ATIYAH–FLOER CONJECTURE

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The Atiyah–Floer Conjecture relates the instanton Floer homology of suitable three-manifolds with the symplectic Floer homology of moduli spaces of flat connections over surfaces, and hence with the quantum cohomology of such moduli spaces. It was originally stated by M. Atiyah for homology 3-spheres in [**?**]. The extension of the conjecture to the case of mapping cylinders was prompted by A. Floer and solved in this case by S. Dostoglou and D. Salamon in [**?**].

Instanton Floer homology for 3-manifolds was introduced by A. Floer in [**?**]. Let (Y, P_Y) be a pair of a closed oriented 3-manifold Y and an $SO(3)$-bundle $P_Y \to Y$. If either Y is a homology 3-sphere or $b_1(Y) > 0$ and the second Stiefel–Whitney class $w_2(P_Y) \neq 0$, then the instanton Floer homology $HF_*^{\text{inst}}(Y, P_Y)$ is defined as the homology of the Morse-type complex constructed out of the Chern–Simons functional. The critical points are flat connections and the connecting orbits are anti-self-dual connections on $P_Y \times \mathbb{R} \to Y \times \mathbb{R}$ decaying exponentially to flat connections $A^{\pm}$ when $t \to \pm\infty$.

The symplectic Floer homology for Lagrangian intersections was introduced by A. Floer in [**?**]. Let (M, ω) be a symplectic manifold which is monotone and simply connected. Let L_0 and L_1 be Lagrangian submanifolds of M. Then there are Floer homology groups $HF_*^{\text{symp}}(M, L_0, L_1)$. Now the critical points are the intersection poings $x \in L_0 \cap L_1$ and the connecting orbits are J-holomorphic strips $u : [0,1] \times \mathbb{R} \to M$ with $u(0,t) \in L_0$, $u(1,t) \in L_1$ and $\lim_{t\to\pm\infty} u(s,t) = x^{\pm}$, where $x^{\pm} \in L_0 \cap L_1$ and J is an almost complex structure compatible with the symplectic form.

Let Σ be a closed oriented surface of genus $g \geq 1$ and let $P \to \Sigma$ be the trivial $SO(3)$-bundle. Then the moduli space $\mathcal{M}(P)$ of flat connections on P is symplectic and smooth except at the trivial connection. Now let $Y = Y_0 \cup_\Sigma Y_1$ be a Heegard splitting of a homology 3-sphere and consider the trivial $SO(3)$-bundle P_Y on Y. Then the flat connections on Σ which extend to Y_0 define a Lagrangian subspace $\mathcal{L}_0 \subset \mathcal{M}(P)$, and analogously we have $\mathcal{L}_1 \subset \mathcal{M}(P)$. Taking care of the singularity one may define $HF_*^{\text{symp}}(\mathcal{M}(P), \mathcal{L}_0, \mathcal{L}_1)$. The Atiyah–Floer conjecture reads

$$HF_*^{\text{inst}}(Y, P_Y) \xrightarrow{\cong} HF_*^{\text{symp}}(\mathcal{M}(P), \mathcal{L}_0, \mathcal{L}_1). \tag{1}$$

This was originally conjectured by M. Atiyah in [**?**]. An overview of the problem appears in [**?**]. This is still an open problem.

The symplectic Floer homology for a symplectic map was introduced by A. Floer in [**?**]. Let (M, ω) be a symplectic manifold which is monotone and simply connected. Let $\phi : M \to M$ be a symplectomorphism. Then the symplectic Floer homology $HF_*^{\text{symp}}(M, \phi)$ can be defined as the Morse-type theory where the critical points are the fixed points of ϕ and the connecting orbits are J-holomorphic strips $u : [0,1] \times \mathbb{R} \to M$ with $u(1,t) = \phi(u(0,t))$ which converge to fixed points $x^{\pm}$ of ϕ as $t \to \pm\infty$. In the case that $\phi = \text{id}$, A. Floer proved [**?**] that $HF_*^{\text{symp}}(M, \text{id}) \cong H^*(M)$. Moreover, there is a natural ring structure for the symplectic Floer homology [**?**] and in [**?**] it is

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proved that there is an isomorphism of rings $HF_*^{\rm symp}(M, {\rm id}) \cong QH^*(M)$, where $QH^*(M)$ is the quantum cohomology of M.

Let Σ be a closed oriented surface of genus $g \geq 1$ and let $Q \to \Sigma$ be the non-trivial $SO(3)$-bundle. The moduli space of flat connections $\mathcal{M}(Q)$ is a smooth symplectic manifold. Consider the mapping cylinder Y_f of a diffeomorphism $f : \Sigma \to \Sigma$. This Y_f fibers over the circle $\mathbb{S}^1$ with fibre Σ. Lift f to a bundle map $\tilde{f} : Q \to Q$. This gives an $SO(3)$-bundle $Q_{\tilde{f}} \to Y_f$. On the other hand $\tilde{f}$ induces a map $\phi_{\tilde{f}} : \mathcal{M}(Q) \to \mathcal{M}(Q)$. The Atiyah–Floer conjecture for mapping cylinders was proposed by Floer [**?**] and reads

$$HF_*^{\rm inst}(Y_f, Q_{\tilde{f}}) \stackrel{\simeq}{\to} HF_*^{\rm symp}(\mathcal{M}(Q), \phi_{\tilde{f}}). \tag{2}$$

In [**?**] S. Dostoglou and D. Salamon prove the existence of an isomorphism between these two Floer homologies by constructing an isomorphism at the chain level and identifying the boundary operators. The idea is named adiabatic limit and consists of stretching Y_f in the direction orthogonal to Σ.

A very important case is that of $\tilde{f} = {\rm id}$. Then $Y_{\rm id} = \Sigma \times \mathbb{S}^1$ and $Q_{\rm id} = Q \times \mathbb{S}^1 \to \Sigma \times \mathbb{S}^1$ is the $SO(3)$-bundle with $w_2(Q_{\rm id}) = PD[\mathbb{S}^1]$. Therefore

$$HF_*^{\rm inst}(\Sigma \times \mathbb{S}^1, Q \times \mathbb{S}^1) \stackrel{\simeq}{\to} HF_*^{\rm symp}(\mathcal{M}(Q), {\rm id}) \cong QH^*(\mathcal{M}(Q)). \tag{3}$$

Both Floer homologies have natural product structures introduced by S. Donaldson (see [**?**]). A stronger version of the Atiyah–Floer conjecture establishes that (**??**) is an isomorphism of rings.

The existence of such isomorphism has been proved by V. Muñoz in [**?**][**?**] by giving an explicit presentation of both rings in terms of the natural generators of the cohomology of $\mathcal{M}(Q)$ and using the relationship of instanton Floer homology of 3-manifolds with Donaldson invariants of 4-manifolds [**?**]. Also in [**?**] D. Salamon proves that the adiabatic limit isomorphism is indeed a ring isomorphism.

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