Formality of Kähler orbifolds and Sasakian manifolds

(joint work with I. Biswas, M. Fernández and A. Tralle)
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Kähler manifolds and almost Kähler manifolds

Let M be a (compact) smooth manifold of dimension $2n$.
An almost complex structure is a tensor $J\colon TM \to TM$, $J^2 = -\mathrm{id}$.
Nijenhuis tensor: $N_J(X,Y) = [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY]$.

Theorem (Newlander-Niremberg)

M is a complex manifold $\iff$ $N_J(X,Y) = 0$ for all vector fields X, Y.

Hermitian metric: $h = \sum h_{jk} dz_j \otimes d\bar{z}_k$
Riemannian metric: $g(u,v) = \mathrm{Re}\, h(u,v)$
Kähler form: $\omega(u,v) = g(Ju,v)$
$\omega = \frac{i}{2} \sum h_{jk} dz_j \wedge d\bar{z}_k$

Definition

M is Kähler if M is complex and $d\omega = 0$ ($\iff$ $\nabla J = 0$).

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M is almost Kähler if $d\omega = 0$ (but we do not require $N_J = 0$).

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Topology properties of Kähler manifolds

Theorem

There are (compact) almost Kähler manifolds which do not admit a Kähler structure.

Topological reasons:

- Fundamental groups. The fundamental groups of Kähler manifolds are very special (Kähler groups).
- The betti numbers $b_1, b_3, b_5, \ldots$ are even. This is a consequence of Hodge theory $H^k(M,\mathbb{C}) = \oplus_{p+q=k} H^{p,q}(M)$.
- Hard Lefschetz property. $[\omega]^k : H^{n-k}(M) \to H^{n+k}(M)$ is an isomorphism for $0 \leq k \leq n$.
- Rational homotopy groups. Formality means that the groups $\pi_k(M) \otimes \mathbb{Q}$ are determined by $H^k(M,\mathbb{Q})$. Kähler manifolds are formal (Deligne-Griffiths-Morgan-Sullivan, 1975).

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Topology properties of almost Kähler manifolds

- Any finitely presented group can be the fundamental group of an almost Kähler manifold (Gompf, 1994).
- The Betti numbers $b_1, b_3, b_5, \ldots$ of symplectic manifolds can be arbitrary. The first non-Kähler symplectic manifold is a nilmanifold (Thurston, 1976) with $b_1 = 3$. Simply-connected examples with $b_3 = 3$ constructed via symplectic blow-up (McDuff, 1984).
- Symplectic manifolds may not satisfy the hard Lefschetz property: nilmanifolds, Gompf examples, McDuff examples.
- There are non-formal symplectic manifolds: nilmanifolds, simply connected examples via symplectic blow-up (Babenko-Taimanov, 2000) of dimension ≥ 10, simply connected examples of dimension 8 via symplectic resolutions (Fernández-Muñoz, 2008), hard-Lefschetz examples (Cavalcanti, 2007), hard-Lefschetz simply-connected examples (Cavalcanti-Fernández-Muñoz, 2008).

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Almost contact manifolds

Let M be a $(2n+1)$-dimensional manifold.
An *almost contact metric structure* is given by(η, ξ, ϕ, g), where:

- η is a 1-form, $\mathcal{D} = \ker \eta$ codimension one distribution
- ξ is a nowhere vanishing vector field with $\eta(\xi) = 1$. So $TM = \mathcal{D} \oplus \langle \xi \rangle$.
- $\phi : TM \to TM$, $\phi^2 = -\mathrm{id} + \xi \otimes \eta$. So $\phi(\xi) = 0$ and $\phi|_{\mathcal{D}}$ is an almost-complex structure.
- g is a Riemannian metric with $g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$. Thus $TM = \mathcal{D} \oplus \langle \xi \rangle$ is orthogonal, and ϕ is isometric on $\mathcal{D}$.

The fundamental 2-form is $F(X, Y) = g(\phi X, Y)$.
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Equivalently, M is almost contact if and only if TM has a reduction of structure group to $\mathrm{U}(n) \times \{1\} \subset \mathrm{SO}(2n+1)$.

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The fundamental 2-form is $F(X, Y) = g(\phi X, Y)$.
So $F(\phi X, \phi Y) = F(X, Y)$ and $\eta \wedge F^n \neq 0$.

Equivalently, M is almost contact if and only if TM has a reduction of structure group to $\mathrm{U}(n) \times \{1\} \subset \mathrm{SO}(2n+1)$.

Almost contact manifolds

Let M be a $(2n+1)$-dimensional manifold.
An *almost contact metric structure* is given by(η, ξ, ϕ, g), where:

- η is a 1-form, $\mathcal{D} = \ker \eta$ codimension one distribution
- ξ is a nowhere vanishing vector field with $\eta(\xi) = 1$. So $TM = \mathcal{D} \oplus \langle \xi \rangle$.
- $\phi : TM \to TM$, $\phi^2 = -\mathrm{id} + \xi \otimes \eta$. So $\phi(\xi) = 0$ and $\phi|_{\mathcal{D}}$ is an almost-complex structure.
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K-contact and Sasakian manifolds

The almost contact structure (η, ξ, ϕ, g) is *contact metric* if $F = d\eta$ (so η is a contact form, i.e., $\eta \wedge (d\eta)^n \neq 0$).

1. K-contact if it is contact and ξ is a Killing vector field, i.e., $\mathcal{L}_\xi g = 0$.
2. Sasakian if it is contact and the Nijenhuis tensor N_ϕ satisfies $N_\phi = -d\eta \otimes \xi$, where $N_\phi(X, Y) := \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y]$.

Remark

Let $X = M \times \mathbb{R}$.
Let $J : TX \to TX$, $J|_{\mathcal{D}} = \phi$, $J(\xi) = \frac{\partial}{\partial t}$.
J is integrable $\iff$ $N_\phi = -d\eta \otimes \xi$.
(M, g) is Sasakian $\iff$ $(M \times \mathbb{R}^+, h = t^2 g + dt^2)$ is Kähler.

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Minimal models

Rational homotopy deals with spaces up rational homotopy equivalence, in particular, with

- Rational homotopy groups: $\pi_n(X) \otimes \mathbb{Q}$.
- Rational (co)homology: $H_n(X, \mathbb{Q})$, $H^n(X, \mathbb{Q})$.

(Here, $\mathbb{Q}$ may be replaced by $\mathbb{R}$ or $\mathbb{C}$)

If X is a smooth manifold, we consider the differential forms $(\Omega X, d)$. This is a graded-commutative differential algebra (GCDA for short). We extract an "invariant" from it.
Consider the equivalence relation $\sim$ between GCDAs generated by quasi-isomorphisms, $\psi : (A_1, d_1) \longrightarrow (A_2, d_2)$, i.e. morphisms inducing isomorphisms

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Minimal models

Minimal model

There is a canonical representative, called the *minimal model*, for any (A, d). The minimal model $(\mathcal{M}, d)$ of (A, d) satisfies:

- $\mathcal{M} = \bigwedge(x_1, x_2, \ldots)$ is free.
  $\bigwedge$ means the "graded-commutative algebra freely generated by"
- $dx_i \in \bigwedge(x_1, \ldots, x_{i-1})$.
- dx_i contains no linear term.
- $(\mathcal{M}, d) \longrightarrow (A, d)$ is a quasi-isomorphism.

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Minimal models

Theorem (Sullivan, 1977)

If either X is simply-connected or X is a nilpotent space, then the minimal model $(\mathcal{M}_X, d) \longrightarrow (\Omega X, d)$ codifies the rational homotopy of X. More specifically, $\mathcal{M}_X = \bigwedge V$, $V = \bigoplus_{n \geq 1} V^n$, where V^n is the vector space given by the degree n generators. Then

$$V^n \cong (\pi_n(X) \otimes \mathbb{R})^*,$$

and

$$H^n(\bigwedge V, d) = H^n(\Omega(X), d) = H^n(X).$$

Definition

A CDGA (A, d) is formal if $(A, d) \sim (H, 0)$.

Clearly, it is $H = H(A, d)$. So there are quasi-isomorphisms

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So the minimal model can be deduced *formally* from $H = H(A, d)$.
All the information is in the cohomology algebra.

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Massey products

There is a quick way to check non-formality.

Let $a_1, a_2, a_3 \in H^*(X)$ be cohomology classes such that $a_1 \cup a_2 = 0$ and $a_2 \cup a_3 = 0$. Take forms α_i in X with $a_i = [\alpha_i]$ and write

$$\alpha_1 \wedge \alpha_2 = d\xi, \ \alpha_2 \wedge \alpha_3 = d\zeta .$$

Then

$$d(\alpha_1 \wedge \zeta - (-1)^{|a_1|}\xi \wedge \alpha_3) = (-1)^{|a_1|}(\alpha_1 \wedge \alpha_2 \wedge \alpha_3 - \alpha_1 \wedge \alpha_2 \wedge \alpha_3) = 0 .$$

The Massey product of the classes a_i is defined as

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If $0 \notin \langle a_1, a_2, a_3 \rangle$ then X is non-formal.

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Higher Massey products

The Massey product $\langle a_1, a_2, \dots, a_t \rangle$, $a_i \in H^{|a_i|}(X)$, $1 \leq i \leq t$, $t \geq 3$, is defined as follows.

Choose $\alpha_{ij} \in \mathcal{A}$ inductively, for $i \leq j$, $(i, j) \neq (1, t)$, with

- $a_1 = [\alpha_{11}]$, $a_2 = [\alpha_{22}]$, ...
- $d\alpha_{12} = (-1)^{|\alpha_{11}|}\alpha_{11} \wedge \alpha_{22}$, $d\alpha_{23} = (-1)^{|\alpha_{22}|}\alpha_{22} \wedge \alpha_{33}$, ...
- $d\alpha_{13} = (-1)^{|\alpha_{11}|}\alpha_{11} \wedge \alpha_{23} + (-1)^{|\alpha_{12}|}\alpha_{12} \wedge \alpha_{33}$, ...
- $d\alpha_{14} = (-1)^{|\alpha_{11}|}\alpha_{11} \wedge \alpha_{24} + (-1)^{|\alpha_{12}|}\alpha_{12} \wedge \alpha_{34} + (-1)^{|\alpha_{13}|}\alpha_{13} \wedge \alpha_{44}$, ...

The Massey product is

$$\langle a_1, a_2, \dots, a_t \rangle = \left\{ \left[\sum_{k=1}^{t-1} (-1)^{|\alpha_{1k}|} \alpha_{1k} \wedge \alpha_{k+1,t} \right] \right\} \subset H^{|a_1| + \cdots + |a_t| - (t-2)}(X).$$

We say that the Massey product is trivial if $0 \in \langle a_1, a_2, \dots, a_t \rangle$.

Theorem

If X is formal then all (higher) Massey products of $(\Lambda V_X, d)$ are zero.

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Orbifolds

Definition

An orbifold X is a topological space with charts $(U, \widetilde{U}, \Gamma, \varphi)$ where $\Gamma \subset GL(n, \mathbb{R})$ is a finite group, $\widetilde{U} \subset \mathbb{R}^n$ is invariant under Γ, $U \subset X$, $\varphi : \widetilde{U} \longrightarrow U$ is a Γ-invariant map with $\widetilde{U}/\Gamma \xrightarrow{\cong} U$ an homeomorphism.

$\Omega^p_{orb}(X)$ denote the orbifold p-forms on X.
Fix a Riemannian (orbifold) metric. The complex

$$\Omega^0_{orb}(X) \xrightarrow{d} \Omega^1_{orb}(X) \xrightarrow{d} \Omega^2_{orb}(X) \xrightarrow{d} \dots \xrightarrow{d} \Omega^n_{orb}(X)$$

is elliptic. There is a Hodge isomorphism

$$H^k(X) \cong \mathcal{H}^k(X) = \ker(\Delta : \Omega^k_{orb}(X) \longrightarrow \Omega^k_{orb}(X)),$$

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Definition

A hermitian metric h has associated fundamental form $\omega \in \Omega^{1,1}_{orb}(X)$.
We say that (X, J, h) is a Kähler orbifold if $d\omega = 0$.

Theorem

For a (compact) Kähler orbifold, $\Delta = 2\Delta_{\overline{\partial}}$. Therefore $\mathcal{H}^k(X) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}(X)$.

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Lemma ($\partial\bar\partial$-lemma)

1. *Take $\alpha \in \Omega^{p,q}_{orb}(X)$ with $\partial\alpha = 0$. If $\alpha = \bar\partial\beta$ for some β, then there exists ψ such that $\alpha = \partial\bar\partial\psi$.*
2. *Take $\alpha \in \Omega^{p,q}_{orb}(X)$ with $\bar\partial\alpha = 0$. If $\alpha = \partial\beta$ for some β, then there exists ψ such that $\alpha = \partial\bar\partial\psi$.*

Proof.

Let $G = G_{\bar\partial}$ be the Green operator, let $H : \Omega_{orb}(X) \to \mathcal{H}(X)$ projection onto the harmonic forms.

$\alpha = H\alpha + \Delta_{\bar\partial} G\alpha = H\alpha + \bar\partial\bar\partial^* G\alpha + \bar\partial^*\bar\partial G\alpha$

$\alpha = \bar\partial\beta \implies H\alpha = 0$ and $\bar\partial G\alpha = G\bar\partial\alpha = 0.$

Hence $\alpha = \bar\partial\bar\partial^* G\alpha = \bar\partial G(\bar\partial^*\alpha)$.

$\bar\partial^* = \sqrt{-1}[\Lambda, \partial]$, where $\Lambda = L_\omega^*$ and $L_\omega(\beta) = \omega\wedge\beta$

$\partial\alpha = 0 \implies \bar\partial^*\alpha = -\sqrt{-1}\partial\Lambda\alpha$

$\implies \alpha = \bar\partial G(-\sqrt{-1}\partial\Lambda\alpha) = \partial\bar\partial(\sqrt{-1}\, G\Lambda\alpha).$

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Theorem

Let M be a (compact) Kähler orbifold. Then M is formal.

Proof.

1. $\imath : (\ker \partial, \overline{\partial}) \hookrightarrow (\Omega^*_{orb}(X), d)$ is a quasi-isomorphism.
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Let M be a (compact) Kähler orbifold. Then M is formal.

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Quasi-regularity

A Sasakian structure on M is *quasi-regular* if the orbits of the flow of ξ are circles.

This means that the flow of ξ is given as $S^1 \times M \to M$.

So $S^1 \hookrightarrow M \longrightarrow N$, where N is a (compact) Kähler orbifold.

The bundle has Euler class $[\omega] \in H^2_{orb}(M, \mathbb{Z})$, with contact form η such that $d\eta = \pi^*(\omega)$, where ω is an orbifold Kähler form.

Proposition

If M is a (compact) Sasakian manifold, then it admits also a quasi-regular Sasakian structure.

K-contact manifolds

If M is a K-contact manifold, then $S^1 \hookrightarrow M \longrightarrow N$, where N is a compact symplectic orbifold.

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Theorem

Let M be a simply connected compact Sasakian manifold. Then the higher order ($r \geq 4$) Massey products for M are zero.

Proof.

M admits a Sasakian structure $\Longrightarrow$ M admits a quasiregular Sasakian structure $\Longrightarrow$ M is a S^1-bundle over a Kähler orbifold $S^1 \to M \to N$.
N is formal, i.e., a model for N is $(H = H^*(N), 0)$.
A model for M is $H \otimes \bigwedge(x)$, with $|x| = 1$, $dx = \omega$.
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- For $r \geq 5$, $\sum_{k=1}^{r-1}(-1)^{|\xi_{1,k}|}\xi_{1,k} \cdot \xi_{k+1,r} = 0$.
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Let M be a simply connected compact Sasakian manifold. Then the higher order ($r \geq 4$) Massey products for M are zero.

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K-contact non-Sasakian manifolds

Theorem

Let M be a simply connected compact symplectic manifold of dimension $2k$ with an integral symplectic form ω. Suppose that there is a non-trivial quadruple Massey product in $H^*(M)$. There exists a sphere bundle $S^{2m+1} \to E \to M$, for $m+1 > k$, such that E is K-contact, but E does not admit any Sasakian structure.

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Let $S^1 \to P \to M$ be the principal S^1-bundle corresponding to $[\omega] \in H^2(M, \mathbb{Z})$. Consider the associated S^{2m+1}-bundle $S^{2m+1} \to E = P \times_{S^1} S^{2m+1} \to M$. Using the K-contact structure of S^{2m+1}, one can construct a K-contact structure on E.

The model for E is $(\bigwedge V_M \otimes \bigwedge(z), D)$, where z has degree $2m+1$. Since $2m+2 > 2k = \dim M$, $D(z) = 0$.

Then a non-zero quadruple Massey product for M gives a non-zero quadruple Massey product for E. Therefore E is not Sasakian. □

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K-contact non-Sasakian manifolds

Remark

Using an 8-dimensional simply connected compact symplectic manifold M with a non-zero quadruple Massey product (Fernández-Muñoz, 2008), one gets a 17-dimensional K-contact non-Sasakian simply connected compact manifold.
It would be desirable to construct lower dimensional examples.

Examples of Sasakian manifolds

1. **Not simply connected and not formal.** Consider M the circle bundle over T^2. Then the minimal model is $\bigwedge(x_1, x_2, x_3)$, with $|x_i| = 1$, $dx_1 = dx_2 = 0$, $dx_3 = x_1 x_2$. Then the Massey product $\langle [x_1], [x_1], [x_2] \rangle = \{[x_1 x_3]\}$ is non-zero.
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3. **Not simply connected and formal.** Consider M the circle bundle over $T^2 \times S^2$. This is also a bundle $S^3 \to M \to T^2$. So the minimal model is $\bigwedge(x_1, x_2, y)$, with $|x_i| = 1$, $|y| = 3$, $dx_i = 0$, $dy = 0$ by degree reasons. So M is formal.
4. **Simply connected and not formal.** Let M be the circle bundle over $S^2 \times S^2 \times S^2$. This is also a bundle $S^3 \to M \to S^2 \times S^2$. Its Euler class is non-zero. $S^2 \times S^2$ is formal, so a model is $H = \langle 1, y_1, y_2, y_1 y_2 \rangle$, where $|y_i| = 2$, $d = 0$. A model for M is $H \otimes \bigwedge(z)$, $|z| = 3$, $dz = y_1 y_2$. There is a non-zero Massey product $\langle y_1, y_2, y_2 \rangle = \{[z y_2]\}$, so M is not formal.

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2. **Simply connected and formal.** S^{2n+1} is the total space of the Hopf fibration $S^{2n+1} \longrightarrow \mathbb{CP}^n$. The minimal model is $\bigwedge(z)$ with $|z| = 2n+1$, $dz = 0$. So S^{2n+1} is formal.
3. **Not simply connected and formal.** Consider M the circle bundle over $T^2 \times S^2$. This is also a bundle $S^3 \to M \to T^2$. So the minimal model is $\bigwedge(x_1, x_2, y)$, with $|x_i| = 1$, $|y| = 3$, $dx_i = 0$, $dy = 0$ by degree reasons. So M is formal.
4. **Simply connected and not formal.** Let M be the circle bundle over $S^2 \times S^2 \times S^2$. This is also a bundle $S^3 \to M \to S^2 \times S^2$. Its Euler class is non-zero. $S^2 \times S^2$ is formal, so a model is $H = \langle 1, y_1, y_2, y_1 y_2 \rangle$, where $|y_i| = 2$, $d = 0$. A model for M is $H \otimes \bigwedge(z)$, $|z| = 3$, $dz = y_1 y_2$. There is a non-zero Massey product $\langle y_1, y_2, y_2 \rangle = \{[z y_2]\}$, so M is not formal.

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Remark

- The last example is not formal, but it has the same cohomology algebra as $M' = (S^2 \times S^5)\#(S^2 \times S^5)$, which is formal, being the connected sum of two formal manifolds.
- However M' cannot admit a Sasakian structure. This can be proved via minimal models.
- $(S^2 \times S^3)\#(S^2 \times S^3)$ has a Sasakian structure, whereas $(S^2 \times S^5)\#(S^2 \times S^5)$ does not.
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