifold and $\omega$ is a closed non-degenerate 2–form on $M$. The form $\omega$ is called a *symplectic form*. Darboux's theorem states that any sufficiently small neighborhood in a symplectic manifold is symplectomorphic to an open set in $\mathbb{R}^{2n}$ with the canonical skew-symmetric bilinear form $\sum_{i=1}^{n} dx^i \wedge dx^{n+i}$.

Any symplectic manifold $(M, \omega)$ carries an *almost complex structure* $J$ compatible with the symplectic form $\omega$, which means that $\omega(X, Y) = \omega(JX, JY)$ for any $X$, $Y$ vector fields on $M$ (see [22, 23]). In particular, if $(M, \omega)$ possesses an *integrable* almost complex structure $J$ compatible with the symplectic form $\omega$, such that the Riemannian metric $g$ given by $g(X, Y) = -\omega(JX, Y)$ is positive definite, then $(M, \omega, J)$ is said to be a Kähler manifold with *Kähler metric* $g$. Thus, one can think of a symplectic manifold as a generalization of a Kähler manifold, and it is natural to ask: Which manifolds carry symplectic forms but not Kähler metrics?

Several geometric methods to construct symplectic manifolds were given by different authors (see for example [3, 4, 6, 9, 13, 16, 20, 21]). Many of the symplectic manifolds there presented do not admit a Kähler metric since either they are not formal or do not satisfy the hard Lefschetz theorem, or they fail both properties of compact Kähler manifolds.

In order to find more classes of symplectic manifolds, especially some with no Kähler metric, we generalize the construction of [11]. There the authors show examples of compact symplectic solvmanifolds $M^6(k)$, of dimension six, each one of which is formal and hard Lefschetz, but it does not possess Kähler metrics because its fundamental group cannot be the fundamental group of any compact Kähler manifold according to the results given by Campana in [5]. In Section 3 we present the compact symplectic manifolds $M^{2(n+1)}(k)$ as a generalization to higher dimension of $M^6(k)$ and, in Proposition 3.1, we prove that each manifold $M^{2(n+1)}(k)$ is formal and hard Lefschetz. Again, each one of the manifolds $M^8(k)$ does not have Kähler metrics since it fails the properties of the fundamental group of a compact Kähler manifold proved by Campana in [5]. But, we do not know whether or not $M^{2(n+1)}(k)$, for $n \geq 4$, admits Kähler metrics. However we show that, when $n$ is even, all of them have indefinite Kähler metrics.

On the other hand, a symplectic form $\omega$ on $M$ is said to be *symplectically aspherical* if the restriction $[\omega]|_{\pi_2(M)} = 0$, that is,

$$\int_{S^2} f^*\omega = 0$$

for every map $f\colon S^2 \to M$. In this case, the symplectic manifold $(M, \omega)$ is said to be *symplectically aspherical*. Such manifolds have been very relevant in the study of the Arnold conjecture [12]. Clearly, any symplectic manifold $(M, \omega)$ with second fundamental group $\pi_2(M) = 0$ is symplectically aspherical. Examples of Kähler and non-Kähler 4–dimensional symplectically aspherical manifolds with nontrivial $\pi_2$ were obtained by Gompf in [14]. There, it is mentioned that J. Kollár produced, in an unpublished paper, another construction of symplectically aspherical Kähler manifolds with $\pi_2 \neq 0$. Recently in [19] examples of symplectically aspherical symplectic manifolds are given by using Donaldson symplectic submanifolds [9].

In Section 4 we construct compact symplectically aspherical symplectic manifolds of dimension 4 with $\pi_2 \neq 0$ by using the symplectic submanifolds obtained by Auroux in [3] as an extension to higher rank bundles of the symplectic submanifolds constructed by Donaldson in [9].

In Theorem 4.3 we prove that any 4–dimensional Auroux symplectic submanifold of the manifolds $M^{2(n+1)}(k)$ is a symplectically aspherical manifold with $\pi_2 \neq 0$ and does not admit Kähler metrics for $n \leq 3$.

## 2 Preliminaries

In this section, we recall some definitions and results about formal manifolds and the hard Lefschetz property, that we will need in the next sections.

A *differential algebra* $(A, d)$ is a graded commutative algebra $A$ over the real numbers, with a differential $d$ which is a derivation, i.e., $d(a \cdot b) = (da) \cdot b + (-1)^{\deg(a)} a \cdot (db)$, where $\deg(a)$ is the degree of $a$.

A differential algebra $(A, d)$ is said to be *minimal* if it satisfies: *a*) $A$ is free as an algebra, that is, $A$ is the free algebra $\bigwedge V$ over a graded vector space $V = \oplus V^i$, and *b*) there exists a collection of generators $\{a_\tau, \tau \in I\}$, for some well ordered index set $I$, such that $\deg(a_\mu) \leq \deg(a_\tau)$ if $\mu < \tau$ and each $da_\tau$ is expressed in terms of preceding $a_\mu$ ($\mu < \tau$). This implies that $da_\tau$ does not have a linear part, i.e., it lives in $\bigwedge V^{>0} \cdot \bigwedge V^{>0} \subset \bigwedge V$.

Morphisms between differential algebras are required to be degree preserving algebra maps which commute with the differentials. Given a differential algebra $(A, d)$, we denote by $H^*(A)$ its cohomology. $A$ is *connected* if $H^0(A) = \mathbb{R}$, and $A$ is *one-connected* if, in addition, $H^1(A) = 0$.

A differential algebra $(\mathcal{M}, d)$ is said to be a *minimal model* of the differential algebra $(A, d)$ if $(\mathcal{M}, d)$ is minimal and there exists a morphism of differential graded algebras $\rho\colon (\mathcal{M}, d) \longrightarrow (A, d)$ inducing an isomorphism $\rho^*\colon H^*(\mathcal{M}) \longrightarrow H^*(A)$ on cohomology. Halperin in [17] proved that any connected differential algebra $(A, d)$ has a minimal model unique up to isomorphism.

A minimal model $(\mathcal{M}, d)$ is said to be *formal* if there is a morphism of differential algebras $\psi\colon (\mathcal{M}, d) \longrightarrow (H^*(\mathcal{M}), d = 0)$ that induces the identity on cohomology.

A *minimal model* of a connected differentiable manifold $M$ is a minimal model $(\bigwedge V, d)$ for the de Rham complex $(\Omega M, d)$ of differential forms on $M$. If $M$ is a simply connected manifold, then the dual of the real homotopy vector space $\pi_i(M) \otimes \mathbb{R}$ is isomorphic to $V^i$ for any $i$. We shall say that $M$ is *formal* if its minimal model is formal or, equivalently, the differential algebras $(\Omega M, d)$ and $(H^*(M), d = 0)$ have the same minimal model. (For details see [15] for example.)

An algebraic-topological condition for the formality of a manifold $M$ is the existence of a morphism $\rho\colon (H^*(M), d = 0) \longrightarrow (\bigwedge^* V, d)$ of differential algebras inducing the identity on cohomology. Consider a map $\rho$ defined by choosing closed forms representatives for each cohomology class of $M$. But notice that, in general, the map $\rho$ is not a morphism of algebras.

In [10] the condition of the *hard Lefschetz property* for a symplectic manifold is weaken to the *s–Lefschetz property* as follows.

**Definition 2.1** *Let $(M, \omega)$ be a compact symplectic manifold of dimension $2n$. We say that $M$ is s–Lefschetz with $s \leq (n-1)$ if*

$$[\omega]^{n-i} : H^i(M) \longrightarrow H^{2n-i}(M)$$

*is an isomorphism for all $i \leq s$.*

Note that $M$ is $(n-1)$–Lefschetz if and only if $M$ satisfies the hard Lefschetz theorem.

# 3 The manifolds $M^{2(n+1)}(k)$

Let $G^{2n+1}(k)$ be the connected completely solvable Lie group of dimension $2n+1$ consisting of matrices of the form

$$a = \begin{pmatrix} E_{2n} & {}^tO_{2n} & A_{2n} \\ O_{2n} & 1 & z \\ O_{2n} & 0 & 1 \end{pmatrix},$$

where $z \in \mathbb{R}$, $O_{2n}$ is the $1 \times 2n$ matrix with all the entries equal to zero, ${}^tO_{2n}$ denotes the transposed matrix of $O_{2n}$, $A_{2n}$ is the $2n \times 1$ matrix $(x_1, y_1, x_2, y_2, \cdots, x_n, y_n)$ with $x_i, y_i \in \mathbb{R}$ $(1 \leq i \leq n)$, and $E_{2n}$ is the diagonal $2n \times 2n$ matrix whose principal diagonal is the vector $(e^{kz}, e^{-kz}, e^{kz}, e^{-kz}, \cdots, e^{kz}, e^{-kz})$, of length $2n$, being $k$ a real number different from zero. Then a global system of coordinates $x_i, y_i, z$ $(1 \leq i \leq n)$ for $G^{2n+1}(k)$ is given by $x_i(a) = x_i$, $y_i(a) = y_i$, $z(a) = z$. A standard calculation shows that a basis for the right invariant 1–forms on $G^{2n+1}(k)$ consists of

$$\{dx_i - k\, x_i\, dz,\ dy_i + k\, y_i\, dz,\ dz \mid 1 \leq i \leq n\}.$$

Alternatively, the Lie group $G^{2n+1}(k)$ may be described as a semidirect product $G^{2n+1}(k) = \mathbb{R} \ltimes_\psi \mathbb{R}^{2n}$, where $\psi(z)$ is the linear transformation of $\mathbb{R}^{2n}$ given by the diagonal matrix $E_{2n}$

for any $z \in \mathbb{R}$. Thus, $G^{2n+1}(k)$ has a discrete subgroup $\Gamma^{2n+1}(k)$ such that the quotient space $N^{2n+1}(k) = \Gamma^{2n+1}(k)\backslash G^{2n+1}(k)$ is compact. Therefore the forms $dx_i - k\, x_i\, dz$, $dy_i + k\, y_i\, dz$, $dz$ $(1 \leq i \leq n)$ descend to 1–forms $\alpha_i, \beta_i, \gamma$ $(1 \leq i \leq n)$ on $N^{2n+1}(k)$ satisfying

$$d\alpha_i = -k\, \alpha_i \wedge \gamma, \quad d\beta_i = k\, \beta_i \wedge \gamma, \quad d\gamma = 0,$$

where $1 \leq i \leq n$, and such that at each point of $N^{2n+1}(k)$, the collection $\{\alpha_i, \beta_i, \gamma \mid 1 \leq i \leq n\}$ is a basis for the 1–forms on $N^{2n+1}(k)$. Using Hattori's theorem [18] we compute the real cohomology of $N^{2n+1}(k)$ :

$$\begin{aligned}
H^0(N^{2n+1}(k)) &= \langle 1 \rangle, \\
H^1(N^{2n+1}(k)) &= \langle [\gamma] \rangle, \\
H^2(N^{2n+1}(k)) &= \langle [\alpha_i \wedge \beta_j] \mid 1 \leq i, j \leq n \rangle, \\
H^3(N^{2n+1}(k)) &= \langle [\alpha_i \wedge \beta_j \wedge \gamma] \mid 1 \leq i, j \leq n \rangle, \\
H^4(N^{2n+1}(k)) &= \langle [\alpha_i \wedge \beta_j \wedge \alpha_k \wedge \beta_r] \mid 1 \leq i < k \leq n, 1 \leq j < r \leq n \rangle, \\
&\vdots \\
H^{2n+1}(N^{2n+1}(k)) &= \langle [\alpha_1 \wedge \beta_1 \wedge \alpha_2 \wedge \beta_2 \wedge \cdots \wedge \alpha_m \wedge \beta_m \wedge \gamma] \rangle.
\end{aligned}$$

In general for $p \geq 2$ we have

$$\begin{aligned}
H^{2p}(N^{2n+1}(k)) &= \langle [\alpha_{i_1} \wedge \beta_{j_1} \wedge \alpha_{i_2} \wedge \beta_{j_2} \wedge \cdots \wedge \alpha_{i_p} \wedge \beta_{j_p}] \mid \\
&\qquad 1 \leq i_1 < i_2 < \cdots i_p \leq n,\ 1 \leq j_1 < j_2 < \cdots j_p \leq n \rangle, \\
H^{2p+1}(N^{2n+1}(k)) &= \langle [\alpha_{i_1} \wedge \beta_{j_1} \wedge \alpha_{i_2} \wedge \beta_{j_2} \wedge \cdots \wedge \alpha_{i_p} \wedge \beta_{j_p} \wedge \gamma] \mid \\
&\qquad 1 \leq i_1 < i_2 < \cdots i_p \leq n,\ 1 \leq j_1 < j_2 < \cdots j_p \leq n \rangle.
\end{aligned}$$

Next let us consider the manifold $M^{2(n+1)}(k) = N^{2n+1}(k) \times S^1$. Hence there are 1–forms $\alpha_i, \beta_i, \gamma, \eta$ on $M^{2(n+1)}(k)$ such that

$$d\alpha_i = -k\, \alpha_i \wedge \gamma, \quad d\beta_i = k\, \beta_i \wedge \gamma, \quad d\gamma = d\eta = 0,$$

where $1 \leq i \leq n$, and such that at each point of $M^{2(n+1)}(k)$, $\{\alpha_i, \beta_i, \gamma, \eta \mid 1 \leq i \leq n\}$ is a basis for the 1–forms on $M^{2(n+1)}(k)$.

**Proposition 3.1** *The manifold $M^{2(n+1)}(k)$ is formal, and has a symplectic form $\omega$ such that $(M^{2(n+1)}(k), \omega)$ satisfies the hard Lefschetz property.*

**Proof :** We define a morphism $\rho\colon (H^*(M^{2(n+1)}(k)), d = 0) \longrightarrow (\Omega^*(M^{2(n+1)}(k)), d)$ by linearly choosing closed forms representatives for each cohomology class; that is, $\rho([\gamma]) = \gamma$, $\rho([\eta]) = \eta$, etc. One can check that $\rho$ is multiplicative and then it is a homomorphism of differential graded algebras which induces the identity on cohomology. Therefore, the manifold $M^{2(n+1)}(k)$ is formal.

The collection $\{\alpha_i \wedge \beta_j, \gamma \wedge \eta \mid 1 \leq i, j \leq n\}$ is a basis for the closed 2–forms on $M^{2(n+1)}(k)$. Thus the 2–form $\omega$ on $M^{2(n+1)}(k)$ defined by

$$\omega = \sum_{i=1}^{n} (\alpha_i \wedge \beta_i) + \gamma \wedge \eta.$$

is a symplectic form on $M^{2(n+1)}(k)$.

Now, a straightforward calculation shows that the map

$$[\omega]^{n+1-i}\colon H^i(M^{2(n+1)}(k)) \longrightarrow H^{2(n+1)-i}(M^{2(n+1)}(k))$$

is an isomorphism for any $1 \leq i \leq n+1$, and so $(M^{2(n+1)}(k), \omega)$ satisfies the hard Lefschetz property.

$\boxed{QED}$

**Remark 3.2** *We must notice that the formality of the manifolds $M^{2(n+1)}(k)$ must be understood only in the sense of existence of the morphism $\rho\colon (H^*(M^{2(n+1)}(k)), d=0) \longrightarrow (\Omega^*(M^{2(n+1)}(k)), d)$, defined in the previous Proposition, such that$\rho$ induces an isomorphism on cohomology, but it does not directly relate to rational homotopy theory.*

**Theorem 3.3** *$M^{2(n+1)}(k)$ does not admit Kähler metrics for $n \leq 3$.*

**Proof :** It is similar to that given in [11] for the manifolds $M^6(k)$. In fact, to show that $M^8(k)$ does not admit Kähler metric, recall that $\Gamma^8(k) = \pi_1(M^8(k))$ is a semidirect product $\mathbb{Z}^2 \ltimes \mathbb{Z}^6$. Moreover, its abelianization is $H_1(M^8(k); \mathbb{Z})$ and thus it has rank 2. We shall see that $\Gamma^8(k)$ cannot be the fundamental group of any compact Kähler manifold.

The exact sequence

$$0 \longrightarrow \mathbb{Z}^6 \longrightarrow \Gamma^8(k) \longrightarrow \mathbb{Z}^2 \longrightarrow 0, \tag{1}$$

shows that $\Gamma^8(k)$ is solvable of class 2, i.e., $D^3\Gamma^8(k) = 0$. Moreover its rank is 8 by additivity (see [1] for details).

Assume now that $\Gamma^8(k) = \pi_1(X)$, where $X$ is a compact Kähler manifold. According to Arapura–Nori's theorem (see Theorem 3.3 of [2]), there exists a chain of normal subgroups

$$0 = D^3\Gamma^8(k) \subset Q \subset P \subset \Gamma,$$

such that $Q$ is torsion, $P/Q$ is nilpotent and $\Gamma^8(k)/P$ is finite. The exact sequence (1) implies that $\Gamma^8(k)$ has no torsion, and so $Q = 0$. As $\Gamma^8(k)/P$ is torsion, thus finite, we have $\operatorname{rank} P = \operatorname{rank} \Gamma^8(k) = 8$. Now, the finite inclusion $P \subset \Gamma^8(k)$ defines a finite cover $p: Y \to X$ that is also compact Kähler and it has fundamental group $P$.

We show that $P$ cannot be the fundamental group of any compact Kähler manifold. For this, we use Campana's result (see Corollary 3.8, page 313, in [5]) that states that *if $G$ is the fundamental group of a Kähler manifold such that $G$ is nilpotent and non-abelian, then $G$ has rank* $\geq 9$.

Since $P$ is the fundamental group of the Kähler manifold $Y$, $P$ is nilpotent and has rank $< 9$, it has to be abelian. This is impossible since any pair of non-zero elements $e \in \mathbb{Z}^2 \subset \Gamma^8(k) = \mathbb{Z}^2 \ltimes \mathbb{Z}^6$, $f \in \mathbb{Z}^6 \subset \Gamma^8(k)$ do not commute. $\boxed{QED}$

**Remark 3.4** *Notice that the previous proof fails for $n \geq 4$ since we would have $\operatorname{rank} P \geq 10$, and so we cannot use the result of Campana mentioned before. For $n \geq 4$ we do not know whether or not $M^{2(n+1)}(k)$ possesses Kähler metrics. In [7] Dardié and Médina prove that any* *completely solvable, unimodular and Kähler Lie algebra is abelian. This fact implies that if $\omega$ is an* invariant *symplectic form on $M^{2(n+1)}(k)$ (for arbitrary $n$) and $J$ is an* invariant *integrable almost complex structure compatible with $\omega$, then $(J, \omega)$ does not define a (positive definite) Kähler metric on $M^{2(n+1)}(k)$.*

Next, we shall construct an indefinite Kähler metric on $M^{2(n+1)}(k)$ when $n$ is *even*. Let $\{X_i, Y_i, Z, T \mid 1 \leq i \leq n\}$ be a basis of (global) vector fields on $M^{2(n+1)}(k)$ dual to the basis of 1–forms $\{\alpha_i, \beta_i, \gamma, \eta \mid 1 \leq i \leq n\}$. Then

$$[X_i, Z] = k\, X_i, \quad [Y_i, Z] = -k\, Y_i, \quad (1 \leq i \leq n),$$

and the other brackets are zero.

Define an almost complex structure $J$ on $M^{2(n+1)}(k)$ (recall that $n$ is even) by

$$JX_{2i-1} = X_{2i}, \quad JY_{2i-1} = Y_{2i}, \quad JZ = T,$$

where $1 \leq i \leq n$. A direct computation shows that the Nijenhuis tensor of $J$ vanishes. Consequently, $J$ is complex. A basis $\{\lambda_{2i-1}, \mu_{2i-1}, \nu \mid 1 \leq i \leq n\}$ for the forms of bidegree $(1, 0)$ is given by

$$\lambda_{2i-1} = \alpha_{2i-1} + \sqrt{-1}\, \alpha_{2i}, \quad \mu_{2i-1} = \beta_{2i-1} + \sqrt{-1}\, \beta_{2i}, \quad \nu = \gamma + \sqrt{-1}\, \eta.$$

Thus we have

$$d\lambda_{2i-1} = -\frac{k}{2}\, \lambda_{2i-1} \wedge (\nu + \bar{\nu}), \quad d\mu_{2i-1} = \frac{k}{2}\, \mu_{2i-1} \wedge (\nu + \bar{\nu}), \quad d\nu = 0.$$

Define

$$\Omega = \sum_{i=1}^{n} (\lambda_{2i-1} \wedge \bar{\mu} + \bar{\lambda}_{2i-1} \wedge \mu_{2i-1}) + \sqrt{-1}\, \nu \wedge \bar{\nu}.$$

Then $\Omega$ is a symplectic form of bidegree $(1, 1)$ on $M^{2(n+1)}(k)$, and so the metric $g$ given by $g(U, V) = \Omega(U, JV)$, for vector fields $U, V$ on $M^{2(n+1)}(k)$, it is an *indefinite Kähler metric.*

# 4 Symplectically aspherical manifolds with nontrivial $\pi_2$ and with no Kähler metrics

In this section we show a method to construct symplectically aspherical manifolds. Those of dimension 4 have nontrivial $\pi_2$ and do not admit Kähler metrics. For this, we use the symplectic submanifolds constructed by Auroux in [3].

Let $(M, \omega)$ be a compact symplectic manifold of dimension $2n$ with $[\omega] \in H^2(M)$ admitting a lift to an integral cohomology class, and let $E$ be any hermitian vector bundle over $M$ of rank $r$. In [3] Auroux constructs symplectic submanifolds $Z_r \hookrightarrow M$ of dimension $2(n - r)$ whose Poincaré dual are $\mathrm{PD}[Z_r] = k^r[\omega]^r + k^{r-1}c_1(E)[\omega]^{r-1} + \ldots + c_r(E)$ for any integer number $k$ large enough, where $c_i(E)$ denotes the $i^{th}$ Chern class of the vector bundle $E$. Moreover, these submanifolds satisfy a *Lefschetz theorem in hyperplane sections*, which means that the inclusion

$j\colon Z_r \hookrightarrow M$ is $(n-r)$–connected, i.e., the map there $j^*\colon H^i(M) \to H^i(Z_r)$ is an isomorphism for $i < n-r$ and a monomorphism for $i = n-r$.

Also Auroux proves [3, Proposition 5] that the Euler characteristic of $Z_r$ is given by $\chi(Z_r) = (-1)^{n-r}\binom{n-1}{n-r}\omega^n k^n + O(k^{n-1})$. Therefore for $k$ large enough, $H^{n-r}(Z_r)$ is of very large dimension. In particular $H^{n-r}(M) \to H^{n-r}(Z_r)$ is not an isomorphism.

The formality and the hard Lefschetz theorem for Auroux symplectic submanifolds were studied by the authors in [11]. There it is proved the following theorem:

**Theorem 4.1** *[11]. If $(M,\omega)$ is formal and/or hard Lefschetz, any Auroux symplectic submanifold $Z_r$ is formal and/or hard Lefschetz. Let $j\colon Z_r \hookrightarrow M$ be the inclusion, for $[z] \in H^p(M)$, where $p \geq n-r+1$, and $\dim M = 2n$, we have*

$$j^*[z] = 0 \iff [z] \cup c_r(E \otimes L^{\otimes k}) = 0. \tag{2}$$

Regarding to the cohomology of $Z_r$ we have

**Proposition 4.2** *Let $M$ be a compact symplectic manifold of dimension $2n$, and let $Z_r \hookrightarrow M$ be an Auroux submanifold of dimension $2(n-r)$. Let us suppose that $M$ is $s$–Lefschetz with $s \leq (n-r-1)$. Then, for each $p = 2(n-r)-i$ with $i \leq s$, there is an isomorphism*

$$H^p(Z_r) \cong \frac{H^p(M)}{\ker(c_r(E \otimes L^{\otimes k}) : H^p(M) \twoheadrightarrow H^{p+2r}(M))}.$$

**Proof :** From (2), we know that there is an inclusion

$$\frac{H^p(M)}{\ker(c_r(E \otimes L^{\otimes k}) : H^p(M) \to H^{p+2r}(M))} \hookrightarrow H^p(Z_r).$$

To prove the reverse inclusion, let us consider an arbitrary metric on $H^*(M)$; for example, the $L^2$–metric on harmonic forms. Let $S \subset H^i(M)$ be the unitary sphere, and denote by $K$ an upper bound of

$$\{a \cup [\omega]^{n-i-q} \cup c_q(E) \mid a \in S,\ q = 1, \ldots, r\}.$$

On the other hand, the $s$–Lefschetz property of $M$ implies that $S \cup [\omega]^{n-i} \subset H^{2n-i}(M)$ does not contain zero. Therefore, there is a lower bound $c > 0$ of the set

$$\{a \cup [\omega]^{n-i} \mid a \in S\}.$$

Now, for any $[z] \in S$, we obtain

$$[z] \cup [\omega]^{n-r-i} \cup (k^r[\omega]^r + k^{r-1}[\omega]^{r-1} \cup c_1(E) + \cdots + c_r(E)) \neq 0$$

taking $k > (r-1)K/c$.

The $s$–Lefschetz property guarantees an isomorphism $[\omega]^{n-i} : H^i(M) \to H^{p+2r}(M)$. Suppose that $c_r(E \otimes L^{\otimes k}) : H^p(M) \to H^{p+2r}(M)$ is not surjective. Then let $\alpha \in H^{p+2r}(M)$ be an element of norm one in the perpendicular of its image. There exists $\beta \in H^p(M)$ such that $k^r[\omega]^r \cup \beta = \alpha$. So, the norm of $\beta$ is at most $c^{-1}k^{-r}$. Then the norm of $c_r(E \otimes L^{\otimes k})\beta - \alpha$ is less or equal than $(r-1)K/ck$. Choosing $k$ large enough we see that this is a contradiction. Now computing dimensions, we have $b^p(M) - (b^p(M) - b^{p+2r}(M)) = b^{p+2r}(M) = b^i(M) = b^i(Z_r) = b^p(Z_r)$, which completes the proof. $\boxed{QED}$

Let us identify the de Rham cohomology group $H^2(M)$ with the group of the homomorphisms $\mathrm{Hom}(H_2(M), \mathbb{R})$, and let $h_M: \pi_2(M) \to H_2(M)$ be the Hurewicz homomorphism for $M$. Suppose that $(M, \omega)$ is a compact symplectic manifold, and denote by $[\omega]$ the de Rham cohomology class defined by the symplectic form $\omega$. We say that $\omega$ is a *symplectically aspherical* form if $[\omega] \circ h_M = 0$, i.e., $[\omega]|_{\pi_2(M)} = 0$, which means that

$$\int_{S^2} f^*\omega = 0$$

for every map $f: S^2 \to M$. In this case, $(M, \omega)$ is said to be a *symplectically aspherical* manifold.

**Theorem 4.3** *Let $(M, \omega)$ be a symplectically aspherical compact manifold. Then any Auroux symplectic submanifold $Z_r \hookrightarrow M$ is also symplectically aspherical. Moreover any 4–dimensional Auroux symplectic submanifold $Z_{n-1} \hookrightarrow M^{2(n+1)}(k)$ is formal, hard Lefschetz and $\pi_2(Z_{n-1}) \neq 0$, and the submanifolds $Z_2 \hookrightarrow M^8(k)$ do not admit Kähler metrics.*

**Proof :** First we note that any symplectic submanifold $j: (N, j^*\omega) \hookrightarrow (M, \omega)$ is also symplectically aspherical. In fact, we have $[j^*\omega]|_{\pi_2(N)} = 0$ since $(j^*\omega)(h_N(\pi_2(N))) = \omega(j_*(h_N(\pi_2(N)))) = \omega(h_M(j_*(\pi_2(N)))) = 0$, where we denote with the same symbol $j_*$ the maps $H_2(N) \longrightarrow H_2(M)$ and $\pi_2(N) \longrightarrow \pi_2(M)$ induced by the inclusion $j$. In particular, if $(M, \omega)$ is a symplectically aspherical manifold, any Auroux symplectic submanifold $Z_r \hookrightarrow M$ is also symplectically aspherical. (Notice that without loss of generality we can assume that $\omega$ is an integral simplectically aspherical form since, according to Proposition 1.4 in [19], *any compact symplectically aspherical manifold has an integral simplectically aspherical form.*)

Next let us consider the compact symplectic manifolds $(M^{2(n+1)}(k), \omega)$ which are symplectically aspherical since $\pi_2(M^{2(n+1)}(k)) = 0$. Now, from Proposition 3.1 and Theorem 4.1 it follows that any Auroux symplectic submanifold $Z_r \hookrightarrow M^{2(n+1)}(k)$ is formal and hard Lefschetz. Consequently, any 4–dimensional Auroux symplectic submanifold $Z_{n-1} \hookrightarrow M^{2(n+1)}(k)$ is formal and hard Lefschetz. Also it satisfies $\pi_2(Z_{n-1}) \neq 0$ since $H_2(Z_{n-1})$ and $H_2(M^{2(n+1)}(k))$ are not isomorphic.

Moreover, for any Auroux symplectic submanifold $Z_2 \hookrightarrow M^8(k)$, a similar argument to the one given in Theorem 3.3 proves that the fundamental group $\pi_1(Z_2) = \pi_1(M^8(k))$ cannot be be the fundamental group of any compact Kähler manifold, and so the submanifolds $Z_2 \hookrightarrow M^8(k)$ do not admit Kähler metrics. This completes the proof. $\boxed{QED}$

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