Cubo A Mathematical Journal
Vol.05/N$^{\underline{o}}$03 – OCTOBER 2003

Leray-Serre Spectral Sequence for Quasi-Fibrations

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ABSTRACT. Let $f : X \to Y$ be a map between two spaces for which Y is stratified such that f is a fibration over each open stratum. We find some spectral sequences to compute the homology of X in terms of the homology of Y and that of the fibers. We apply this to give a Lefschetz theorem for the degeneracy loci of a morphism between holomorphic bundles on a complex manifold.

1 Homotopy and Homology

To understand the topology of a space X, one of the primary issues is the computation of the homotopy groups $\pi_n(X)$. Even in the simplest examples of spaces, homotopy groups turn out to be very difficult to compute. On the other hand, simplicial homology $H_n(X)$ is a good alternative being easy to compute due to the existence of a Mayer-Vietoris principle. This allows to divide the space X and "glue" the homology of the pieces. The good news

2000 *Mathematics Subject Classification.* 55N10, 32Q55.
Key words and phrases. singular homology, homotopy, spectral sequence, Lefschetz theorem, degeneracy loci.
Partially supported by The European Contract Human Potential Programme, Research Training Network HPRN-CT-2000-00101.

is that Hurewicz theorem relates both up to some extent, which may help us to compute homotopy groups.

Let us first review the classical notions about homotopy from [Sp]. Consider a topological space X with a fixed base point $* \in X$. Then $\pi_n(X)$ consists of equivalence classes of maps $\alpha : \mathbb{S}^n \to X$ from the n-sphere sending a base point $x_0 \in \mathbb{S}^n$ to $*$, modulo the equivalence relation $\simeq$ given by $\alpha_0 \simeq \alpha_1$ if there is a homotopy $H : \mathbb{S}^n \times [0,1] \to X$ with $H(x,0) = \alpha_0(x)$, $H(x,1) = \alpha_1(x)$, for all $x \in \mathbb{S}^n$ and $H(x_0, t) = *$, for all $t \in [0,1]$. A space X is said to be n-connected if $\pi_i(X) = 0$ for $i \leq n$. For the special case $n = 0$, we have that $\pi_0(X) = 0$ if X is arc-wise connected. Also $\pi_1(X)$ is the fundamental group of X, so that X is 1-connected means that it is simply-connected.

There is a relative version of the homotopy groups: if $A \subset X$ is a subspace containing the base point, then $\pi_n(X, A)$ consists of equivalence classes of maps $\alpha : (D^n, \partial D^n) \to (X, A)$, this meaning that α is a map from the disk D^n to X sending the boundary to A. The equivalence relation is through homotopies that send the boundaries into A.

For notions in relation with homology one can look at [Ma]. Let X be a topological space. Let $I = [0,1]$ be the unit interval. A n-cube is a continuous map $T : I^n \to X$, and it is degenerate if it is independent of one of its variables. The complex of singular chains in X is defined as the free abelian group $C_n(X)$ generated by the n-cubes, modulo the degenerate ones. The boundary operator ∂ associates to every n-cube the $(n-1)$-chain which consists in the sum of its faces with an appropriate sign given by its orientation. The homology $H_*(X)$ of X is given by the homology of $(C_*(X), \partial)$.

For the relative version, let $A \subset X$. Then the complex $C_*(X, A)$ is the quotient complex $C_*(X)/C_*(A)$ with the induced boundary operator. The homology of this complex is the relative homology $H_*(X, A)$.

There is a morphism, the Hurewicz homomorphism,

$$\pi_n(X, A) \longrightarrow H_n(X, A),$$

which sends the class of the map $\alpha : (D^n, \partial D^n) \to (X, A)$ to the cycle $\alpha \in C_*(X, A)$ using the homeomorphism $I^n \cong D^n$ and since $\partial\alpha \in C_*(A)$. For the following theorem see [Sp, §7].

Hurewicz theorem. *Let (X, A) be a pair of 1-connected spaces. Then $\pi_i(X, A) = 0$ for $i \leq n-1$ implies that $\pi_n(X, A) \cong H_n(X, A)$. In plain terms, all the homotopy and homology groups are zero up to the first non-zero ones, which are isomorphic.*

This means that (X, A) is n-connected, i.e., $\pi_i(A) \xrightarrow{\cong} \pi_i(X)$, for $i < n$, and $\pi_n(A) \twoheadrightarrow \pi_n(X)$. If the spaces A and X are CW-complexes then, up to homotopy equivalence, X is constructed out of A by attaching cells of dimension $n+1$ and over.

2 Spectral Sequence for a Filtration

Suppose that X is a space and $A \subset X$, then we have the long exact sequence of the pair

$$\cdots \to H_n(A) \to H_n(X) \to H_n(X, A) \xrightarrow{\partial_*} H_{n-1}(A) \to \cdots$$

This comes from the exact sequence of chain complexes $C_*(A) \to C_*(X) \to C_*(X, A)$. The connecting homomorphism ∂_* is defined as follows. Let $[z] \in H_*(X, A)$ with a representative $z \in C_*(X, A)$. Lift z to $\tilde{z} \in C_*(X)$, then $\partial\tilde{z} \in C_*(A)$ and set $\partial_*[z] = [\partial\tilde{z}]$.

We may think of this case as a two terms filtration $X_0 = A \subset X_1 = X$. What happens when we have a longer filtration $X_0 \subset X_1 \subset \cdots X_d = X$? Then $C_*(X)$ is a filtered complex with filtration given by $C_*(X_q) \subset C_*(X)$. This gives rise to *a spectral sequence*. A spectral sequence is a sequence of double complexes (E^r_{pq}, d^r) with differentials $d^r : E^r_{pq} \to E^r_{p+r-1,q-r}$ such that the homology of the r-th term is the following one $H(E^r, d^r) = E^{r+1}$. We say that it converges if for each fixed (p, q) the terms E^r_{pq} stabilizes for r large enough. The limiting one is denoted as E^∞_{pq}. For specific material on spectral sequences, the reader is recommended the nice book [Mc] or [BT, §14].

In our case we generate a spectral sequence with

$$E^0_{pq} = C_{p+q}(X_q)/C_{p+q}(X_{q-1}) = C_{p+q}(X_q, X_{q-1}).$$

This has an induced differential d^0 which is the standard boundary in chains. The homology is

$$E^1_{pq} = H_{p+q}(X_q, X_{q-1}).$$

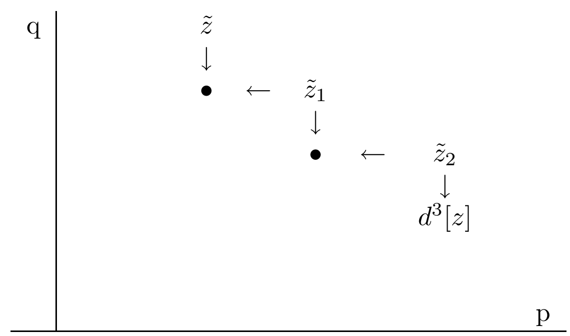
The general machinery on spectral sequences tells us that the differential

$$d^1 : H_{p+q}(X_q, X_{q-1}) \to H_{p+q-1}(X_{q-1}, X_{q-2})$$

is as follows: let $[z] \in H_{p+q}(X_q, X_{q-1})$. Select a representative $z \in C_{p+q}(X_q, X_{q-1})$ and lift it to $\tilde{z} \in C_{p+q}(X_q)$. Then consider the image of $\partial\tilde{z} \in C_{p+q-1}(X_{q-1})$ in $C_{p+q-1}(X_{q-1}, X_{q-2})$, and set $d_1[z] = [\partial\tilde{z}] \in H_{p+q-1}(X_{q-1}, X_{q-2})$. In particular, it follows that $d^1[z] = 0$ means that there exists $\tilde{z}_1 \in C_{p+q}(X_{q-1})$ such that $\partial\tilde{z} + \partial\tilde{z}_1 \in C_*(X_{q-2})$.

In general, a class $[z]$ survives to E^r, i.e., $d^1[z] = 0, \ldots, d^{r-1}[z] = 0$ if there are $\tilde{z}_i \in$

$C_{p+q}(X_{q-i})$ such that $\partial(\tilde{z} + \tilde{z}_1 + \cdots + \tilde{z}_{r-1})$ belongs to $C_{p+q-1}(X_{q-r})$. The latter element defines $d^r[z] \in E^r$. Moreover, the class $[z]$ is zero in E^r if there are $\tilde{w}_i \in C_{p+q+1}(X_{q+i})$ with $\partial(\tilde{w}_1 + \tilde{w}_2 + \cdots + \tilde{w}_r) - \tilde{z} \in C_*(X_{q-1})$.



The E^∞-term is the homology $H_*(X)$ in the following sense. There is a filtration of the homology $H_*(X)$ given by the subspaces $F_q H_*(X) = \operatorname{im}(H_*(X_q) \to H_*(X))$. Then

$$E^\infty_{pq} = \frac{F_q H_{p+q}(X)}{F_{q-1} H_{p+q}(X)}.$$

If we work with homology over a field, e.g. the reals, we have $\bigoplus_{p+q=n} E^\infty_{pq} \otimes \mathbb{R} \cong H_n(X) \otimes \mathbb{R}$ as vector spaces.

Now we are going to look to the case of a pair (X, B). For any $A \subset X$, we have an exact sequence

$$\cdots \to H_n(A, A \cap B) \to H_n(X, B) \to H_n(X, A \cup B) \to H_{n-1}(A, A \cap B) \to \cdots$$

This comes from the exact sequence of complexes

$$\frac{C_*(A)}{C_*(A \cap B)} \hookrightarrow \frac{C_*(X)}{C_*(B)} \twoheadrightarrow \frac{C_*(X)}{C_*(A) + C_*(B)}$$

For proving this we need that $C_*(A) + C_*(B) \subset C_*(A \cup B)$ induces isomorphism in homology [Ma, page 151]. This is termed as $\{A, B\}$ is an excisive couple, i.e., that the Mayer-Vietoris holds for $A \cup B$, A and B. This is true when the interiors of A and B cover $A \cup B$ in the relative topology of $A \cup B$. The proof consists on a process of subdivision of the cycles in $A \cup B$. But also it is true when A and B are CW-subcomplexes of a CW-complex X, since then there are open sets $U \supset A$, $V \supset B$ in $A \cap B$ such that U retracts to A, V retracts to B and $U \cap V$ retracts to $A \cap B$.

Theorem (spectral sequence for a filtration). *Let (X, B) be a pair of CW-complexes, $X_0 \subset X_1 \subset \cdots X_d = X$ a filtration by CW-subcomplexes. Then there is a spectral sequence with $E^1_{pq} = H_{p+q}(X_q, X_{q-1} \cup (B \cap X_q))$ converging to $H_n(X, B)$.*

We look at the filtration given by

$$\frac{C_*(X_q)}{C_*(B \cap X_q)} \subset \frac{C_*(X)}{C_*(B)}.$$

The E^0 term is the cokernel in

$$\frac{C_*(X_{q-1})}{C_*(B \cap X_{q-1})} \hookrightarrow \frac{C_*(X_q)}{C_*(B \cap X_q)} \twoheadrightarrow \frac{C_*(X_q)}{C_*(X_{q-1}) + C_*(B \cap X_q)}.$$

Under the assumptions, the various couples are excisive, so we get a spectral sequence as before.

3 Spectral Sequences for Quasi-Fibrations

Let us now turn our attention to a different issue. Very typically spectral sequences are used for a fibration $f : E \to B$ with fiber F. Suppose that B is 1-connected. Then the homology of the different fibers are identified in a canonical way. Therefore $H_*(F)$ is a system of (constant!) coefficients over the base B. We have the following standard relationship between the homology of the base B and fiber F and that of the total space E (see [Mc] and [BT]).

Leray-Serre spectral sequence. *Suppose that B is 1-connected. There is a spectral sequence whose second term is $E^2_{pq} = H_q(B; H_p(F))$ and converging to $H_n(E)$.*

We prove this result for CW-complexes. Take a CW-decomposition for B and consider the skeleta $B_n = B^{(n)}$. This is a filtration of the space B. (It may be an infinite filtration $B = \cup_{b \geq 1} B_n$, but the theory works because B has the weak topology with respect to skeleta). Now for any group G of coefficients, the homology $H_*(B; G)$ can be computed as the homology of the $H_n(B_n, B_{n-1}) \otimes G$. This is true since the spectral sequence $E^1_{pq} = H_{p+q}(B_q, B_{q-1}; G)$ is only non-zero for $n = p + q = q$, i.e., for $p = 0$. Therefore the only non-trivial differential is $d^1 : E^1_{pq} \to E^1_{p,q-1}$. All the other ones have to vanish and hence $H_*(B; G) = H_*(E^1, d^1)$.

Consider now the following filtration $E_n = f^{-1}(B_n)$ of the total space E and the corresponding spectral sequence for the filtration. The fibration over each n-cell in (E_n, E_{n-1}) is trivial. This means that the complexes $C_*(E_n, E_{n-1})$ and $C_*(B_n, B_{n-1}) \otimes C_*(F)$ are almost the same, in the sense that their homologies coincide, $H_*(E_n, E_{n-1}) = H_*(B_n, B_{n-1}) \otimes H_*(F)$, which is the E^1-term of the spectral sequence of the filtration. The E^2-term is the homology $H_*(B; H_*(F))$. Note that the first non-zero differential will be a map $d^r : H_q(B; H_p(F)) \to H_{q-r}(B; H_{p+r-1}(F))$.

Instead it is more common to have a map $f : X \to Y$ which is not a fibration but can be decomposed into pieces where it actually is. We want to extend the Leray-Serre spectral sequence to this case, relating the homology of X with that of Y.

Definition. A map $f : X \to Y$ between two spaces is called a *quasi-fibration* if there is a filtration Y_q of Y by closed subspaces such that denoting by $X_q = f^{-1}(Y_q)$ the corresponding filtration for X,

1. the restriction of f to $X_q - X_{q-1}$, $f_q : X_q - X_{q-1} \to Y_q - Y_{q-1}$ is a fibration with fiber F_q, for every q.
2. For each q, there is an open neighborhood U of Y_{q-1} in Y_q that retracts to Y_{q-1}, such that $f^{-1}(U)$ is a neighborhood of X_{q-1} in X_q that retracts to X_{q-1}.

Before going on further, let us say some words on the hypothesis on f. The condition of U being a neighborhood of Y_{q-1} in Y_q retracting to Y_{q-1} is technically referred to as NDR=Neighbourhood Deformation Retract. It occurs very often, for instance when Y_{q-1} is a CW-subcomplex of a CW-complex Y_q (e.g. a submanifold, even singular, in a differentiable manifold).

If we are dealing with CW-complexes (the filtration given by CW-subcomplexes) and f is cellular, then condition (ii) above is satisfied assuming f is proper: Take a retracting neighborhood U of Y_{q-1} in Y_q. We consider the neighborhood $f^{-1}(U)$ of X_{q-1} and take a retracting neighborhood $V \subset f^{-1}(U)$ of X_{q-1} in X_q. Then since f is proper, f is closed, so there exists U' (in fact, $U' \subset Y - f(X - V)$) with $U' \subset U$ and $f^{-1}(U') \subset V$. Moreover we may assume (since X_q is a CW-complex) that U also retracts to $\overline{U'}$. Let $H : U \times [0,1] \to U$ be the map with $H(y,0) = y$, $H(y',t) = y'$ for $y' \in \overline{U'}$ and $H(y,1) \in \overline{U'}$ for $y \in U$. Easily we may also assume that $H(y,t) \in U - Y_{q-1}$ for any $y \in U - Y_{q-1}$, $t \in [0,1]$.

By the fibration property, we lift the retraction to $\tilde{H} : (f^{-1}(U) - X_{q-1}) \times [0,1] \to f^{-1}(U) - X_{q-1}$ with $\tilde{H}(x,0) = x$, and $\tilde{H}(x,1) \in f^{-1}(\overline{U'})$. Now take a continuous function $u : U \to [0,1]$ which is 0 in a neighborhood $W \subset U'$ of X_{q-1} and 1 in $f^{-1}(U - U')$. Then $K(x,t) = \tilde{H}(x, u(x)t)$ is a retraction of $f^{-1}(U) - X_{q-1}$ to $f^{-1}(\overline{U'}) - X_{q-1}$ which is the identity in $f^{-1}(W) - X_{q-1}$. Therefore it may be extended with $K(x,t) = x$ for $x \in X_{q-1}$. Follow this retraction with the retraction form V to X_{q-1} to get a retraction from $f^{-1}(U)$ to X_{q-1} as required.

Theorem. *Let $f : X \to Y$ be a quasi-fibration with all of the Y_q being 1-connected. Then for a fixed integer q, there is a spectral sequence whose E^2 term is $H_*(Y_q, Y_{q-1}; H_*(F_q))$ and converging to $H_*(X_q, X_{q-1})$.*

Under the assumptions we have

$$H_*(X_q, X_{q-1}) = H_*(X_q, f^{-1}(U)) = H_*(X_q - X_{q-1}, f^{-1}(U - Y_{q-1})).$$

the last equality by excision. Also $H_*(Y_q, Y_{q-1}) = H_*(Y_q, U) = H_*(Y_q - Y_{q-1}, U - Y_{q-1})$. Since

$$(X_q - X_{q-1}, f^{-1}(U - Y_{q-1})) \to (Y_q - Y_{q-1}, U - Y_{q-1})$$

is a fibration with fiber F_q we have the result from the Leray-Serre spectral sequence.

For pairs we have a similar result which we state in the context of CW-complexes.

Theorem. *Let $f : X \to Y$ be a proper map between CW-complexes and let (X, B) be a pair of CW-complexes. Let Y_q be a filtration of Y by CW-subcomplexes and let $X_q = f^{-1}(Y_q)$, $B_q = f^{-1}(Y_q) \cap B$ be the corresponding filtrations of X and B. Suppose that $(X_q - X_{q-1}, B_q - B_{q-1}) \to Y_q - Y_{q-1}$ is a fibration (of a pair of spaces) with fiber (F_q, G_q). Then for fixed q, there is a spectral sequence with $E^2 = H_*(Y_q, Y_{q-1}; H_*(F_q, G_q))$ and converging to $E^\infty = H_*(X_q, X_{q-1} \cup B_q)$.*

4 The Fundamental Group on a Quasi-Fibration

The Leray-Serre spectral sequence (at least in the simplified version we have stated it in section 3) leaves aside an important issue of the behaviour of the topology of spaces under a quasi-fibration, namely the question relative to the fundamental group. Let us deal with this by hand.

Theorem. Let $f : X \to Y$ be a quasi-fibration between 0-connected spaces, such that the fibers F_q are connected. Then

$$f_* : \pi_1(X) \to \pi_1(Y)$$

is surjective.

To see this, take a loop $\gamma : [0,1] \to Y$. Consider a sub-interval $[0, t_1]$ such that $\gamma([0, t_1))$ is included in $Y_q - Y_{q-1}$ and $\gamma(t_1) \in Y_{q-1}$. This allows to write $\gamma = \gamma_1 * \gamma_2$ as a juxtaposition of $\gamma_1 = \gamma|_{[0,t_1]}$ and $\gamma_2 = \gamma|_{[t_1,1]}$.

Now consider a neighborhood U of Y_{q-1} in Y_q that retracts to Y_{q-1} such that $f^{-1}(U)$ retracts to X_{q-1}. Choose some $\varepsilon > 0$ with $\gamma_1([t_1 - \varepsilon, t_1]) \subset U$. Then we use the fibration property for $X_q - X_{q-1} \to Y_q - Y_{q-1}$ to lift $\gamma_1|_{[0,t_1-\varepsilon]}$ to a path $\tilde{\gamma}_1$. With the retraction, we may join $\tilde{\gamma}_1(t_1 - \varepsilon)$ with a point in X_{q-1} by a path $\tilde{\delta}$ in $f^{-1}(U)$. Let $\delta = f \circ \tilde{\delta}$ be the induced path. The path $\delta^{-1} * \gamma_1|_{[t_1-\varepsilon,t_1]}$ is in U with end-points in Y_{q-1} so it is homotopic (via the retraction) to a path α in Y_{q-1}. This means that $\delta * \alpha \simeq \gamma_1|_{[t_1-\varepsilon,t_1]}$. So we decompose

$$\gamma = \big((\gamma_1|_{[0,t_1-\varepsilon]}) * \delta\big) * (\alpha * \gamma_2)\,.$$

The first loop is lifted to a path $\tilde{\gamma}_1 * \tilde{\delta}$ with end-point in X_{q-1}.

We perform the above construction simultaneously in every portion of γ in Y_q. Specifically, suppose $\gamma([0,1]) \subset Y_q$. Write $\gamma = \gamma_1 * \cdots * \gamma_r$ with $\gamma_i = \gamma|_{[t_{i-1},t_i]}$ satisfying that either $\gamma_i([t_{i-1}, t_i]) \subset Y_{q-1}$ or $\gamma_i((t_{i-1}, t_i)) \subset Y_q - Y_{q-1}$ and $\gamma_i(\{t_{i-1}, t_i\}) \subset Y_{q-1}$ (except for possibly $\gamma_1(0)$ and $\gamma_r(1)$). Let $I \subset \{1, \ldots, r\}$ be the set of indices of those γ_i satisfying the second condition.

The method above allows to change γ by $\gamma \simeq \gamma_1' * \cdots * \gamma_r'$ such that γ_i' is liftable to $\tilde{\gamma}_i'$ for $i \in I$, and γ_i' is in Y_{q-1} for $i \notin I$. Since γ_i' is included in Y_{q-1}, for $i \notin I$, by induction on q we may suppose that they are liftable to $\tilde{\gamma}_i'$. If the end points of consecutive $\tilde{\gamma}_i'$ do not match then we use that the fibers are arc-wise connected to join them.

Therefore there is a loop $\tilde{\gamma} = \tilde{\gamma}_1' * \cdots * \tilde{\gamma}_r'$ such that $f \circ \tilde{\gamma} \simeq \gamma$, as required to have surjectivity of f_*.

Obviously the condition on the fibers F_q being 0-connected is necessary. Just think of the projection

$$\begin{aligned} f : \mathbb{S}^1 = \{(x,y) | x^2 + y^2 = 1\} &\to [-1,1] \\ (x,y) &\mapsto x \end{aligned}$$

which is a quasi-fibration, but not surjective in the fundamental groups.

5 Degeneracy Loci

An important point of a mathematical theory is to have applications which prove its usefulness. Our application touches the world of algebraic geometry. Start with a complex manifold M of (complex) dimension n. This means that M is covered by charts which are open sets of $\mathbb{C}^n$ and the change of charts are bi-holomorphisms. A holomorphic vector bundle $E \to M$ of rank e is a complex vector bundle which admits local trivializations where the transition functions are holomorphic [We].

For two holomorphic vector bundles E and F over M, of ranks e and f respectively, let $\phi : E \to F$ be a holomorphic morphism. This means that ϕ is holomorphic as a section of $\mathrm{Hom}(E, F)$. The r-degeneracy locus of ϕ is the subset

$$D_r(\phi) = \{x \in M \bigm| \mathrm{rk}\, \phi(x) \le r\}.$$

This is a complex submanifold of M but it is not smooth in general. It is usually said that $D_r(\phi)$ has a determinantal structure [ACGH], which forces $D_r(\phi)$ to be singular along $D_{r-1}(\phi)$.

The way to understand the degeneracy loci is the following. Construct the grassmannian bundle $G = \mathrm{Gr}(e-r, E)$ over M. This consists of putting the grassmannian $\mathrm{Gr}(e-r, E_x)$ of $(e-r)$-subspaces of the fiber E_x over every $x \in M$ in a differentiable varying way. The canonical projection of this fibration to M will be denoted by π. There is a universal bundle $\mathcal{U}$ over G. Every point $V \in G$ is actually an $(e-r)$-dimensional subspace $V \subset E_x$ for $x = \pi(V)$. So the bundle $\mathcal{U}$ has fiber over $V \in G$ the space V itself. Since $V \subset E_{\pi(V)}$, we get $\mathcal{U} \subset \pi^* E$. Composing this inclusion with the map $\pi^*(\phi) : \pi^* E \to \pi^* F$ we get a map $\phi_r : V \to \pi^* F$. Now ϕ_r is zero at a point $V \in G$ if and only if $\phi(V) = 0$. This means that at $x = \pi(V)$ there is an $(e-r)$-dimensional subspace of E_x in the kernel of $\phi(x)$ and therefore $\mathrm{rk}(\phi(x)) \le r$. So denoting by $Z(\phi_r) \subset G$ the zero set of ϕ_r, we have

$$\pi(Z(\phi_r)) = D_r(\phi).$$

Moreover this map is one to one over $D_r(\phi) - D_{r-1}(\phi)$. We have an obvious filtration of $D_r(\phi)$ given by those $D_{r-i}(\phi)$ with $i \ge 0$. Over $D_{r-i}(\phi) - D_{r-i-1}(\phi)$ the fiber of the map $\pi : Z(\phi_r) \to D_r(\phi)$ is $\mathrm{Gr}(e-r, e-r+i)$. So

$$\pi : Z(\phi_r) \to D_r(\phi)$$

is a quasi-fibration.

When the section ϕ_r of $\mathrm{Hom}(\mathcal{U}, \pi^* F)$ is transversal, $Z(\phi_r)$ are smooth subvarieties of

G, so that they can be considered as a desingularization of $D_r(\phi)$. Moreover $\dim G = \dim M + 2r(e-r)$ and $\operatorname{rank}_{\mathbb{C}}(U^* \otimes \pi^* F) = f(e-r)$, so that the dimension of $Z(\phi_r)$ is $2n - 2(e-r)(f-r)$. If this is negative then $D_r(\phi)$ is empty. Finally let

$$\rho(r) = n - (e-r)(f-r).$$

Note that it must be $r \leq e$ and $r \leq f$ since otherwise we have $D_r(\phi) = M$.

We pose ourselves a question: what can be said about the topology of the degeneracy loci $D_r(\phi)$?

6 A Lefschetz Theorem

Let M be a complex manifold and let $E \to M$ be a rank e holomorphic vector bundle. For a holomorphic section s of E, let $W = Z(s)$ be the zero set of s. The topology of W can be controlled under an extra assumption on E.

We say that E is a positive vector bundle [We, page 223][Gr, Chapter 0] if there exists an hermitian metric h_E in the fibers of E such that the curvature Θ associated to the hermitian connection induced by the holomorphic structure satisfies that $\sqrt{-1}\Theta$ is a positive 2-form: $\langle s, \sqrt{-1}\Theta(u, Ju)s\rangle > 0$ for any non-zero $u \in T_pM$ and non-zero $s \in E_p$, where Ju is the vector obtained by multiplying u by i in the complex space $T_pM \cong \mathbb{C}^n$. To go into details of the definition of curvature and consequences of positivity see [We]. When E is of rank 1 this produces a Kähler form for M.

Lefschetz theorem on hyperplane sections. *Let E be a rank e positive vector bundle over a compact complex manifold M and let s be a holomorphic section of E. Let $W = Z(s)$ be the zero set of s and $\rho = n - e$. If the section s is transverse then W has dimension 2ρ. In any case, the pair (M, W) is ρ-connected.*

The result is originally given in [AF][Bo]. We also have a Lefschetz theorem for degeneracy loci given in [MP] (see [De] for the homology version built up on considerations of connectedness of degeneracy loci in [FL]).

Lefschetz theorem for degeneracy loci. *Let ϕ be a morphism between the holomorphic vector bundles E and F, of ranks e and f respectively, over a compact complex manifold M, such that $\operatorname{Hom}(E, F)$ is positive. Let $W = Z(\phi_r) \subset G = \operatorname{Gr}(e-r, E)$ and $\rho = n - (e-r)(f-r)$. Then (G, W) is ρ-connected.*

The theorem does not follows from the previous Lefschetz theorem, since the positivity of $\mathrm{Hom}(E,F)$ does *not* imply that $\mathrm{Hom}(\mathcal{U},\pi^*F)$ is positive. Actually the proof of the theorem in [MP] goes through the way of proving that $\mathrm{Hom}(\mathcal{U},\pi^*F)$ keeps "part of" the positivity of $\mathrm{Hom}(E,F)$, namely the positivity along the section ϕ_r.

Our application is the following information on the topology of $D_r(\phi)$, using the map $\pi : G \to M$.

Theorem. *Let ϕ be a morphism between the holomorphic vector bundles E and F over a compact complex manifold M, such that* $\mathrm{Hom}(E,F)$ *is positive. Suppose that $\rho(r+k) \geq 0$, for some $k \geq 0$. Let $\varepsilon = 0$ if $\rho(r+k) > 0$ and $\varepsilon = 1$ if $\rho(r+k) = 0$. Then the pair $(M, D_r(\phi))$ is $(2k+1-\varepsilon)$-connected.*

Proof. We prove this by induction on k. For $k = 0$ it is very easy: if $\rho(r) = 0$ then $(G, Z(\phi_r))$ is 0-connected, i.e., every connected component of G has at least a point of $Z(\phi_r)$. Since the connected components of G are in 1-1 correspondence with those of M and $Z(\phi_r)$ surjects onto $D_r(\phi)$, the same happens to $(M, D_r(\phi))$. If $\rho(r) > 0$ then $(G, Z(\phi_r))$ is 1-connected. The commutative diagram

$$\begin{array}{ccc} \pi_1(Z(\phi_r)) & \twoheadrightarrow & \pi_1(G) \\ \downarrow & & \| \\ \pi_1(D_r(\phi)) & \rightarrow & \pi_1(M) \end{array} \tag{1}$$

implies that the arrow in the bottom row is a surjection. Therefore $(M, D_r(\phi))$ is also 1-connected.

Now let us turn to the case $k > 0$. We may suppose that M is connected from now on, working on every connected component of M if necessary. If $\rho(r-1) \geq 0$ then $\rho(r) \geq (e-r)+(f-r)+1 \geq 2$, unless $e = f = r$ in which case $D_r(\phi) = M$ and there is nothing to prove. If $\rho(r) \geq 2$ then $(G, Z(\phi_r))$ is 2-connected. Then the top row in diagram (1) is an isomorphism. By the result in section 4, $\pi_1(Z(\phi_r)) \twoheadrightarrow \pi_1(D_r(\phi))$ is a surjection. This implies that the bottom row in (1) must be an isomorphism. So the fundamental groups of $Z = Z(\phi_r)$, G, M and $D_r = D_r(\phi)$ all coincide. Let now $p : \widetilde{M} \to M$ be the universal covering space. Also $\tilde{p} : p^*G \to G$ is a universal covering space, and $\widetilde{Z} = \tilde{p}^{-1}(Z)$, $\widetilde{D}_r = p^{-1}(D_r)$ are the universal covering spaces of Z and D respectively. Pulling back all our construction by p we see that to prove that $(M, D_r(\phi))$ is n-connected is equivalent to prove that $(\widetilde{M}, \widetilde{D}_r(\phi))$ is n-connected.

The above considerations imply that we can work on $\widetilde{M}$. To simplify notation, we denote $\widetilde{M}$ by M, i.e., we assume to start with that M is simply-connected and keep the notations without tildes. Under this condition, Hurewicz theorem in section 1 says that (M, D_r) is $(2k+1-\varepsilon)$-connected if we manage to prove that

$$H_i(M, D_r) = 0, \qquad i \leq 2k+1-\varepsilon.$$

By Lefschetz theorem (G, Z) is $\rho(r)$-connected. Now $\rho(r-k) \geq 0$ implies that $\rho(r) = (e-r)k + (f-r)k + k^2 \geq 2k+1$ (again if either $e = r$ or $f = r$ then $D_r(\phi) = M$ and there is nothing to prove). So $H_n(G, Z) = 0$ for $n \leq 2k+1$.

Let us stratify M with $M \supset D_r \supset D_{r-1} \supset \cdots \supset D_{r-k}$ and let G be stratified with $G_{r-i} = \pi^{-1}(D_{r-i})$ and $Z_{r-i} = Z \cap G_{r-i}$. With this notations in place, we have that

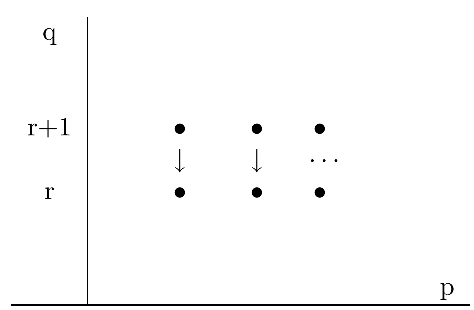
$$(G_{r-i} - G_{r-i-1}, Z_{r-i} - Z_{r-i-1}) \to D_{r-i} - D_{r-i-1}$$

is a fibration with fiber $(\mathrm{Gr}(e-r, e), \mathrm{Gr}(e-r, e-r+i))$. Let $i > 0$. The last theorem in section 3 implies that there is a spectral sequence with $E^2 = H_*(D_{r-i}, D_{r-i-1}) \otimes H_*(\mathrm{Gr}(e-r, e), \mathrm{Gr}(e-r, e-r+i))$ converging to $E^\infty = H_*(G_{r-i}, G_{r-i-1} \cup Z_{r-i})$. (We can take the homology of the grassmannian out of the 'coefficient group' since it is free.) Now the pair $(\mathrm{Gr}(e-r, e), \mathrm{Gr}(e-r, e-r+i))$ is $(2i+1)$-connected. By induction hypothesis, the pair (M, D_{r-i}) is $(2(k-i)+1-\varepsilon)$-connected. Therefore by the long exact sequence of the pair, $H_n(D_{r-i}, D_{r-i-1}) = 0$ for $n \leq (2(k-i)-\varepsilon)$. Hence $E^2_{pq} = 0$ for $p+q \leq 2(k-i)-\varepsilon+(2i+1)+1$. This implies that $H_n(G_{r-i}, G_{r-i-1} \cup Z_{r-i}) = 0$ for $n \leq 2k+2-\varepsilon$.

The spectral sequence for the filtration $G \supset G_r \supset G_{r-1} \supset \cdots \supset G_{r-k}$ has

$$\begin{aligned} E^2_{p,r+1} &= H_{p+r+1}(G, G_r), \\ E^2_{p,r-i} &= H_{p+q}(G_{r-i}, G_{r-i-1} \cup Z), \quad i \geq 0, \end{aligned}$$

and converges to $E^\infty = H_n(G, Z)$. By the above considerations $E^2_{pq} = 0$ for $q < r$ and $p+q \leq 2k+2-\varepsilon$. Looking at the limit, $H_n(G, Z) = 0$ for $n \leq 2k+1$. We have that it must be $d_1 = \partial_* : H_n(G, G_r) \to H_{n-1}(G_r, G_{r-1} \cup Z)$ an isomorphism, for $n \leq 2k+2$.



Now there are two spectral sequences: $E^2_{ij} = H_j(M, D_r) \otimes H_i(\mathrm{Gr}(e-r, e))$ converging to the $H_n(G, G_r)$, and $\bar{E}^2_{ij} = H_{j-1}(D_r, D_{r-1}) \otimes H_i(\mathrm{Gr}(e-r, e), pt)$ converging to $H_{n-1}(G_r, G_{r-1} \cup Z)$, where pt is the base point. Here we have 'moved' the second spectral sequence one unit to the top so that $f = \partial_*$ induces a map of bidegree $(0,0)$ between the two spectral sequences, i.e., a map $f^r : E^r \to \bar{E}^r$ for every step, such that $f^{r+1} = H(f^r)$. In the limit, $f^\infty : E^\infty \to \bar{E}^\infty$ must be an isomorphism.

Look at $f^2_{ij} : E^2_{ij} \to \bar{E}^2_{ij}$. For $i + j \leq 2k + 2 - \varepsilon$ it is surjective, since $H_j(M, D_r) \to H_{j-1}(D_r, D_{r-1})$ is surjective for $j \leq 2k - \varepsilon$. Also f^2_{ij} is injective for $i > 0$, $j \leq 2k - 1 - \varepsilon$, since $H_j(M, D_r) \to H_{j-1}(D_r, D_{r-1})$ is injective for $j \leq 2k - 1 - \varepsilon$. Now let us prove that $f^r : E^r_{ij} \to \bar{E}^r_{ij}$ is isomorphism for $i + j \leq 2k + 1 - \varepsilon$, $i > 0$ and an epimorphism for $i + j \leq 2k + 2 - \varepsilon$, by induction on r. We consider the diagram

$$\begin{array}{ccccc}
E^r_{i-r+1,j+r} & \xrightarrow{d_r} & E^r_{ij} & \xrightarrow{d_r} & E_{i+r-1,j-r} \\
f^r_{i-r+1,j+r} \downarrow & & f^r_{ij} \downarrow & & f^r_{i+r-1,j-r} \downarrow \\
\bar{E}^r_{i-r+1,j+r} & \xrightarrow{d_r} & \bar{E}^r_{ij} & \xrightarrow{d_r} & \bar{E}_{i+r-1,j-r}
\end{array}$$

For $i = 0$ we have $\bar{E}^r_{0j} = 0$, so it must be $d_r : E^r_{0j} \to E^r_{r-1,j-r}$ the zero map for $j \leq 2k + 2 - \varepsilon$. Hence $E^{r+1}_{0j} \cong E^r_{0j}$.

For $i + j < 2k + 1 - \varepsilon$ and $i > 0$, $i \neq r - 1$, the vertical arrows are isomorphisms, so f^{r+1}_{ij} is an isomorphism. For $i = r - 1$ the result also holds using that $d_r : E^r_{0j} \to E^r_{r-1,j-r}$ is the zero map. For $i + j = 2k + 1 - \varepsilon$, $i > 0$, the first vertical arrow is an epimorphism, the second and third are isomorphisms, so f^{r+1}_{ij} is an isomorphism. Finally for $i + j = 2k + 2 - \varepsilon$, f^r_{ij} is an epimorphism and $f^r_{i+r-1,j-r}$ is an isomorphism, hence f^{r+1}_{ij} is an epimorphism.

Looking at the E^∞ stage, $f^\infty_{0j} : E^\infty_{0j} \to \bar{E}^\infty_{0j} = 0$ is an isomorphism for $j \leq 2k + 1 - \varepsilon$. This means that $E^\infty_{0j} = E^2_{0j} = H_j(M, D_r) = 0$ for $j \leq 2k + 1 - \varepsilon$ as desired. $\square$

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