SYMPLECTIC GEOMETRY

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1. Introduction

1.1. **Classical Mechanics.** Let $U \subset \mathbb{R}^n$ be an open subset of the n-dimensional space, where a particle of mass m moves subject to a force $F(x)$. By Newton's equation, the trajectory $x(t)$ of the particle satisfies the differential equation

$$m\,\ddot{x}(t) = F(x(t))\,.$$

Using coordinates $(x,y) \in U \times \mathbb{R}^n$, where $(x,y) = (x, \dot{x})$, we have the equivalent equations:

$$\left\{ \begin{array}{l} \dot{x}(t) = y(t), \\ \dot{y}(t) = \frac{1}{m} F(x(t))\,. \end{array} \right. \tag{1}$$

Note that the space $U \times \mathbb{R}^n$ is the tangent bundle TU.

We say that the force is conservative if $F = -\nabla V$, for some function $V(x)$ on U, called the *potential function* of the mechanical system. In this case, a particle $x(t)$ has kinetic energy $K = \frac{1}{2} m\,|\dot{x}(t)|^2$ and potential energy $V = V(x(t))$. The total energy is

$$E(x,y) = K + V = \frac{1}{2} m\,|y(t)|^2 + V(x(t))\,.$$

It is easy to see that E is constant along time, by computing

$$\frac{dE}{dt} = m\langle y(t), \dot{y}(t)\rangle + \langle \nabla V, \dot{x}\rangle = \langle y, F\rangle - \langle F, y\rangle = 0.$$

The metric $v \mapsto m|v|^2$ gives an isomorphism between the tangent and cotangent bundles, $TU \to T^*U$, $v \mapsto p = mv$. We may consider the particle as a trajectory on T^*U, $(x(t), p(t))$, satisfying the equations

$$\left\{ \begin{array}{l} \dot{x}(t) = \frac{1}{m} p(t)\,, \\ \dot{p}(t) = F(x(t))\,. \end{array} \right.$$

The function $H = K + V = \frac{1}{2m}|p|^2 + V(x)$ defined on the cotangent bundle, is known as the *Hamiltonian* of the mechanical system. The equations of the movement get rewritten as

$$\left\{ \begin{array}{l} \dot{x}_i(t) = \frac{\partial H}{\partial p_i}, \\ \dot{p}_i(t) = -\frac{\partial H}{\partial x_i}\,. \end{array} \right.$$

Partially supported through Special Action grant from MICINN (Spain) PCI2006-A9-0586.

Therefore a particle follows the flow of the vector field

$$X_H = \sum_i \left(\frac{\partial H}{\partial p_i} \frac{\partial}{\partial x_i} - \frac{\partial H}{\partial x_i} \frac{\partial}{\partial p_i} \right) \tag{2}$$

on the cotangent bundle.

This vector field can be computed intrinsically: T^*U has a canonical 1-form λ (called the Liouville form), given by $\lambda_{(x,\alpha)} = \pi^*\alpha$, for $(x,\alpha) \in T^*U$ (i.e. $x \in U$, $\alpha \in T_x^*U$), $\pi : T^*U \to U$. Locally, in coordinates (x,p), it is $\lambda = \sum p_j \, dx_j$. The 2-form $\omega = -d\lambda$ on T^*U is well-defined and has local expression

$$\omega = \sum_j dx_j(t) \wedge dp_j(t) \, .$$

Then X_H is the unique vector field satisfying $\omega(X_H, \cdot) = dH$.

If we want to develop mechanics intrinsically, we have to use a smooth manifold M in place of U. We have used a metric at several points: to compute the gradient of V, for the kinetic energy, etc. However, the formulation with the hamiltonian does not use this extra information. If we have the hamiltonian H, then X_H in (2) can be extracted in an intrinsic way. Just note that T^*M comes equipped with a 1-form λ and a 2-form $\omega = -d\lambda$, and X_H satisfies $\omega(X_H, \cdot) = dH$.

Now, even we may forget about T^*M and take any $2n$-dimensional manifold Q with a 2-form like ω. This is enough for theoretical purposes. Such pair (Q, ω) is known as a symplectic manifold. Thus, geometric mechanics take place in a symplectic manifold. It uses a function H whose symplectic flow (that is, the flow of X_H) leaves ω invariant (that is, $L_{X_H}\omega = 0$). So the particles follow a flow which is by symplectomorphisms.

Now our interest is to understand what type of manifolds can be symplectic.

1.2. **Complex manifolds.** A complex manifold M of dimension n is a Hausdorff topological space endowed with an atlas $\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}$ consisting of charts $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha) \subset \mathbb{C}^n$, which are homeomorphisms onto open sets of $\mathbb{C}^n$, and whose changes of charts $\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$ are biholomorphisms (holomorphic maps whose inverses are also holomorphic).

The primary examples of complex manifolds are the smooth algebraic projective varieties: take the complex projective space $\mathbb{CP}^N$, with projective coordinates $[z_0 : \ldots : z_N]$. Let $f_1(z_0, \ldots, z_N), \ldots, f_r(z_0, \ldots, z_N)$ be a collection of homogeneous polynomials and consider the zero set

$$Z(f_1, \ldots, f_r) = \{[z_0 : \ldots : z_N] \in \mathbb{CP}^N \,|\, f_1(z_0, \ldots, z_N) = \cdots = f_r(z_0, \ldots, z_N) = 0\} \subset \mathbb{CP}^N \, .$$

This is a complex manifold of dimension $N - r$ when it is smooth. Smoothness happens if the Jacobian $\frac{\partial(f_1, \ldots, f_r)}{\partial(z_0, \ldots, z_N)}$ has (maximal) rank r at every non-zero $(z_0, \ldots, z_N)$.

There are other ways to construct complex manifolds: for instance, if M is a complex manifold, and Γ is a group acting discretely and freely on M by biholomorphisms, then the quotient M/Γ is again a complex manifold. For instance, take $M = \mathbb{C}^2 - \{0\}$, and the group $\mathbb{Z}$ generated by $\varphi : M \to M$, $\varphi(z,w) = (2z, 2w)$. The quotient $M/\mathbb{Z}$ is a smooth manifold which is known as the *Hopf surface*. It is compact and diffeomorphic to $S^1 \times S^3$.

There is a way to understand a complex manifold from the point of view of differential geometry. Let M be a complex manifold of complex dimension n. Then M is a smooth manifold of dimension $2n$. At each point $p \in M$, the tangent space T_pM is a $2n$-dimensional real vector space which is actually a complex vector space. From linear algebra, this is equivalent to having a complex structure $J_p : T_pM \to T_pM$ (a linear map satisfying $J_p^2 = -\mathrm{Id}$). This gives a tensor $J \in \mathrm{End}\,(TM)$ such that $J^2 = -\mathrm{Id}$.

In general, a pair (M,J), where $J \in \mathrm{End}\,(TM)$, $J^2 = -\mathrm{Id}$, is called an *almost complex manifold*. To recover a complex atlas from (M,J) (and hence, for M to be a complex manifold), it is necessary that J satisfies an extra condition, called the *integrability* condition. The Nijenhuis tensor is defined as

$$N_J(X,Y) = [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY]\,,$$

for vector fields X, Y. The famous theorem of Newlander-Niremberg says that (M,J) is a complex manifold if and only if $N_J = 0$.

One of the most prominent questions in complex geometry is the following: given a smooth manifold M, does there exist J such that (M,J) is a complex manifold?

(1) The existence of J giving an almost-complex structure is a topological question.
(2) Assuming the existence of an almost-complex structure, to find an integrable one is an analytical problem.

1.3. **Kähler manifolds.** Let M be a complex manifold. A hermitian metric h is a tensor $h : TM \times \overline{TM} \to \mathbb{C}$, which is complex-linear in the first variable, satisfies $h(v,u) = \overline{h(u,v)}$, and $h(u,u) > 0$ for all non-zero u. Any complex manifold admits hermitian structures.

It is worth to note that the hermitian metric and the complex structure give rise to other two interesting tensors on M:

- a Riemannian metric: $g(u,v) = \mathrm{Re}\,h(u,v)$. Note that $g(Ju,Jv) = g(u,v)$.
- a 2-form $\omega(u,v) = g(u,Jv)$. Equivalently, $\omega(u,v) = \mathrm{Im}\,h(u,v)$. Note that ω is maximally non-degenerate, that is, $\omega(u,v) = 0, \forall v \implies u = 0$. This is equivalent to $\omega^n \neq 0$.

Locally, in coordinates $(z_1, \ldots, z_n)$, if the metric is written as $h = \sum h_{i\bar{j}}\, dz_i \cdot d\bar{z}_j$, then the fundamental 2-form is $\omega = \frac{\mathbf{i}}{2} \sum h_{i\bar{j}}\, dz_i \wedge d\bar{z}_j$. The non-degeneracy is obtained from

$$\frac{\omega^n}{n!} = \det(h_{i\bar{j}}) \left(\frac{\mathbf{i}}{2}\right)^n dz_1 \wedge d\bar{z}_1 \wedge \ldots \wedge dz_n \wedge d\bar{z}_n = \mathrm{vol}_g.$$

We say that a $(1,1)$-form α is positive if $\alpha = \frac{\mathbf{i}}{2}\sum a_{i\bar{j}}\, dz_i \wedge d\bar{z}_j$, where $(a_{i\bar{j}})$ is a positive definite hermitian matrix. Equivalently (and more intrinsically), if $\alpha(Ju,u) > 0$, for all non-zero u.

Note that with ω and J we can recover g by $g(u,v) = \omega(Ju,v)$ and h by $h(u,v) = g(u,v) + i\omega(u,v)$.

The projective space $\mathbb{CP}^n$ has a very natural hermitian metric, which is obtained as follows. Fix a basis on $\mathbb{C}^{n+1}$, and take a hermitian metric h_0 on $T_{p_0}\mathbb{CP}^n$ for a base-point $p_0 = [1 : 0 : \ldots, 0]$. Then consider the (unique) metric h on $\mathbb{CP}^n$ which is $U(n+1)$-invariant (obtained by moving h_0 with the matrices in this group). This can be obtained more intrinsically as follows: as $T_p\mathbb{CP}^n = \mathrm{Hom}\,(l_p, \mathbb{C}^{n+1}/l_p)$ (where $l_p \subset \mathbb{C}^{n+1}$ is the line defined by p), we can give it the metric induced by that of $\mathbb{C}^{n+1}$, for any $p \in \mathbb{CP}^n$. Take affine coordinates, that is, consider the open set $U = \{[z_0 : z_1 : \ldots : z_n] \,|\, z_0 \neq 0\} \subset \mathbb{CP}^n$. Then $U \cong \mathbb{C}^n$ where the point $[1 : z_1 : \ldots : z_n]$ has coordinates $z = (z_1, \ldots, z_n)$. Then the metric is written as

$$h(z) = \frac{(1+|z|^2)\sum dz_i \cdot d\bar{z}_i - \sum_{i,j} z_j \bar{z}_i dz_i \cdot d\bar{z}_j}{(1+|z|^2)^2}.$$

So the fundamental 2-form is

$$\omega = \frac{\mathbf{i}}{2}\frac{(1+|z|^2)\sum dz_i \wedge d\bar{z}_i - \sum_{i,j} z_j \bar{z}_i dz_i \wedge d\bar{z}_j}{(1+|z|^2)^2}.$$

It is easy to calculate that

$$\omega = \frac{\mathbf{i}}{2}\partial\bar{\partial}\log(1+|z|^2).$$

So we have $d\omega = 0$.

To sum up, ω is a positive closed $(1,1)$-form. This means that $[\omega]$ is a 2-cohomology class. Moreover, $[\omega]^n \in H^{2n}(\mathbb{CP}^n)$ is non-zero, since it is a multiple of the volume form. In particular, $[\omega] \neq 0$.

Definition 1. A Kähler manifold (M, J, ω) is a complex manifold (M, J) together with a closed positive $(1,1)$-form ω.

For a Kähler manifold, J is parallel, i.e. $\nabla J = 0$.

Note that if $S \subset \mathbb{CP}^n$ is a smooth algebraic complex manifold, then S is Kähler. This is easy to see: just take as hermitian metric h_S the restriction of the Fubini-Study metric of $\mathbb{CP}^n$ to S. Then the fundamental 2-form is $\omega_S = \omega|_S$. Therefore, it is a positive $(1,1)$-form and $d\omega_S = 0$.

Moreover, $[\omega_S] \in \bar{H}^2(S, \mathbb{Z})$ (where we denote by $\bar{H}^2(S,\mathbb{Z})$ the image of $H^2(S,\mathbb{Z}) \to H^2(S,\mathbb{R})$, which is a lattice). Moreover, the converse is true:

Theorem 2 (Kodaira, see [We])**.** *If (M, J, ω) is a compact Kähler manifold and $[\omega] \in \bar{H}^2(M, \mathbb{Z})$, then there is a holomorphic embedding $M \hookrightarrow \mathbb{CP}^N$, for some large N.*

1.4. **Topological properties.** Let (M, J, ω) be a compact Kähler manifold of complex dimension n. Then M satisfies very striking topological properties:

(1) There is a Hodge decomposition of the cohomology of M,
$$H^k(M, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(M)$$
and $\overline{H^{p,q}(M)} = H^{q,p}(M)$. Therefore, the Hodge numbers $h^{p,q} = \dim H^{p,q}(M)$ satisfy that $h^{q,p} = h^{p,q}$. So for k odd, the Betti numbers
$$b^k(M) = \sum_{p+q=k} h^{p,q} = \sum_{p=0}^{(k-1)/2} \left(h^{p,k-p} + h^{k-p,p} \right) = 2 \sum_{p=0}^{(k-1)/2} h^{p,k-p}$$
are even.
The Hopf surface has $b_1 = 1$, hence it is not Kähler.

(2) Hard-Lefschetz theorem. The map
$$[\omega]^k : H^{n-k}(M) \to H^{n+k}(M)$$
is an isomorphism for each $k = 1, \dots, n$. This implies in particular that the map $[\omega]^i$ on $H^{n-k}(M)$ is injective for $1 \le i \le k$, and so that $b^{i+2} \ge b^i$ for $0 \le i \le n-2$.

(3) Fundamental group. When M is not simply-connected, there are striking results restricting the nature of the fundamental group of M. These type of results can be found in the nice book [ABCKT]. For instance a free product like $\mathbb{Z} * \mathbb{Z}$ cannot be the fundamental group of a compact Kähler manifold. A group that can be the fundamental group of a compact Kähler manifold is called a *Kähler group*.

(4) Complex surfaces. If $n = \dim_{\mathbb{C}} M = 2$, then the classification of complex surfaces [BPV] gives you exactly the diffeomorphism types of 4-manifolds M admitting a complex (or Kähler) structure. This is a very short list (except for the *surfaces of general type* which are not fully understood). For instance, a consequence of the classification is the following result: if M is complex and has even Betti number b_1, then it admits a Kähler structure (maybe changing the complex structure).

(5) If $M \subset \mathbb{CP}^N$ is a smooth projective manifold, then there exist complex submanifolds in any dimension. By Bertini's theorem, we may intersect with a generic linear subspace H^r of codimension r so that $Z = M \cap H^r$ is a smooth complex submanifold of complex dimension $n-r$. Moreover, the Lefschetz theorem on hyperplane sections says that
$$H^i(M) \to H^i(Z)$$
is an isomorphism for $i = 0, 1, \dots, n-r-1$, and an epimorphism for $i = n-r$.

(6) Homotopy groups. Suppose M is compact Kähler and simply-connected. Then there are extra properties on the rational homotopy groups $V_k = \pi_k(M) \otimes \mathbb{Q}$ of M. The most relevant property is that of *formality*. To explain it, we have to explain some notions of rational homotopy theory. We dedicate the next section to this.

1.5. **Rational homotopy of simply-connected manifolds.** A *differential graded algebra* (dga, for short) (A,d) is a positively graded commutative algebra A over the rational (or real) numbers, with a differential d such that $d(a\cdot b)=(da)\cdot b+(-1)^{\deg(a)}a\cdot(db)$. We shall assume that $A^0=\mathbb{Q}$, $A^1=0$. A dga (A,d) is said to be *minimal* if:

(1) A is free as an algebra, that is, A is the free algebra $\bigwedge V$ over a graded vector space $V=\oplus V^i$, and
(2) there exists a collection of generators $\{x_\tau\}$ such that dx_τ is expressed in terms of preceding x_μ, $\mu<\tau$. This implies that dx_τ does not have a linear part, i.e., it lives in $\bigwedge V^{>0}\cdot\bigwedge V^{>0}\subset\bigwedge V$.

A *minimal model* of a dga (A,d) is a minimal dga $(\mathcal{M},d)$ with a dga morphism $\rho:(\mathcal{M},d)\to(A,d)$ which is a quasi-isomorphism (quism), that is, $\rho^*\colon H^*(\mathcal{M})\to H^*(A)$ is an isomorphism on cohomology.

In the category of dga's, there is an equivalence relation $\sim$ generated by quisms (that is, $\sim$ is the minimal equivalence relation such that if $\rho:(A_1,d_1)\to(A_2,d_2)$ is a quism, then $(A_1,d_1)\sim(A_2,d_2)$). Then the minimal model of a dga (A,d) is the canonical representative of the quism equivalence class of (A,d).

A minimal model of a simply-connected differentiable manifold M is a minimal model $(\bigwedge V,d)$ of the de Rham complex $(\Omega M,d)$ of differential forms on M (to be precise here, we have to work over the field of real numbers). By the work of Sullivan, this contains the information of the rational homotopy of M:

$$V^k\cong(\pi_k(M)\otimes\mathbb{R})^*,\ \text{for all } k\geq 2\,.$$

Now we are ready to introduce the notion of *formality*. A minimal model $(\mathcal{M},d)$ is *formal* if there is a quism $\psi\colon(\mathcal{M},d)\to(H^*(\mathcal{M}),0)$. A dga (A,d) is formal if its minimal model is so, that is, if $(A,d)\sim(H^*(A),0)$. And a simply-connected manifold M is formal if its minimal model is formal.

Theorem 3 ([DGMS]). *Let M be a simply-connected compact Kähler manifold. Then M is formal.*

The proof of this uses essentially the fact that the dga of differential forms $(\Omega(M),d)$ of a Kähler manifold has a bigrading given by the (p,q)-forms, together with some analytical properties of the Laplacian.

1.6. **Symplectic manifolds.** (See [MS] for general results on symplectic geometry.)

A symplectic manifold (M,ω) consists of a smooth manifold together with a 2-form $\omega\in\Omega^2(M)$ satisfying:

- $d\omega=0$,
- ω is non-degenerate at every point.

The second condition implies that M is of even dimension $2n$. Then $\mathrm{vol}_M = \frac{1}{n!}\omega^n$ defines a volume form (in particular, M is naturally oriented). Note that ω works as an antisymmetric metric, as it gives an isomorphism $TM \to T^*M$, $X \mapsto i_X\omega = \omega(X, \cdot)$.

Let (M, ω) be a symplectic manifold of dimension $2n$. A symplectic submanifold $N \subset M$ is a submanifold such that $\omega|_N$ is non-degenerate (hence a symplectic form on N). A Lagrangian submanifold $L \subset M$ is a submanifold of dimension n such that $\omega|_L = 0$ (n is the maximum possible dimension for such situation to occur).

A symplectomorphism $f : (M_1, \omega_1) \to (M_2, \omega_2)$ is a smooth map such that $f^*\omega_2 = \omega_1$. For surfaces $\dim M = 2$, a symplectic form on M is an area form, and a symplectomorphism is an area preserving map.

Note that a small perturbation of a symplectic ω, that is, a closed 2-form ω' such that $|\omega - \omega'|_{C^1} < \epsilon$ (for some small $\epsilon > 0$), is still symplectic. If the cohomology class does not change, we have the following result.

Lemma 4 (Moser's stability)**.** *If $\{\omega_t, t \in [0,1]\}$ is a family of symplectic forms on M with $\omega_0 = \omega$, and $[\omega_t] = [\omega]$, then there exists a family $\psi_t : (M, \omega) \to (M, \omega_t)$ of sympletomorphisms. (Moreover, if $\omega_t = \omega$ over some subset A, then $\psi_t = \mathrm{Id}$ over A.)*

Proof. As $[\omega_t] = cte$, $\frac{d[\omega_t]}{dt} = 0$. So there are 1-forms α_t such that $\frac{d\omega_t}{dt} = d\alpha_t$. Using the non-degeneracy of the symplectic forms, there are vector fields X_t such that $i_{X_t}\omega_t = -\alpha_t$. Take the flow ψ_t produced by $\{X_t\}$. Let us check that $\psi_t^*\omega_t = \omega$, that is, that $\psi_t^*\omega_t = cte$. We compute

$$\begin{aligned}
\frac{d}{dt}(\psi_t^*\omega_t) &= \psi_t^* L_{X_t}\omega_t + \psi_t^*\frac{d\omega_t}{dt} \\
&= \psi_t^*(di_{X_t}\omega_t + i_{X_t}d\omega_t) + \psi_t^*(d\alpha_t) \\
&= \psi_t^*(-d\alpha_t) + \psi_t^*(d\alpha_t) = 0\,,
\end{aligned}$$

as required. □

Locally, a symplectic manifold (M, ω) has a standard form, which is given by the well-known Darboux theorem.

Theorem 5 (Darboux)**.** *Let $p \in M$. Then there is a chart around p, $(x_1, y_1, \ldots, x_n, y_n)$, on which ω is written $\omega = dx_1 \wedge dy_1 + \ldots + dx_n \wedge dy_n$.*

Proof. To get Darboux theorem, we apply Moser's stability lemma in a ball B centered at 0 to the given 2-form ω and to the family $\omega_t = f_t^*\omega$, where $f_t : B \to B$, $f_t(x) = (1-t)x$, $t \in [0,1]$. Note that for $t = 1$, ω_1 is a constant symplectic form over B, as $\omega_1(x) = \omega(0)$, $\forall x \in B$. In a suitable basis, it can be written $\omega(0) = \sum_{i=1}^n dx_i \wedge dy_i$ (this is the canonical form for a constant coefficient symplectic form). □

This result says that there are no local invariants for a symplectic form. Therefore there are only global invariants.

Let $B \subset M$ be a Darboux ball. This means that $\omega = dx_1 \wedge dy_1 + \ldots + dx_n \wedge dy_n$. We can use complex coordinates for the ball B, given as $z_j = x_j + \mathbf{i}\, y_j$, $j = 1, \ldots, n$. In this way, we understand that $B \subset \mathbb{C}^n$ and the symplectic form gets rewritten as

$$\omega = \frac{\mathbf{i}}{2}(dz_1 \wedge d\bar{z}_1 + \ldots + dz_n \wedge d\bar{z}_n).$$

In particular, we see that we can arrange an almost complex structure J in the ball such that (B, J, ω) is a standard ball of $\mathbb{C}^n$.

2. Hard methods: Analysis

In this section we are going to describe results which extend properties known in the Kähler setting to the symplectic world. These methods usually require techniques of PDEs on manifolds.

Let (M, ω) be a compact symplectic manifold. We say that an almost-complex structure J on M is compatible with ω if $g(u, v) = \omega(Ju, v)$ defines a metric on M.

The relevant topological result is that there exists ω-compatible J's and that the space $\mathcal{J}_\omega = \{J \,|\, J \text{ is } \omega\text{-compatible}\}$ is contractible. This means that the choice of J is in some sense unique.

This is proved as follows: ω produces a reduction of the structure group of TM from $\mathrm{GL}(2n, \mathbb{R})$ to $\mathrm{Sp}(2n, \mathbb{R})$. But $\mathrm{U}(n) \subset \mathrm{Sp}(2n, \mathbb{R})$ is the maximal compact subgroup. Therefore there are reductions to $\mathrm{U}(n)$ and they are given by the sections of the associated bundle with fiber $\mathrm{Sp}(2n, \mathbb{R})/\mathrm{U}(n)$, which is contractible. Note that giving a reduction of the structure group to $\mathrm{U}(n)$ is equivalent to giving an ω-compatible J. Finally, note that $\mathrm{U}(n) = \mathrm{Sp}(2n, \mathbb{R}) \cap \mathrm{SO}(2n)$, so we have a reduction to $\mathrm{SO}(2n)$, i.e. a metric.

Now we have an *almost-Kähler* manifold (M, J, ω). This means that (M, J) is an almost-complex manifold with a symplectic form ω compatible with J.

2.1. **Gromov-Witten invariants.** Let (M, ω) be a symplectic manifold and $J \in \mathcal{J}_\omega$. Fix a compact Riemann surface Σ of some genus $g \geq 0$. This has a complex structure which we denote by j. A map $u : \Sigma \to M$ is *pseudo-holomorphic* if $du : T\Sigma \to TM$ is complex linear. This means that $du \circ j = J \circ du$. We can write $du \in \Omega^1(\Sigma, u^*TM)$, where u^*TM is a complex bundle. Decompose du into $(1,0)$ and $(0,1)$ components, $du = \partial_J u + \bar{\partial}_J u$, where

$$\partial_J u = \frac{1}{2}(du - J \circ du \circ j)\,,\; \bar{\partial}_J u = \frac{1}{2}(du + J \circ du \circ j)\,.$$

Let $A \in H_2(M, \mathbb{Z})$ be a 2-homology class. The moduli space of pseudo-holomorphic curves in the class A is

$$\mathcal{M}_A^g = \{u : \Sigma \to M \,|\, u_*[\Sigma] = A, du \circ j = J \circ du\} = \{u \in \mathrm{Map}(\Sigma, M) \,|\, u_*[\Sigma] = A, \bar{\partial}_J u = 0\}\,.$$

This has a nice structure:

- $\bar{\partial}_J u = 0$ is an elliptic PDE on $\mathrm{Map}(\Sigma, M)$. $\bar{\partial}_J$ is a section of the vector bundle $\mathcal{E} \to \mathbb{M} = \mathrm{Map}(\Sigma, M)$, $\mathcal{E}_u = \Omega^{0,1}(u^*TM)$, $u \in \mathbb{M}$. Note that $\mathbb{M}$ is of infinite dimension, and $\mathcal{E}$ is of infinite rank. The ellipticity amounts to say that the difference of these two infinities, the dimension of $\mathcal{M} = Z(\partial_J) \subset \mathbb{M}$, is finite.
- After a suitable perturbation, $\bar{\partial}_J$ is transverse to zero. So $\mathcal{M}$ is a smooth manifold.
- One has to quotient out by the automorphisms of u (which creates singularities). To avoid this, one takes pointed curves $(\Sigma, x_1, \ldots, x_r)$. So the automorphism group is just the identity. This gives rise to moduli spaces $\mathcal{M}_A^{g,r}$.
- If j is allowed to vary, we have moduli spaces $\widetilde{\mathcal{M}}_A^{g,r}$ parametrizing $(u, j, x_1, \ldots, x_r)$.
- The theory of Gromov allows to compactify $\mathcal{M}_A^{g,r}$ by adding trees of curves with bubbles and different reducible components. This gives rise to a compact space $\overline{\mathcal{M}}_A^{g,r}$. The boundary is of high codimension is good cases.

There are well-defined evaluation maps

$$\begin{array}{rcl} ev_{x_i} : \overline{\mathcal{M}}_A^{g,r} & \longrightarrow & M \\ u & \mapsto & u(x_i)\,. \end{array}$$

Fix genus $g = 0$ and $r = 3$, so we deal with $\Sigma = \mathbb{CP}^1$ with three marked points, say $0, 1, \infty \in \mathbb{CP}^1$. We define the *Gromov-Witten invariants* of (M, ω) as

$$GW_A(\alpha, \beta, \gamma) = \langle ev_0^*\alpha \cup ev_1^*\beta \cup ev_\infty^*\gamma, [\overline{\mathcal{M}}_A^{0,3}] \rangle\,, \text{ for } \alpha, \beta, \gamma \in H^*(M)\,.$$

Fix a basis $\{\alpha_i\}$ of $H^*(M)$. Then define the operation $*$ on $H^*(M)$ by $n_{A,ijk} = GW_A(\alpha_i, \alpha_j, \alpha_k)$, and

$$\langle \alpha_i * \alpha_j, \alpha_k \rangle = \sum_{A,k} n_{A,ijk}\, q^A\,,$$

where we have to enlarge the coefficients to add extra variables q^A, for each $A \in H^2(M, \mathbb{Z})$.

In good situations, the boundary $\overline{\mathcal{M}}_A^{g,r} - \mathcal{M}_A^{g,r}$ has high codimension. For instance, this happens if M is positive, i.e. if $c_1(M) = \lambda\omega$, $\lambda > 0$. In such cases, we have the following:

Theorem 6 ([RT]). *The operation $*$ is a ring structure on $H^*(M)$, called the quantum multiplication. The quantum cohomology $QH^*(M) = (H^*(M), *)$ is an invariant of the symplectomorphism type of (M, ω).*

This ring serves to distinguish symplectic structures. In this way, one can construct symplectic 6-manifolds which are diffeomorphic, but not symplectomorphic nor deformation symplectically equivalent.

By work of Taubes, a version of the Gromov-Witten invariants for 4-manifolds is equivalent to the Seiberg-Witten invariants (which are invariants of the diffeomorphism type of a 4-manifold, constructed using gauge-theoretic techniques). This has allowed to construct two homeomorphic 4-manifolds M_1, M_2, such that M_1 is Kähler and M_2 is symplectic but non-Kähler: M_2 is taken as a symplectic fiber connected sum (see Section 3.3) of a suitable Kähler surface M_1 along a symplectically knotted surface $\Sigma \subset M_1$. Then M_2 has Seiberg-Witten invariants different from those of all Kähler surfaces (here it is used the Kodaira classification of complex surfaces).

2.2. **Asymptotically holomorphic theory.** Let (M, ω) be a symplectic $2n$-manifold such that $[\omega] \in \bar{H}^2(M, \mathbb{Z})$ and fix an ω-compatible almost-complex structure J.

By the integrality of $[\omega]$, there is a complex line bundle $L \to M$ whose first Chern class is $c_1(L) = [\omega]$. This allows to construct a connection A on L whose curvature is $F_A = \frac{\mathbf{i}}{2\pi}\omega$. By construction, this connection has positive curvature.

We expand the metric by considering $g_k = kg$ for integers $k \geq 1$. g_k is associated to $\omega_k = k\omega$, which is the curvature of the connection kA on $L^{\otimes k}$. Donaldson [Do1] developed the asymptotically holomorphic theory in the search for substitutes of holomorphic sections of $L^{\otimes k}$. Let $s_k \in \Gamma(L^{\otimes k})$, then $\bar{\partial} s_k \in \Omega^{0,1}(L^{\otimes k})$. If J is not integrable, then we cannot expect to get holomorphic sections s_k, but we may try to find sequences of sections (s_k) such that $|\bar{\partial} s_k| \to 0$. More concretely, we call a sequence of sections (s_k) asymptotically holomorphic (A.H.) if $|\bar{\partial} s_k|_{C^r} \leq C k^{-1/2}$.

It is not very difficult to construct plenty of such sections. Actually, there are A.H. sections concentrated around any given point $p \in M$ (these play the role of approximations of the Dirac delta). Here is where the positivity of the curvature plays a fundamental role.

Let B be a Darboux ball around p with coordinates $(z_1, \ldots, z_n)$, and $\omega = \frac{\mathbf{i}}{2}(dz_1 \wedge d\bar{z}_1 + \ldots + dz_n \wedge d\bar{z}_n)$. We can take $A = d + \frac{\pi}{2}\sum(z_j d\bar{z}_j - \bar{z}_j dz_j)$. So the section $s = e^{-\pi k |z|^2/2}$ *is holomorphic* with respect to the standard (integrable) complex structure J_0 on B and the connection kA. From $|N_J|_g \leq C$, we have $|N_J|_{g_k} \leq C k^{-1/2}$, so $|\bar{\partial}_J s| \leq C k^{-1/2}$. As s has Gaussian decay, it can be multiplied by a suitable bump function so that it can be extended to the whole of M and still satisfies $|\bar{\partial}_J s_k| \leq C k^{-1/2}$.

The next step in Donaldson's programme is to construct a section s_k such that its zero-set gives a smooth (symplectic) submanifold $W_k = Z(s_k)$. Typically, for this one needs transversality of the section to the zero section, that is:

$$\nabla s_k : T_p M \to L_p^{\otimes k} \tag{3}$$

is surjective for each $p \in W_k$. In this case, W_k is smooth and $T_p W_k = \ker(\nabla_p s_k)$ at $p \in W_k$.

As we have only approximate holomorphicity, we need to require a stronger statement (or expressed in a different way, some amount of transversality). We say that s_k is η-transverse if (3) does multiply the norm of vectors at least by $\eta > 0$. Therefore, as

$$\nabla s_k = \partial s_k + \bar{\partial} s_k \, ,$$

when $\bar{\partial} s_k$ is very small (for k large), ∂s_k is the main contribution, and it is non-zero. Moreover, $\partial s_k : T_p M \to L_p^{\otimes k} \cong \mathbb{C}$ is complex linear, its zero set is a complex subspace and

$\ker(\nabla s_k)$ is very close to $\ker(\partial s_k)$. Therefore $W_k = Z(s_k)$ is smooth and close to being complex. More accurately,

$$\angle(T_pW_k, J(T_pW_k)) \leq Ck^{-1/2}\,.$$

In particular, W_k is a symplectic submanifold, and $PD[W_k] = c_1(L^{\otimes k})$.

Donaldson constructs η-transverse sections in a very clever way: first, he proves that in a ball $B_{g_k}(x, r_0)$, $r_0 > 0$ fixed, one can perturb an A.H. sequence of sections by adding concentrated sections around x, in such a way to obtain σ-transversality on the ball, by perturbing an amount δ, with σ and δ related by

$$\sigma = \delta(\log \delta^{-1})^{-p}$$

(p is a large integer, but fixed and independent of δ and k).

Then the process has to be iterated, but there is one complication. Due to the expansion of the metric, $g_k = kg$, the balls $B_{g_k}(x, 1) = B_g(x, k^{-1/2})$ get smaller, so we need $C\,k^n$ balls to cover M. At each step we get transversality of lower amount, and after going over all the balls, we get some transversality ϵ_k, with $\epsilon_k \to 0$, as $k \to \infty$. To avoid this, the balls are sorted out in finitely many groups. In each group the balls are separated by a large g_k-distance $D > 0$, and all the perturbations in the same group can be realized simultaneously, since the Gaussian decay implies that the perturbation in one ball does not reduce significatively the amount of transversality in the rest. Thus the number of times the perturbation process is carried out is independent of k. This reduces the amount of transversality but keeps it over some $\epsilon > 0$, independently of k.

Let us see another application of A.H. theory, which appears in [MPS]. This is the extension of the Kodaira embedding theorem to the symplectic setting. Let (M, ω) be a symplectic manifold with $[\omega] \in \bar{H}^2(M, \mathbb{Z})$, and $J \in \mathcal{J}_\omega$. We look for A.H. sequences of sections

$$s_k^0, \ldots, s_k^N \in \Gamma(L^{\otimes k})\,,$$

(N is large, say $N \geq 2n+1$) such that:

- $s_k = (s_k^0, \ldots, s_k^N)$ is a section of $\underline{\mathbb{C}}^{N+1} \otimes L^{\otimes k}$ which is η-transverse to zero. As $n = \dim_{\mathbb{C}} M < N+1$, we cannot have that ∇s_k is surjective, so it must be $|s_k| > \eta$ over all of M. Thus there is a well-defined map
$$\psi_k : M \to \mathbb{CP}^N, \quad \psi_k(x) = [s_k^0 : \ldots : s_k^N]\,.$$
- The complex linear map $\partial\psi_k : TM \to T\mathbb{CP}^N$ is η-transverse. This means that $\partial\psi_k$ multiplies the length of vectors at least by η. In particular, ψ_k is an immersion, and $M_k = \psi_k(M) \subset \mathbb{CP}^N$ satisfies that it is close to being complex.
- ψ_k is injective (this is a generic position argument).

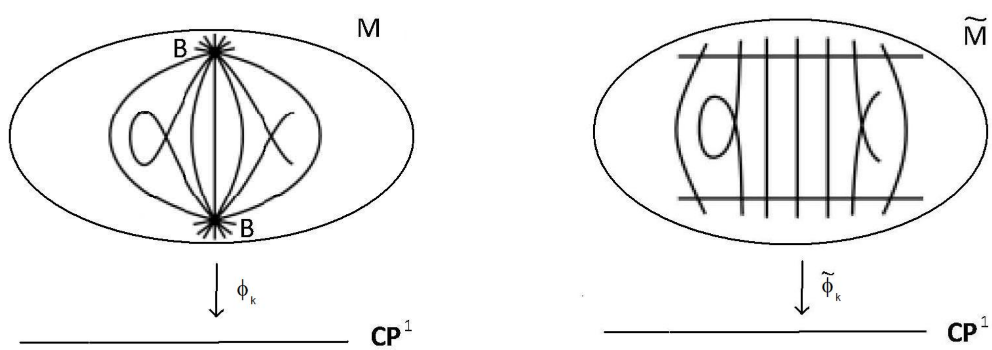
In particular, $\psi_k : M \to \mathbb{CP}^N$ are symplectic embeddings (after an extra small deformation).

2.3. **Lefschetz pencils.** Let (M, ω) be a symplectic $2n$-manifold such that $[\omega] \in \bar{H}^2(M, \mathbb{Z})$ is an integral class. Fix an ω-compatible almost-complex structure J. Donaldson [Do2] constructs a symplectic Lefschetz fibration for M which extends the Lefschetz fibrations for algebraic projective varieties. Using A.H. theory, he looks for A.H. sequences of sections

$$s_k^0, s_k^1 \in \Gamma(L^{\otimes k})$$

which satisfy the following transversality properties:

- (s_k^0, s_k^1) is an η-transverse sequence of sections. In particular, the zero set $B_k = Z(s_k^0, s_k^1)$ is a symplectic submanifold of codimension 4. There is a well-defined map
$$\phi_k = [s_k^0, s_k^1] : M - B_k \to \mathbb{CP}^1 .$$
- s_k^0 is η-transverse to zero, so that the fiber of ϕ_k over ∞ is smooth, and removing it we have a map $\phi_k(x) = \frac{s_k^1}{s_k^0} : M - F_\infty \to \mathbb{C}$. Note that the fibers $F_\lambda = \phi_k^{-1}(\lambda)$, $\lambda \in \mathbb{C}$, can be compactified to $\overline{F_\lambda} = F_\lambda \cup B_k$, which is smooth along B_k. These are codimension 2 symplectic submanifolds (off the singular locus of ϕ_k).
- The singular locus of ϕ_k (where ϕ_k is not submersive) consists of finitely many points $\Delta_k \subset M - B_k$. At each $p \in \Delta_k$, a transversality requirement for the holomorphic Hessian $\partial\partial\phi_k$ of ϕ_k allows to achieve (after an extra perturbation) a local model as follows: there are Darboux coordinates $(z_1, \ldots, z_n) \in B \subset \mathbb{C}^n$ around p such that
$$\phi_k = \lambda + z_1^2 + \ldots + z_n^2.$$
This gives a double point singularity at p for the fiber F_λ, $\lambda = \phi_k(p)$.



These sequences of A.H. Lefschetz pencils have a very nice behaviour (stability, asymptotic uniqueness, some type of homogeneity, etc.). What is more relevant is that the manifold M can be recovered from the Lefschetz pencil as follows: blow up the base locus B_k to get a symplectic manifold $\widetilde{M} = Bl_{B_k} M$ (see Section 3.4 for symplectic blow-ups). Then there is a well defined map (called *Lefschetz fibration*)

$$\tilde{\phi}_k : \widetilde{M} \longrightarrow \mathbb{CP}^1 .$$

Consider a base point $* \in \mathbb{CP}^1$ whose corresponding fiber F is smooth. We have a true fibration with fiber F (symplectic of dimension $2n-2$) over $\mathbb{CP}^1 - Crit(\phi_k)$, where $Crit(\phi_k) = \phi_k(\Delta_k)$ are the critical values (we assume that the images in $\mathbb{CP}^1$ of the critical points are different; this is easily achieved with a small perturbation). When we approach a critical value $\lambda \to \lambda_0 \in Crit(\phi_k)$, the fiber F_{λ_0} is obtained by contracting some Lagrangian sphere $S^{n-1} \subset F_\lambda = F$ (called the vanishing cycle). Moreover, the monodromy of a small loop around λ_0 is a symplectomorphism of F called a *Dehn twist* around S^{n-1} (it consists of cutting along S^{n-1} and regluing it with a twist).

This allows to recover M out of some algebraic information extracted from the Lefschetz pencil. Theoretically this would allow to classify symplectic structures (at least on low dimension, $2n = 4$). In practice the problem gets intractable since we have to allow $k \gg 0$ for A.H. Lefschetz pencils.

3. Soft methods: Topology

The topological methods address the following question:

Can we construct symplectic manifolds and detect that they do not admit Kähler metrics?

One of the ways to achieve this is to see that there are very few topological restrictions for a symplectic manifold to exist, whereas there are very strong restrictions on the topological properties of Kähler manifolds. This produces symplectic manifolds M such that there is no Kähler manifold M' which is homotopy equivalent to M.

We would like to remind here that in the case of 4-manifolds there are also gauge-theoretic results which allow to produce pairs of manifolds M_1, M_2 which are homeomorphic but not diffeomorphic, M_1 being Kähler, M_2 being symplectic but without Kähler structure.

3.1. **Nilmanifolds.** A nilpotent group is a simply-connected Lie group G satisfying the following nilpotency condition: if we define the nested sequence of subgroups G_i by $G_1 = G$, $G_i = [G_{i-1}, G]$ for $i \geq 2$, then there is some $N > 0$ such that $G_N = 0$. Any nilpotent group of dimension n is diffeomorphic to $\mathbb{R}^n$, as a differentiable manifold.

A *nilmanifold* is a compact n-dimensional manifold of the form $M = G/\Gamma$, where Γ is a discrete (cocompact) subgroup of a nilpotent group G. The Lie algebra $\mathfrak{g}$ of G is a nilpotent Lie algebra: if we define the nested sequence of Lie subalgebras $\mathfrak{g}_i$ by $\mathfrak{g}_1 = \mathfrak{g}$, $\mathfrak{g}_i = [\mathfrak{g}_{i-1}, \mathfrak{g}]$ for $i \geq 2$, then there is some $N > 0$ such that $\mathfrak{g}_N = 0$ (note that $\mathfrak{g}_i$ is the Lie algebra of G_i).

The right invariant forms of M, $\Omega^*_{inv}(M) \subset \Omega^*(M)$, are in bijective correspondence with the forms at the neutral element of G, that is, $\Omega^*_{inv}(M) \cong \bigwedge^*(\mathfrak{g}^*)$. By a theorem of Nomizu [No], the inclusion $(\Omega^*_{inv}(M), d) \hookrightarrow (\Omega^*(M), d)$ is a quism. Therefore, $(\mathcal{M}, d) = (\bigwedge^*(\mathfrak{g}^*), d)$ is the minimal model of M. Write $\mathfrak{g} = \langle e_1, \dots, e_n \rangle$ and $\mathfrak{g}^* = \langle x_1, \dots, x_n \rangle$. Then if $[e_i, e_j] =$

$\sum_{k>i,j} a_{ij}^k e_k$, we have that

$$(\bigwedge^*(x_1,\dots,x_n),d), \qquad dx_k = -\sum_{i,j<k} a_{ij}^k\, x_i\wedge x_j\,,$$

is the minimal model of M.

A symplectic form can be obtained by a 2-form $\omega=\sum a_{ij}\, x_i\wedge x_j$ satisfying $d\omega=0$ and $\omega^{n/2}=c\, x_1\wedge x_2\wedge\cdots\wedge x_n$, $c\neq 0$. Note that the volume form is $\mathrm{vol}=x_1\wedge\cdots\wedge x_n$.

A nilmanifold M can't be formal unless it is a torus. If there is a quism

$$\psi:(\bigwedge^*(\mathfrak{g}^*),d)\longrightarrow H^*(\bigwedge^*(\mathfrak{g}^*),d)\,,$$

then arrange $x_1,\dots,x_n$ in increasing order so that the closed elements are $x_1,\dots,x_r$, and $\psi(x_{r+1})=\ldots=\psi(x_n)=0$. This would imply that $\psi(\mathrm{vol})=0$ (which is impossible), unless $r=n$. This means that M is a torus.

The Kodaira–Thurston manifold. Let H be the Heisenberg group, that is, the connected nilpotent Lie group of dimension 3 consisting of matrices of the form

$$a=\begin{pmatrix} 1 & x & z\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix},$$

where $x,y,z\in\mathbb{R}$. Let Γ be the discrete subgroup of H consisting of matrices whose entries are integer numbers. So the quotient space $M=\Gamma\backslash G$ is compact.

A basis of right invariant vector fields is $\{\frac{\partial}{\partial z},\frac{\partial}{\partial y}+x\frac{\partial}{\partial z},\frac{\partial}{\partial x}\}$. Therefore the generators of $\mathfrak{g}^*$ are $\alpha=dx$, $\beta=dy$, $\gamma=dz-x\,dy$. Note that

$$d\gamma=-dx\wedge dy=-\alpha\wedge\beta\,.$$

The Kodaira–Thurston manifold KT is the product $KT=M\times S^1$. Then, there are 1–forms α,β,γ,η on KT such that $d\alpha=d\beta=d\eta=0$, $d\gamma=-\alpha\wedge\beta$. Note that

$$\omega=\alpha\wedge\gamma+\beta\wedge\eta$$

defines a symplectic form since $d\omega=0$ and $\omega^2=2\,\alpha\wedge\gamma\wedge\beta\wedge\eta\neq 0$.

KT is therefore a non-formal symplectic 4-manifold, hence it cannot admit a Kähler structure. This was the first example of a symplectic manifold not admitting a Kähler structure, as shown by Thurston [Th]. But he showed this by checking that $b_1=3$ (this comes from Nomizu's theorem, which gives $H^1(KT)=\langle[\alpha],[\beta],[\eta]\rangle$). KT was also a manifold introduced by Kodaira as a manifold admitting a complex structure but not a Kähler one.

Massey products. There is an easier way to prove the non-formality of a manifold, which avoids the use (and computation) of minimal models.

Let M be a smooth manifold and let $a_i \in H^{p_i}(M)$, $1 \leq i \leq 3$, be three cohomology classes such that $a_1 \cup a_2 = 0$ and $a_2 \cup a_3 = 0$. Take forms α_i on M with $a_i = [\alpha_i]$ and write $\alpha_1 \wedge \alpha_2 = d\xi$, $\alpha_2 \wedge \alpha_3 = d\eta$. The Massey product of the classes a_i is defined as

$$\langle a_1, a_2, a_3 \rangle = [\alpha_1 \wedge \eta + (-1)^{p_1+1} \xi \wedge \alpha_3] \in \frac{H^{p_1+p_2+p_3-1}(M)}{a_1 \cup H^{p_2+p_3-1}(M) + H^{p_1+p_2-1}(M) \cup a_3}.$$

(The denominator in the quotient group is due to the indeterminacy in the choice of ξ and η.)

The relevant result is that if M has a non-trivial Massey product then M is non-formal. This is due to the following: Massey products can easily be defined on any dga, and then they can be seen to be transferred through quisms. Therefore if M is formal, one can transfer the Massey product through $(\Omega^*(M), d) \sim (H^*(M), 0)$. But in the latter dga, all Massey products are zero, as the differential is zero.

Finally, to prove this way that KT is non-formal, it is enough to compute the Massey product:

$$\langle [\alpha], [\alpha], [\beta] \rangle = [0 \cdot \beta + \alpha \wedge (-\gamma)] = -[\alpha \wedge \gamma] \neq 0 \quad \text{in } H^2(KT).$$

Another example. In a similar way, we can also obtain a symplectic nilmanifold X of dimension 4 which is non-formal but which does not admit any complex structure. Such X has minimal model $(\bigwedge^*(\mathfrak{g}^*), d)$, with $\mathfrak{g}^* = \langle \alpha, \beta, \gamma, \eta \rangle$, $d\gamma = \alpha \wedge \beta$, $d\eta = \alpha \wedge \gamma$. The 2-form $\omega = \alpha \wedge \eta + \beta \wedge \gamma$ is symplectic. The first Betti number of X is even, $b_1 = 2$, as $H^1(X) = \langle [\alpha], [\beta] \rangle$. This implies that if X admits a complex structure then it also has a Kähler structure, by a result of Kodaira (or by the classification of complex surfaces [BPV]). But this is not possible, since X is non-formal as it is a nilmanifold which is not a torus. So X cannot admit a complex structure at all.

3.2. **Symplectic fibrations.** One can construct symplectic structures on fibrations in some situations in which both the fiber and the base have symplectic structures.

Suppose that (F, σ) is a symplectic manifold. A *symplectic fibration* over B with fiber (F, σ) consists of a fibration $F \to M \xrightarrow{\pi} B$ such that there is a cover $\{U_\alpha\}$ of B for which there are trivializations $\psi_\alpha : \pi^{-1}(U_\alpha) \xrightarrow{\cong} F \times U_\alpha$, and the changes of trivializations $\psi_\alpha \circ \psi_\beta^{-1} : F \times (U_\alpha \cap U_\beta) \to F \times (U_\alpha \cap U_\beta)$ are of the form $(f, x) \mapsto (\varphi_x(f), x)$, where $\varphi_x : (F, \sigma) \to (F, \sigma)$ is a symplectomorphism.

It is not automatic that there is a global closed 2-form on M restricting to σ at each fiber $F_x = \pi^{-1}(x)$. At least we need a cohomological condition: that there exists $a \in H^2(M)$ with $a|_{F_x} = [\sigma]$. This justifies condition (1) in the theorem below.

Theorem 7 (Thurston)**.** *Let $(F, \sigma) \to M \to B$ be a symplectic fibration, and assume that:*

1. *there exists $a \in H^2(M)$ with $a|_{F_x} = [\sigma]$,*
2. *(B, ω_B) is a symplectic manifold.*

Then there is a symplectic structure on M.

Proof. First, fix a representative η of a. We want to find another representative restricting to a symplectic form on each fiber. Consider a covering $B = \bigcup U_\alpha$ such that $\pi^{-1}(U_\alpha) \cong F \times U_\alpha$. Let σ_α be the 2-form on $\pi^{-1}(U_\alpha)$ defined as the pull-back of σ from F. Then $\sigma_\alpha = \eta + d\psi_\alpha$, $\psi_\alpha \in \Omega^1(F \times U_\alpha)$. Take a partition of unity $\{\rho_\alpha\}$ on B subordinated to $\{U_\alpha\}$, and consider the closed 2-form

$$\hat{\sigma} = \eta + \sum d(\rho_\alpha \psi_\alpha)\,.$$

For $b \in B$, $\hat{\sigma}|_{F_b} = \eta|_{F_b} + \sum \rho_\alpha(b) d\psi_\alpha|_{F_b} = \sum \rho_\alpha(b)(\eta + d\psi_\alpha)|_{F_b} = \sum \rho_\alpha(b)\sigma_\alpha|_{F_b}$. But $\sigma_\alpha|_{F_b} = \sigma$ for any $b \in U_\alpha$. This implies that $\hat{\sigma}|_{F_b} = \sigma$, for all $b \in B$.

Finally, consider $\omega = \pi^*\omega_B + \epsilon\,\hat{\sigma}$, for small $\epsilon > 0$. It is easy to see that this is symplectic on M. □

This can be used in many situations. For instance, it can be used to prove that KT is a symplectic manifold. Note that KT has (local) coordinates (x, y, z, w), where (x, y, z) are the coordinates for the Heisenberg group and w is the coordinate for the extra S^1 factor. The action of Γ is as follows:

$$\begin{aligned}
(x, y, z, w) &\mapsto (x + 1, y, z + y, w),\\
(x, y, z, w) &\mapsto (x, y + 1, z, w),\\
(x, y, z, w) &\mapsto (x, y, z + 1, w),\\
(x, y, z, w) &\mapsto (x, y, z, w + 1).
\end{aligned}$$

The projection $(x, y, z, w) \mapsto (x, w)$ gives a map $KT \to T^2$ and the fiber is T^2. Note that this is a symplectic fibration, since there is no monodromy when going around the w-direction and the monodromy around the x-direction is the symplectic map $(y, z) \mapsto (y, z + y)$.

Using Theorem 7, with $a = [dy \wedge dz]$, we have a symplectic structure on KT.

3.3. **Fiber connected sum.** Let (M_1, ω_1), (M_2, ω_2) be two symplectic manifolds and suppose that (N, ω) is a symplectic manifold of dimension $2n - 2$, with two symplectic embeddings:

$$\imath_j : (N, \omega) \longrightarrow (M_j, \omega_j)\,,\ j = 1, 2.$$

Let $\nu_j \to N$ be the normal bundle to $\imath_j$. Then if $\nu_1 \cong \nu_2^*$ are isomorphic bundles (where ν_2^* is the dual bundle to ν_2), we can glue M_1 and M_2 along N with the following construction of Gompf [Go].

Consider a neighborhood $U_j \subset M_j$ of $N_j = \imath_j(N)$, which can be identified with the ϵ-disc bundle in ν_j. So there is a fibration $D^2_\epsilon \to U_j \to N_j$, $j = 1, 2$. Now remove the closed δ-disc bundle $V_j \subset U_j$ in ν_j, with $\delta < \epsilon$, to get a fibration by annuli $A_{\epsilon,\delta} = \{z \in \mathbb{C} \,|\, \delta < |z| < \epsilon\}$,

$$A_{\epsilon,\delta} \to U_j - V_j \to N\,.$$

Since a symplectic form for $A_{\epsilon,\delta}$ is just an area form, and a symplectic map is an area-preserving map, there is an orientation-reversing self-symplectomorphism

$$f : A_{\epsilon,\delta} \to A_{\epsilon,\delta}\,,$$

sending the inner boundary onto the outer one and viceversa. This can be done parametrically to produce a diffeomorphism $F : U_1 - V_1 \to U_2 - V_2$. The *fiber connected sum* is defined as

$$M = M_1 \#_N M_2 := (M_1 - V_1) \cup_F (M_2 - V_2)\,.$$

By Moser's stability lemma, one can arrange that the symplectic forms of $U_1 - V_1$ and $U_2 - V_2$ coincide. In this way, M is endowed with a symplectic structure.

This was used by Gompf to produce many examples of symplectic non-Kähler manifolds.

Theorem 8. *Let Γ be any finitely presented group, and let $2n \geq 4$. Then there is a symplectic manifold M of dimension $2n$ with fundamental group $\pi_1(M) \cong \Gamma$.*

Proof. Let us give the main ideas of the proof (for details, see [Go]). For simplicity, focus on the 4-dimensional case. Take a presentation of the group $\Gamma = \langle x_1, \ldots, x_r | r_1, \ldots, r_s \rangle$ with generators and relations. Consider a 4-manifold M whose fundamental group contains a free group of r-elements, e.g. a surface of genus r times a two-torus, $M = \Sigma_r \times T^2$. Therefore Γ is a quotient of $\pi_1(M)$. Then construct tori T_j inside M which are Lagrangian and for which the image of $\pi_1(T_j) \to \pi_1(M)$ generate exactly the kernel of $\pi_1(M) \twoheadrightarrow \Gamma$. These are easily arranged to be disjoint: a loop in Σ_r times a loop in the T^2 factor suffice.

Now note that if $T_j \subset M$ is a Lagrangian 2-torus, then the normal bundle to T_j is trivial (an ω-compatible almost-complex J produces an isomorphism between the tangent and normal bundles to T_j). This allows to perturb the symplectic form on $D^2 \times T_j$ to make T_j symplectic. Once that T_j is symplectic, we do a fiber connected sum of M and some other manifold X along T_j. The chosen symplectic 4-manifold X should contain a symplectic 2-torus $T \subset X$ such that $X - T$ is simply connected (there are plenty of examples with these properties, using e.g. Kähler surfaces). In this way, the resulting 4-manifold $M' = M \#_{T_j = T} X$ has $\pi_1(M') = \pi_1(M)/\pi_1(T_j)$. Repeating the construction over all T_j we eventually get a 4-manifold whose fundamental group is Γ. $\Box$

As a consequence, if Γ is a non-Kähler group, then any compact symplectic manifold (M, ω) with $\pi_1(M) \cong \Gamma$ cannot be Kähler.

3.4. **Symplectic blow-up.** Let (M, ω) be a symplectic manifold. Fix a point $p \in M$. We can define the blow-up at p as follows: consider a Darboux chart B around p, with complex coordinates $(z_1, \ldots, z_n)$ such that $\omega = \frac{\mathbf{i}}{2}(dz_1 \wedge d\bar{z}_1 + \ldots dz_n \wedge d\bar{z}_n)$. Consider the blow-up of B at the origin. This is the manifold

$$\widetilde{B} = \{((z_1, \ldots, z_n), [w_1, \ldots, w_n]) \in B \times \mathbb{CP}^{n-1} | (z_1, \ldots, z_n) = \lambda(w_1, \ldots, w_n),\ \lambda \in \mathbb{C}\}.$$

Note that there is a projection $\pi : \widetilde{B} \to B$. The preimage of any $q \neq 0$ is a point $(q, [q])$. The preimage of 0 is $E = \pi^{-1}(0) = \{0\} \times \mathbb{CP}^{n-1} \cong \mathbb{CP}^{n-1} \subset \widetilde{B}$. This is a codimension 2 submanifold.

Now let us give $\widetilde{B}$ a symplectic form. For $B \times \mathbb{CP}^{n-1}$ we take the symplectic form given as $\beta = \omega + \epsilon\,\Omega$, where ω is the symplectic form on B, Ω is the Fubini-Study form on $\mathbb{CP}^{n-1}$, and $\epsilon > 0$. This is actually a Kähler form for $B \times \mathbb{CP}^{n-1}$. As $\widetilde{B} \subset B \times \mathbb{CP}^{n-1}$ is a complex submanifold, β is a Kähler form (hence symplectic) over $\widetilde{B}$.

Finally, define the blow-up $\widetilde{M}$ by gluing $M - \{p\}$ with $\widetilde{B}$ along the difeomorphism $B - \{p\} \overset{\pi}{\cong} \widetilde{B} - E$. To give a symplectic structure to $\widetilde{M}$, we have to glue the symplectic forms as follows. Consider Ω and note that when restricted to $B - \{p\}$, it is exact, hence $\Omega = d\psi$. Take a bump function of B which is one in a neigbourhood of E, and zero off B, and consider $d(\rho\psi)$. This can be extended as zero off $\widetilde{B}$ and as Ω near E. Now consider

$$\omega_{\widetilde{M}} = \omega + \epsilon\, d(\rho\psi).$$

Near E, this equals the form $\beta = \omega + \epsilon\,\Omega$, which is symplectic. Off $\widetilde{B}$, it coincides with ω, also symplectic. And in the intermediate region, it is a small perturbation of ω, which is still symplectic (for $\epsilon > 0$ small).

The blow-up construction can be extended to blow-up a symplectic manifold along an embedded symplectic submanifold, as proved by McDuff [Mc].

Theorem 9. *Let (M, ω) be a symplectic manifold, and let $N \subset M$ be a symplectic submanifold of codimension $2r \geq 4$. Then there is a well-defined symplectic blow-up of M along N, which is a symplectic manifold $(\widetilde{M} = Bl_N M, \widetilde{\omega})$, together with a map $\pi : \widetilde{M} \to M$ such that:*

- *$E = \pi^{-1}(N) \to N$ is the fibration which is the (complex) projectivization of the normal bundle of N in M.*
- *$\pi : \widetilde{M} - E \to M - N$ is a diffeomorphism. Moreover, $\widetilde{\omega}$ and $\pi^*\omega$ coincide off a neighbourhood of E.*

Proof. To prove this, we have to introduce an ω-compatible almost-complex structure J as follows: first take some J on TN. Then on the symplectic normal bundle $\nu_N \to N$, defined as $\nu_N(p) = (T_pN)^o = \{u \in T_pM \,|\, \omega(u, T_pN) = 0\}$, $p \in N$, we put a complex structure J, so that ν_N becomes a complex vector bundle. Then this J can be extended to a neighborhood of N and later to the whole of M.

Now we consider a neighborhood $U \subset M$ of N which is symplectomorphic to the disc bundle $\nu_\epsilon \subset \nu_N$, whose fibers are ϵ-balls $D_\epsilon \subset \nu_N(p)$. We may blow-up $\nu_\epsilon \to N$ along the zero section, by doing a fiberwise blow-up. The result is a fiber bundle

$$\widetilde{D}_\epsilon \to Bl_N\,\nu_\epsilon \to N\,.$$

This has a map $\pi : Bl_N\,\nu_\epsilon \to \nu_\epsilon$. Finally, we glue $Bl_N\,\nu_\epsilon$ and $M-N$ along $Bl_N\,\nu_\epsilon - \pi^{-1}(N) \cong U-N$. The symplectic forms would be glued in a similar fashion as for the case of blowing-up at a point. □

The symplectic blow-up was used to produce the first examples of compact symplectic manifolds which are *simply-connected* and non-Kähler.

Take a symplectic embedding of the Kodaira-Thurston manifold into a projective space, $KT \subset \mathbb{CP}^n$. Note that it should be $n \geq 5$. Consider the symplectic blow-up

$$M = Bl_{KT}\mathbb{CP}^n\,.$$

It is simply-connected since $\mathbb{CP}^n$ is so, but it cannot be Kähler because it is non-formal. This is seen easily with a Massey product. Note that the exceptional divisor $E \subset M$ is a codimension 2 submanifold and there is a fibration $\mathbb{CP}^{n-3} \to E \xrightarrow{\pi} KT$. The dual form associated to E, $\nu \in \Omega^2(M)$, is a closed 2-form compactly supported in a neighborhood of E. Note that this gives sense to expressions like $\pi^*a \wedge \nu$, for any form $a \in \Omega(KT)$. The following Massey product

$$\langle [\pi^*\alpha \wedge \nu], [\pi^*\alpha \wedge \nu], [\pi^*\beta \wedge \nu] \rangle = -[\pi^*(\alpha \wedge \gamma) \wedge \nu^3]$$

is non-zero (we need $n-2 \geq 3$ for this non-vanishing). Therefore M is not formal. This produces simply-connected symplectic non-formal manifolds on any dimension $2n \geq 10$.

3.5. **Symplectic resolution of singularities.** An *orbifold* (of dimension n) is a topological space M with an atlas with charts modeled on U/G_p, where U is an open set of $\mathbb{R}^n$ and G_p is a finite group acting linearly on U with only one fixed point $p \in U$. An orbifold M contains a discrete set Δ of points $p \in M$ for which $G_p \neq \mathrm{Id}$. The complement $M - \Delta$ has the structure of a smooth manifold. The points of Δ are called singular points of M. For any singular point $p \in \Delta$, a small neighbourhood of p is of the form B/G_p, where B is a ball in $\mathbb{R}^n$.

An orbifold form $\alpha \in \Omega^*_{orb}(M)$ consists of G_p-equivariant forms on U, for each chart U/G_p, with the obvious compatibility condition. A *symplectic orbifold* (M, ω) is a $2n$-dimensional orbifold M together with a 2-form $\omega \in \Omega^2_{orb}(M)$ such that $d\omega = 0$ and $\omega^n \neq 0$ at every point.

Definition 10. A *symplectic resolution* of a symplectic orbifold (M, ω) is a smooth symplectic manifold $(\widetilde{M}, \widetilde{\omega})$ and a map $\pi : \widetilde{M} \to M$ such that:

(a) π is a diffeomorphism $\widetilde{M} - E \to M - \Delta$, where $\Delta \subset M$ is the singular set and $E = \pi^{-1}(\Delta)$ is the *exceptional set*.
(b) The exceptional set E is a union of (possibly intersecting) smooth symplectic submanifolds of $\widetilde{M}$ of codimension at least 2.
(c) $\widetilde{\omega}$ and $\pi^*\omega$ agree on the complement of a small neighbourhood of E.

The following result is in [FM].

Theorem 11. *Any symplectic orbifold has a symplectic resolution.*

Proof. Consider a ball B/G_p around a singular point $p \in \Delta$. Take Darboux coordinates on B, and a complex structure on B. So B/G_p is an (open) complex singular variety. We take a resolution of singularities in the complex setting (say, using Hironaka desingularisation process, which ensures that a suitable sequence of blow-ups yield eventually a smooth complex variety), $\widetilde{B/G_p} \to B/G_p$. This complex variety has a Kähler form. Finally we glue $M - \{p\}$ to $\widetilde{B/G_p}$ along $B/G_p - \{p\}$, and glue the symplectic forms as in the case of the blow-up of a symplectic manifold at a point. $\square$

This result was used in [FM] to produce the first example of a simply-connected symplectic non-formal manifold of dimension 8. This goes as follows.

Consider the complex Heisenberg group $H_{\mathbb{C}}$, that is, the complex nilpotent Lie group of complex matrices of the form

$$a = \begin{pmatrix} 1 & u_2 & u_3 \\ 0 & 1 & u_1 \\ 0 & 0 & 1 \end{pmatrix},$$

and let $G = H_{\mathbb{C}} \times \mathbb{C}$, where $\mathbb{C}$ is the additive group of complex numbers. We denote by u_4 the coordinate function corresponding to this extra factor. In terms of the natural (complex) coordinate functions (u_1, u_2, u_3, u_4) on G, we have that the complex 1-forms $\mu = du_1$, $\nu = du_2$, $\theta = du_3 - u_2\, du_1$, $\eta = du_4$ are right invariant and

$$d\mu = d\nu = d\eta = 0, \quad d\theta = \mu \wedge \nu.$$

Let $\Lambda \subset \mathbb{C}$ be the lattice generated by 1 and $\zeta = e^{2\pi i/3}$, and consider the discrete subgroup $\Gamma \subset G$ formed by the matrices in which $u_1, u_2, u_3, u_4 \in \Lambda$. We define the compact (parallelizable) nilmanifold

$$M = \Gamma \backslash G.$$

We can describe M as a principal torus bundle

$$T^2 = \mathbb{C}/\Lambda \hookrightarrow M \to T^6 = (\mathbb{C}/\Lambda)^3,$$

by the projection $(u_1, u_2, u_3, u_4) \mapsto (u_1, u_2, u_4)$.

Now introduce the following action of the finite group $\mathbb{Z}_3$

$$\begin{array}{rcl} \rho : G & \to & G \\ (u_1, u_2, u_3, u_4) & \mapsto & (\zeta\, u_1, \zeta\, u_2, \zeta^2\, u_3, \zeta\, u_4). \end{array}$$

The complex 2-form

$$\omega = i\, \mu \wedge \bar{\mu} + \nu \wedge \theta + \bar{\nu} \wedge \bar{\theta} + i\, \eta \wedge \bar{\eta}$$

is actually a real form, it is closed and satisfies $\omega^4 \neq 0$. Hence ω is a symplectic form on M. Moreover, ω is $\mathbb{Z}_3$-invariant. Hence the space

$$(\widehat{M} = M/\mathbb{Z}_3, \omega)$$

is a symplectic orbifold. By Theorem 11, this can be desingularized to a smooth symplectic manifold $\widetilde{M}$ of dimension 8.

It remains to see that $\widetilde{M}$ is simply connected and non-formal. This follows from the same assertions for the orbifold $\widehat{M}$. It is not difficult to see that $\widehat{M}$ is simply connected, but we shall content ourselves to see here that $b_1(\widehat{M}) = 0$. By Nomizu's theorem, $H^1(M) = \langle \mathrm{Re}(\mu), \mathrm{Im}(\mu), \mathrm{Re}(\nu), \mathrm{Im}(\nu), \mathrm{Re}(\eta), \mathrm{Im}(\eta) \rangle \cong \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$, and the action of $\mathbb{Z}_3$ is by rotations. So

$$H^1(\widehat{M}) = H^1(M)^{\mathbb{Z}_3} = 0\,.$$

The non-formality of $\widehat{M}$ is more difficult to check. Clearly M is non-formal since it is a nilmanifold which is not a torus. To quotient by $\mathbb{Z}_3$ kills 'part of' the minimal model (for instance, it kills the cohomology of degree 1, where one usually writes down Massey products). One has to check that enough non-formality remains after the $\mathbb{Z}_3$-quotient is performed. Details can be found in [FM].

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