Cohomologically Kähler Manifolds with no Kähler Metrics

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Abstract

We show some examples of compact symplectic solvmanifolds, of dimension greather than four, which are cohomologically Kähler and do not admit Kähler metric since their fundamental groups cannot be the fundamental group of any compact Kähler manifold [8]. Some of the examples that we study were considered by Benson and Gordon in [7]. However whether such manifolds have Kähler metrics was an open question. The formality and the hard Lefschetz property are studied for the symplectic submanifolds constructed by Auroux in [4] and some consequences are discussed.

1 Introduction

A symplectic manifold (M, ω) is a pair consisting of a $2n$–dimensional differentiable manifold M together with a closed 2–form ω which is non-degenerate (that is, ω^n never vanishes). The form ω is called symplectic. By the Darboux theorem, in canonical coordinates, ω can be expressed as $\omega = \sum_{i=1}^{n} dx^i \wedge dx^{n+i}$.

Any symplectic manifold (M, ω) carries an almost complex structure J compatible with the symplectic form ω, which means that $\omega(X, Y) = \omega(JX, JY)$ for any X, Y vector fields on M (see [21, 23]). If (M, ω) has an integrable almost complex structure J compatible with the symplectic form ω, such that the Riemannian metric g given by $g(X, Y) = -\omega(JX, Y)$ is positive definite, then (M, ω, J) is said to be a Kähler manifold with Kähler metric g.

The problem of how compact symplectic manifolds differ topologically from Kähler manifolds led during the last years to the introduction of several geometric methods for constructing symplectic manifolds (see [6, 9, 15, 20, 22]). The symplectic manifolds there presented do not admit a Kähler metric since either they are not formal or do not satisfy hard Lefschetz theorem, or they fail both properties of compact Kähler manifolds.

The purpose of this note is to show that the formality and the hard Lefschetz property of any compact symplectic manifold M are not sufficient conditions to imply the existence of a Kähler metric on M. We describe three families of compact symplectic solvmanifolds $M^6(c)$, $P^6(c)$ and $N^6(c)$ of dimension 6, and a family of compact symplectic solvmanifolds $N^8(c)$ of dimension 8, each of which is formal and satisfies the hard Lefschetz property. Thus, they are cohomologically Kähler, their odd Betti numbers are even (see [19]) and their even Betti numbers are nonzero.

In [12] there are given examples of 4–dimensional compact symplectic manifolds which are cohomologically Kähler but which do not possess complex structures, so admit no Kähler metrics.

This is done by appealing to classification theorems of Kodaira and Yau that are specific to complex dimension 2.

In our case we resort, in Section 3, to the properties of the fundamental group of a compact Kähler manifold given by Campana in [8] to show that none of the manifolds $M^6(c)$, $N^6(c)$, $P^6(c)$ and $N^8(c)$ admit Kähler metrics (see Theorem 3.2 and Theorem 3.3). A similar technique was used in [14] to prove the existence of 4–dimensional Donaldson symplectic submanifolds with no complex structures. The manifolds $N^6(c)$ as well as the manifolds $P^6(c)$ were considered in [7]. There, Benson and Gordon show that they are cohomologically Kähler. However, whether or not they have a Kähler metric was an open question.

On the other hand, in Section 4, we study the formality and the hard Lefschetz property for the symplectic submanifolds obtained by Auroux in [4] as an extension to higher rank bundles of the symplectic submanifolds constructed by Donaldson in [11]. Let (M, ω) be a compact symplectic manifold of dimension $2n$ with $[\omega] \in H^2(M)$ having a lift to an integral cohomology class, and let E be any hermitian vector bundle over M of rank r. In [4] Auroux proves the existence of some integer number k_0 such that for any $k \geq k_0$ there is a symplectic submanifold $Z_r \hookrightarrow M$ of dimension $2(n-r)$ whose homology class realizes the Poincaré dual of $k^r[\omega]^r + k^{r-1}c_1(E)[\omega]^{r-1} + \ldots + c_r(E)$, where $c_i(E)$ denotes the i^{th} Chern class of the vector bundle E. For such manifolds the inclusion $j\colon Z_r \hookrightarrow M$ induces on cohomology:

- an isomorphism $j^*\colon H^i(M) \to H^i(Z_r)$ for $i < n-r$;
- a monomorphism $j^*\colon H^i(M) \hookrightarrow H^i(Z_r)$ for $i = n-r$.

As a consequence of this study, we get some examples of Auroux symplectic submanifolds (in particular, non-parallelizable manifolds) of dimension 6 which are formal and hard Lefschetz, but which do not carry Kähler metrics.

2 Formal manifolds

First, we need some definitions and results about minimal models. Let (A, d) be a *differential algebra*, that is, A is a graded commutative algebra over the real numbers, with a differential d which is a derivation, i.e. $d(a \cdot b) = (da) \cdot b + (-1)^{\deg(a)} a \cdot (db)$, where $\deg(a)$ is the degree of a.

A differential algebra (A, d) is said to be *minimal* if:

(i) A is free as an algebra, that is, A is the free algebra $\bigwedge V$ over a graded vector space $V = \oplus V^i$, and

(ii) there exists a collection of generators $\{a_\tau, \tau \in I\}$, for some well ordered index set I, such that $\deg(a_\mu) \leq \deg(a_\tau)$ if $\mu < \tau$ and each da_τ is expressed in terms of preceding a_μ ($\mu < \tau$). This implies that da_τ does not have a linear part, i.e., it lives in $\bigwedge V^{>0} \cdot \bigwedge V^{>0} \subset \bigwedge V$.

Morphisms between differential algebras are required to be degree preserving algebra maps which commute with the differentials. Given a differential algebra (A, d), we denote by $H^*(A)$ its cohomology. A is *connected* if $H^0(A) = \mathbf{R}$, and A is *one–connected* if, in addition, $H^1(A) = 0$.

We shall say that $(\mathcal{M}, d)$ is a *minimal model* of the differential algebra (A, d) if $(\mathcal{M}, d)$ is minimal and there exists a morphism of differential graded algebras $\rho\colon (\mathcal{M}, d) \longrightarrow (A, d)$

inducing an isomorphism $\rho^*: H^*(\mathcal{M}) \longrightarrow H^*(A)$ on cohomology. Halperin in [17] proved that any connected differential algebra (A, d) has a minimal model unique up to isomorphism.

A minimal model $(\mathcal{M}, d)$ is said to be *formal* if there is a morphism of differential algebras $\psi: (\mathcal{M}, d) \longrightarrow (H^*(\mathcal{M}), d = 0)$ that induces the identity on cohomology. The formality of a minimal model can be distinguished as follows.

Theorem 2.1 *[10]. A minimal model $(\mathcal{M}, d)$ is formal if and only if we can write $\mathcal{M} = \bigwedge V$ and the space V decomposes as a direct sum $V = C \oplus N$ with $d(C) = 0$, d is injective on N and such that every closed element in the ideal $I(N)$ generated by N in $\bigwedge V$ is exact.*

A *minimal model* of a connected differentiable manifold M is a minimal model $(\bigwedge V, d)$ for the de Rham complex $(\Omega M, d)$ of differential forms on M. If M is a simply connected manifold, the dual of the real homotopy vector space $\pi_i(M) \otimes \mathbf{R}$ is isomorphic to V^i for any i. We shall say that M is *formal* if its minimal model is formal or, equivalently, the differential algebras $(\Omega M, d)$ and $(H^*(M), d = 0)$ have the same minimal model. (For details see [10, 16] for example.)

In [14] the condition of *formal* manifold is weaken to *s–formal* manifold as follows.

Definition 2.2 *Let $(\mathcal{M}, d)$ be a minimal model of a differentiable manifold M. We say that $(\mathcal{M}, d)$ is s–formal, or M is a s–formal manifold ($s \geq 0$) if we can write $\mathcal{M} = \bigwedge V$ such that for each $i \leq s$ the space V^i of generators of degree i decomposes as a direct sum $V^i = C^i \oplus N^i$, where the spaces C^i and N^i satisfy the three following conditions:*

(i) $d(C^i) = 0$,

(ii) the differential map $d: N^i \longrightarrow \bigwedge V$ *is injective,*

(iii) any closed element in the ideal $I_s = I_s(\bigoplus_{i \leq s} N^i)$, *generated by* $\bigoplus_{i \leq s} N^i$ *in* $\bigwedge(\bigoplus_{i \leq s} V^i)$, *is exact in* $\bigwedge V$.

The relation between the formality and the s–formality for a manifold is given in the following theorem.

Theorem 2.3 *[14]. Let M be a connected and orientable compact differentiable manifold of dimension $2n$, or $(2n-1)$. Then M is formal if and only if is $(n-1)$–formal.*

3 Formal and hard Lefschetz symplectic manifolds with no Kähler metric

In this section we show the existence of compact symplectic manifolds of dimension > 4, not admitting Kähler metrics even when they are formal and hard Lefschetz.

Example 1 *The manifolds $M^6(c)$ [1].* Let $G(c)$ be the connected completely solvable Lie group of dimension 5 consisting of matrices of the form

$$a = \begin{pmatrix} e^{cz} & 0 & 0 & 0 & 0 & x_1 \\ 0 & e^{-cz} & 0 & 0 & 0 & y_1 \\ 0 & 0 & e^{cz} & 0 & 0 & x_2 \\ 0 & 0 & 0 & e^{-cz} & 0 & y_2 \\ 0 & 0 & 0 & 0 & 1 & z \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where $x_i, y_i, z \in \mathbf{R}$ $(i = 1, 2)$ and c is a real number different from zero. Then a global system of coordinates x_1, y_1, x_2, y_2, z for $G(c)$ is given by $x_i(a) = x_i$, $y_i(a) = y_i$, $z(a) = z$. A standard calculation shows that a basis for the right invariant 1–forms on $G(c)$ consists of

$$\{dx_1 - cx_1dz, dy_1 + cy_1dz, dx_2 - cx_2dz, dy_2 + cy_2dz, dz\}.$$

Alternatively, the Lie group $G(c)$ may be described as a semidirect product $G(c) = \mathbf{R} \ltimes_{\psi} \mathbf{R}^4$, where $\psi(z)$ is the linear transformation of $\mathbf{R}^4$ given by the matrix

$$\begin{pmatrix} e^{cz} & 0 & 0 & 0 \\ 0 & e^{-cz} & 0 & 0 \\ 0 & 0 & e^{cz} & 0 \\ 0 & 0 & 0 & e^{-cz} \end{pmatrix},$$

for any $z \in \mathbf{R}$. Thus, $G(c)$ has a discrete subgroup $\Gamma(c) = \mathbf{Z} \ltimes_{\psi} \mathbf{Z}^4$ such that the quotient space $\Gamma(c)\backslash G(c)$ is compact. Therefore the forms $dx_i - cx_idz, dy_i + cy_idz, dz$ $(i = 1, 2)$ descend to 1–forms $\alpha_i, \beta_i, \gamma$ $(i = 1, 2)$ on $\Gamma(c)\backslash G(c)$.

Now let us consider the manifold $M^6(c) = \Gamma(c)\backslash G(c) \times S^1$. Hence there are 1–forms $\alpha_1, \beta_1, \alpha_2, \beta_2, \gamma, \eta$ on $M^6(c)$ such that

$$d\alpha_i = -c\alpha_i \wedge \gamma, \quad d\beta_i = c\beta_i \wedge \gamma, \quad d\gamma = d\eta = 0,$$

where $i = 1, 2$, and such that at each point of $M^6(c)$, $\{\alpha_1, \beta_1, \alpha_2, \beta_2, \gamma, \eta\}$ is a basis for the 1–forms on $M^6(c)$. Using Hattori's theorem [18] we compute the real cohomology of $M^6(c)$:

$$\begin{aligned}
H^0(M^6(c)) &= \langle 1 \rangle, \\
H^1(M^6(c)) &= \langle [\gamma], [\eta] \rangle, \\
H^2(M^6(c)) &= \langle [\alpha_1 \wedge \beta_1], [\alpha_1 \wedge \beta_2], [\alpha_2 \wedge \beta_1], [\alpha_2 \wedge \beta_2], [\gamma \wedge \eta] \rangle, \\
H^3(M^6(c)) &= \langle [\alpha_1 \wedge \beta_1 \wedge \gamma], [\alpha_1 \wedge \beta_2 \wedge \gamma], [\alpha_2 \wedge \beta_1 \wedge \gamma], [\alpha_2 \wedge \beta_2 \wedge \gamma], \\
&\qquad [\alpha_1 \wedge \beta_1 \wedge \eta], [\alpha_1 \wedge \beta_2 \wedge \eta], [\alpha_2 \wedge \beta_1 \wedge \eta], [\alpha_2 \wedge \beta_2 \wedge \eta] \rangle, \\
H^4(M^6(c)) &= \langle [\alpha_1 \wedge \beta_1 \wedge \alpha_2 \wedge \beta_2], [\alpha_1 \wedge \beta_1 \wedge \gamma \wedge \eta], [\alpha_1 \wedge \beta_2 \wedge \gamma \wedge \eta], \\
&\qquad [\alpha_2 \wedge \beta_1 \wedge \gamma \wedge \eta], [\alpha_2 \wedge \beta_2 \wedge \gamma \wedge \eta] \rangle, \\
H^5(M^6(c)) &= \langle [\alpha_1 \wedge \beta_1 \wedge \alpha_2 \wedge \beta_2 \wedge \gamma], [\alpha_1 \wedge \beta_1 \wedge \alpha_2 \wedge \beta_2 \wedge \eta] \rangle, \\
H^6(M^6(c)) &= \langle [\alpha_1 \wedge \beta_1 \wedge \alpha_2 \wedge \beta_2 \wedge \gamma \wedge \eta] \rangle.
\end{aligned}$$

Therefore the Betti numbers of $M^6(c))$ are

$$b_0(M^6(c)) = b_6(M^6(c)) = 1,$$

$$b_1(M^6(c)) = b_5(M^6(c)) = 2,$$

$$b_2(M^6(c)) = b_4(M^6(c)) = 5,$$

$$b_3(M^6(c)) = 8.$$

Proposition 3.1 *The manifold $M^6(c)$ is 2–formal and so formal. Moreover, $M^6(c)$ has a symplectic form ω such that $(M^6(c), \omega)$ satisfies the hard Lefschetz property.*

Proof : To prove that $M^6(c)$ is 2–formal, we see that its minimal model must be a differential graded algebra $(\mathcal{M}, d)$, being $\mathcal{M}$ the free algebra of the form $\mathcal{M} = \bigwedge(a_1, a_2) \otimes \bigwedge(b_1, b_2, b_3, b_4) \otimes \bigwedge V^{\geq 3}$ where the generators a_i have degree 1, the generators b_j have degree 2, and the differential d is given by $da_i = db_j = 0$ where $i = 1, 2$ and $1 \leq j \leq 4$. The morphism $\rho\colon \mathcal{M} \to \Omega(M)$, inducing an isomorphism on cohomology, is defined by $\rho(a_1) = \gamma$, $\rho(a_2) = \eta$, $\rho(b_1) = \alpha_1 \wedge \beta_1$, $\rho(b_2) = \alpha_1 \wedge \beta_2$, $\rho(b_3) = \alpha_2 \wedge \beta_1$ and $\rho(b_4) = \alpha_2 \wedge \beta_2$.

According to Definition 2.2, we get $C^1 = \langle a_1, a_2 \rangle$ and $N^1 = 0$, thus $M^6(c)$ *is 1–formal.* Moreover $M^6(c)$ *is* 2*–formal* since $C^2 = \langle b_1, b_2, b_3, b_4 \rangle$ and $N^2 = 0$. Now the formality of $M^6(c)$ follows from Theorem 2.3.

Let us define the symplectic form ω on $M^6(c)$ by

$$\omega = \alpha_1 \wedge \beta_1 + \alpha_2 \wedge \beta_2 + \gamma \wedge \eta.$$

Then, the maps $[\omega]\colon H^2(M^6(c)) \longrightarrow H^4(M^6(c))$ and $[\omega]^2\colon H^1(M^c(k)) \longrightarrow H^5(M^6(c))$ are isomorphisms. Thus $(M^6(c), \omega)$ satisfies the hard Lefschetz property.

$\boxed{QED}$

The manifolds $M^6(c)$ were considered in [1]. There the formality of $M^6(c)$ is obtained as consequence of the existence of a morphism $(H^*(M^6(c)), d = 0) \longrightarrow (\Omega^*(M^6(c)), d)$ that induces the identity on cohomology. Such a morphism is defined by linearity choosing closed forms representatives for each cohomology class. However, whether or not $M^6(c)$ has a Kähler metric was an open question.

Theorem 3.2 *$M^6(c)$ does not admit Kähler metrics.*

Proof : In order to show that $M^6(c)$ does not admit Kähler metric, notice that $\Gamma = \pi_1(M^6(c))$ is a product $\Gamma = \Gamma(c) \times \mathbf{Z}$. Moreover, its abelianization is $H_1(M^6(c); \mathbf{Z})$ and thus it has rank 2. We shall see that Γ cannot be the fundamental group of any compact Kähler manifold.

The exact sequence

$$0 \longrightarrow \mathbf{Z}^4 \longrightarrow \Gamma \longrightarrow \mathbf{Z}^2 \longrightarrow 0, \tag{1}$$

shows that Γ is solvable of class 2, i.e., $D^3\Gamma = 0$. Moreover its rank is 6 by additivity (see [2] for details).

Assume now that $\Gamma = \pi_1(X)$, where X is a compact Kähler manifold. According to Arapura–Nori's theorem (see Theorem 3.3 of [3]), there exists a chain of normal subgroups

$$0 = D^3\Gamma \subset Q \subset P \subset \Gamma,$$

such that Q is torsion, P/Q is nilpotent and Γ/P is finite. The exact sequence (1) implies that Γ has no torsion, and so $Q = 0$. As Γ/P is torsion, thus finite, we have $\operatorname{rank} P = \operatorname{rank} \Gamma = 6$. Now, the finite inclusion $P \subset \Gamma$ defines a finite cover $p : Y \to X$ that is also compact Kähler and it has fundamental group P.

We show that P cannot be the fundamental group of any compact Kähler manifold. For this, we use Campana's result (see Corollary 3.8, page 313, in [8]) that states that *if G is the fundamental group of a Kähler manifold such that G is nilpotent and non–abelian, then G has rank* ≥ 9.

Since P is the fundamental group of the Kähler manifold Y, P is nilpotent and has rank < 9, it has to be abelian. This is impossible since any pair of non-zero elements $e \in \mathbf{Z}^2 \subset \Gamma = \mathbf{Z}^2 \ltimes \mathbf{Z}^4$, $f \in \mathbf{Z}^4 \subset \Gamma$ do not commute (see for example [13, page 22]). $\boxed{QED}$

Example 2 *The manifolds* $N^6(c)$. Let us consider the connected completely solvable Lie group $G(c)$ of dimension 3 consisting of matrices of the form

$$a = \begin{pmatrix} e^{cz} & 0 & 0 & x \\ 0 & e^{-cz} & 0 & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $x, y, z \in \mathbf{R}$ $(i = 1, 2)$ and c is a nonzero real number. Then a global system of coordinates x, y, z for $G(c)$ is given by $x(a) = x$, $y(a) = y$, $z = z$. A standard calculation shows that a basis for the right invariant 1–forms on $G(c)$ consists of

$$\{dx - cxdz, dy + cydz, dz\}.$$

Let $\Gamma(c)$ be a discrete subgroup of $G(c)$ such that the quotient space $Sol(3) = \Gamma(c)\backslash G(c)$ is compact (for the existence of such a subgroup $\Gamma(c)$ see [5, page 20]). Hence, the forms $dx - cxdz, dy + cydz, dz$ all descend to 1–forms α, β, γ on $Sol(3)$ such that

$$d\alpha = -c\alpha \wedge \gamma, \quad d\beta = c\beta \wedge \gamma, \quad d\gamma = 0. \tag{2}$$

We use again Hattori's theorem [18] to compute the real cohomology of $Sol(3)$

$$\begin{aligned} H^0(Sol(3)) &= \langle 1 \rangle, \\ H^1(Sol(3)) &= \langle [\gamma] \rangle, \\ H^2(Sol(3)) &= \langle [\alpha \wedge \gamma] \rangle, \\ H^3(Sol(3)) &= \langle [\alpha \wedge \beta \wedge \gamma] \rangle. \end{aligned}$$

Denote by $M^4(c)$ the product $M^4(c) = Sol(3) \times S^1$. In [12], it is proved that $M^4(c)$ is cohomologically Kähler (in fact, it has the same minimal model as $T^2 \times S^2$) and it does not carry complex structures, and so it carries no Kähler metrics. This is done by appealing to classification theorems of Kodaira and Yau that are specific to complex surfaces.

Next we consider other examples in dimensions 6 and 8 related also with $Sol(3)$. Define the manifolds $N^6(c) = Sol(3) \times Sol(3)$, $P^6(c) = Sol(3) \times T^3$ and $N^8(c) = Sol(3) \times Sol(3) \times T^2 = N^6(c) \times T^2$. Those manifolds are formal since they are product of formal manifolds.

From the definition of $N^6(c)$ and from equations (2), one can check that there are 1–forms $\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2$, on $N^6(c)$ such that

$$d\alpha_i = -c\alpha_i \wedge \gamma_i, \quad d\beta_i = c\beta_i \wedge \gamma_i, \quad d\gamma_i = 0,$$

where $i = 1, 2$, and such that at each point of $N^6(c)$, $\{\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2\}$ is a basis for the 1–forms on $N^6(c)$. Let us define the symplectic form ω_1 on $N^6(c)$ by

$$\omega_1 = \alpha_1 \wedge \beta_1 + \alpha_2 \wedge \beta_2 + \gamma_1 \wedge \gamma_2.$$

We use again the equations (2) to show that there is a basis $\{\alpha_1, \beta_1, \gamma_1, \eta_1, \eta_2, \eta_3\}$ for the 1–forms on $P^6(c)$ such that

$$d\alpha_1 = -c\alpha_1 \wedge \gamma_1, \quad d\beta_1 = c\beta_1 \wedge \gamma_i, \quad d\gamma_1 = d\eta_j = 0,$$

for $1 \leq j \leq 3$, since $P^6(c) = Sol(3) \times T^3$. Thus, the 2–form ω_2 defined by

$$\omega_2 = \alpha_1 \wedge \beta_1 + \gamma_1 \wedge \eta_1 + \eta_2 \wedge \eta_3,$$

is a symplectic form on $P^6(c)$.

It is clear that $N^8(c)$ is a symplectic manifold since it is the product of symplectic manifolds. In fact, a symplectic form ω_3 on $N^8(c)$ is given by

$$\omega_3 = \omega_1 + \eta,$$

where η is a symplectic form on the 2–torus T^2.

One can check that the manifolds $N^6(c)$, $P^6(c)$ and $N^8(c)$ are cohomologically Kähler. Now using a similar argument to the one given in Theorem 3.2 we get the following

Theorem 3.3 *The manifolds $N^6(c)$, $P^6(c)$ and $N^8(c)$ are formal and hard Lefschetz but they admit no Kähler metrics.*

We notice that the manifolds $N^6(c)$ and $P^6(c)$ were considered as examples of cohomologically Kähler manifolds by Benson and Gordon in [7]. However, whether or not they have a Kähler metric was an open question.

4 Formality and hard Lefschetz property for Auroux symplectic submanifolds

In this section we study the conditions under which Auroux symplectic manifolds are formal and/or satisfy the hard Lefschetz theorem.

Let (M, ω) be a compact symplectic manifold of dimension $2n$ with $[\omega] \in H^2(M)$ admitting a lift to an integral cohomology class, and let E be any hermitian vector bundle over M of rank r. In [4] Auroux constructs symplectic submanifolds $Z_r \hookrightarrow M$ of dimension $2(n-r)$ whose Poincaré dual is $\mathrm{PD}[Z_r] = k^r[\omega]^r + k^{r-1}c_1(E)[\omega]^{r-1} + \ldots + c_r(E)$ for any integer number k large enough, where $c_i(E)$ denotes the i^{th} Chern class of the vector bundle E. Moreover, these submanifolds satisfy a *Lefschetz theorem in hyperplane sections*, meaning that the inclusion $j\colon Z_r \hookrightarrow M$ is $(n-r)$–connected, i.e., the map there $j^*\colon H^i(M) \to H^i(Z_r)$ is an isomorphism for $i < n-r$ and a monomorphism for $i = n-r$.

In general, let X and Y be compact manifolds. We say that a differentiable map $f\colon X \to Y$ is a *homotopy s–equivalence* ($s \geq 0$) if it induces isomorphisms $f^*\colon H^i(Y) \xrightarrow{\cong} H^i(X)$ on cohomology for $i < s$, and a monomorphism $f^*\colon H^s(Y) \hookrightarrow H^s(X)$ for $i = s$. Therefore, for any Auroux symplectic submanifold, the inclusion $j\colon Z_r \hookrightarrow M$ is a homotopy $(n-r)$–equivalence.

Theorem 4.1 *[14]. Let X and Y be compact manifolds, and let $f\colon X \to Y$ be a homotopy s–equivalence. If Y is $(s-1)$–formal then X is $(s-1)$–formal.*

As a consequence of Theorem 4.1 we get the following corollary.

Corollary 4.2 *Let M be a compact symplectic manifold of dimension $2n$ and let $Z_r \hookrightarrow M$ be an Auroux submanifold of dimension $2(n-r)$. For each $s \leq (n-r-1)$, if M is s–formal then Z_r is s–formal. In particular, Z_r is formal if M is $(n-r-1)$–formal.*

In order to continue the analysis of the Auroux symplectic submanifolds we introduce the following

Definition 4.3 *Let (M, ω) be a compact symplectic manifold of dimension $2n$. We say that M is s–Lefschetz with $s \leq (n-1)$ if*

$$[\omega]^{n-i} : H^i(M) \longrightarrow H^{2n-i}(M)$$

is an isomorphism for all $i \leq s$. By extension, if we say that M is s–Lefschetz with $s \geq n$ then we just mean that M is hard Lefschetz.

Theorem 4.4 *Let (M, ω) be a compact symplectic manifold of dimension $2n$ such that the de Rham cohomology class $[\omega] \in H^2(M)$ has a lift to an integral cohomology class, and let $Z_r \hookrightarrow M$ be an Auroux submanifold of dimension $2(n-r)$. Then, for large enough k and for each $s \leq (n-r-1)$, if M is s–Lefschetz then Z_r is s–Lefschetz. Therefore, Z_r is hard Lefschetz if M is $(n-r-1)$–Lefschetz.*

Proof : From now on, we denote by L the complex line bundle over M whose first Chern class is $c_1(L) = [\omega]$. Let $p = 2(n-r) - i$, where $i \leq (n-r-1)$, and let us consider the map $j^*\colon H^p(M) \to H^p(Z_r)$ induced by the inclusion j on cohomology. First we claim that for $[z] \in H^p(M)$ it holds

$$j^*[z] = 0 \iff [z] \cup c_r(E \otimes L^{\otimes k}) = 0, \tag{3}$$

for large values of the parameter k. This can be shown via Poincaré duality. Clearly $j^*[z] = 0$ if and only if $j^*[z] \cdot a = 0$ for any $a \in H^i(Z_r)$. Since there is an isomorphism $H^i(Z_r) \cong H^i(M)$ for $i \leq (n-r-1)$, we can assume that there exists a closed i–form x on M with $[x|_{Z_r}] = [\hat{x}] = a$, being $\hat{x}$ the differential form on Z_r given by $\hat{x} = j^*(x)$. So

$$j^*[z] \cdot [\hat{x}] = \int_Z \hat{z} \wedge \hat{x} = \int_M z \wedge x \wedge \tilde{c}_r(E \otimes L^{\otimes k}),$$

since $[Z_r] = PD[c_r(E \otimes L^{\otimes k})]$, where $\tilde{c}_r(E \otimes L^{\otimes k})$ is a differential form on M representing $c_r(E \otimes L^{\otimes k})$. Hence $j^*[z] = 0$ if and only if $([z] \cup c_r(E \otimes L^{\otimes k})) \cup [x] = 0$ for all $[x] \in H^i(M)$, from where the claim follows.

Now consider an arbitrary norm on $H^*(M)$; for example, the L^2–norm on harmonic forms. Let $S \subset H^i(M)$ be the unitary sphere, and denote by K an upper bound of

$$||\{a \cup [\omega]^{n-i-q} \cup c_q(E) \,|\, a \in S, q = 1, \ldots, r\}||.$$

On the other hand, the s–Lefschetz property of M implies that $S \cup [\omega]^{n-i} \subset H^{2n-i}(M)$ does not contain zero. Therefore, there is a lower bound $K' > 0$ for the set

$$||\{a \cup [\omega]^{n-i} \,|\, a \in S\}||.$$

Now, for any $[z] \in S$, we obtain

$$[z] \cup [\omega]^{n-r-i} \cup (k^r[\omega]^r + k^{r-1}[\omega]^{r-1} \cup c_1(E) + \cdots + c_r(E)) \neq 0$$

taking $k > (r-1)K/K'$. Thus, $\hat{z} \cup [\hat{\omega}^{n-r-i}] \neq 0$ for any $[\hat{z}] \in H^i(Z_r)$, which proves that Z_r is also s–Lefschetz.

$\boxed{QED}$

Let us now consider the compact symplectic solvmanifolds $N^8(c)$ defined in Example 2 of Section 3. Since $N^8(c)$ has a symplectic form that defines an integral cohomology class, there exist Auroux symplectic submanifolds $Z_r \hookrightarrow N^8(c)$ of dimension $2(4-r)$ for $1 \leq r \leq 3$.

Proposition 4.5 *Any Auroux symplectic submanifold $Z_r \hookrightarrow N^8(c)$ is formal and hard Lefschetz. Moreover, Z_r does not admit Kähler metrics for $r = 1, 2$, and the submanifolds $Z_3 \hookrightarrow N^8(c)$ are Kähler.*

Proof : From Theorem 3.3, Corollary 4.2 and Theorem 4.4 we get that any Auroux symplectic submanifold $Z_r \hookrightarrow N^8(c)$ is formal and hard Lefschetz. Moreover, a similar argument to the one given in Theorem 3.2 proves that the submanifolds Z_r do not admit Kähler metrics for $r = 1, 2$.

$\boxed{QED}$

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