Complex, symplectic and Kähler geometry

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Geometry

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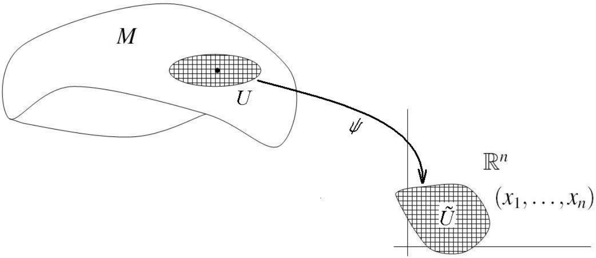
Smooth manifold: topological space such that every point has a neighbourhood (chart).



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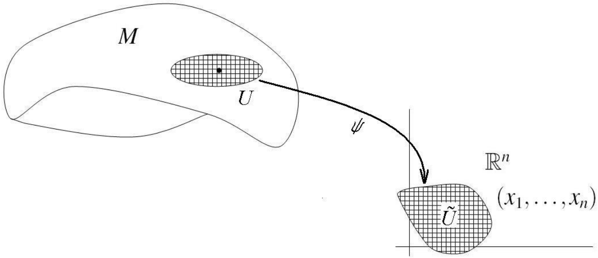

$\rightsquigarrow$ smooth functions on M, (tangent) vectors, etc.

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Main focus

Classify smooth (compact) manifolds with a given structure.

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This is an example of a number of topological obstructions for admitting a geometrical structure.
Topology $\rightsquigarrow$ Geometry.

Algebraic Geometry

Consider the ambient space $\mathbb{C}^n$.
Take $F_1, \ldots, F_m \in \mathbb{C}[z_1, \ldots, z_n]$.
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$S = V(F_1, \dots, F_m)$, $F_i(z_0, \dots, z_n)$ homogeneous polynomials, is a compact complex manifold.

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- If M is a symplectic manifold, does it admit a Kähler structure?

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Analysis (PDEs) on manifolds $\rightsquigarrow$ Topology.

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Harmonic forms:

$$\mathcal{H}^k(M) = \{\alpha \in \Omega^k(M) | \triangle\alpha = 0\} = \{\alpha | d\alpha = 0, d^*\alpha = 0\} \cong \\ \cong \frac{\{\alpha | d\alpha = 0\}}{\{\alpha = d\beta\}} = H^k(M).$$

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Complex coordinates: $z_j = x_{2j-1} + i\, x_{2j}$, $j = 1, \ldots, d$.
$dz_j = dx_{2j-1} + i\, x_{2j}, \qquad d\bar{z}_j = dx_{2j-1} - i\, x_{2j}$
(p, q)-forms: $\alpha = \sum f_{IJ}\, dz_{i_1} \wedge \ldots dz_{i_p} \wedge d\bar{z}_{i_1} \wedge \ldots d\bar{z}_{i_q}$
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$d\alpha = \partial\alpha + \bar{\partial}\alpha$
$\partial : \Omega^{p,q}(M) \to \Omega^{p+1,q}(M)$,
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Corollary

If M is a Kähler manifold then b_{2k+1} is even.

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Then $\triangle : \Omega^{p,q}(M) \to \Omega^{p,q}(M)$.
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Hodge decomposition: $H^k(M) = \bigoplus_{p+q=k} H^{p,q}(M)$.
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In particular, the Betti numbers satisfy:
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Corollary

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Analysis on manifolds $\rightsquigarrow$ Topology.

Kodaira-Thurston manifold

Kodaira, 1964

Complex manifold with $b_1 = 3$. It is given as

$$M = \left\{ \begin{pmatrix} 1 & z & w \\ 0 & 1 & \bar{z} \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, z, w \in \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\, i) \right\}$$

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$\alpha = da, \beta = db \in \Omega^1(T^2)$. Connection 1-form $\eta = dc - b\, da \in \Omega^1(H)$, $d\eta = \alpha \wedge \beta$.

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$\alpha = da, \beta = db \in \Omega^1(T^2)$. Connection 1-form $\eta = dc - b\, da \in \Omega^1(H)$, $d\eta = \alpha \wedge \beta$. Let $N = H \times S^1$, $\gamma = d\theta$.

The symplectic form is $\omega = \alpha \wedge \gamma + \beta \wedge \eta$.

Kähler vs. complex/symplectic geometry

Topological properties of Kähler manifolds

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Question

Does it exist a (compact) manifold M satisfying some topological property (e.g. b_{2k+1} even) admitting complex/symplectic structure but not admitting a Kähler structure?

Constructions of (compact) symplectic manifolds

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- Non-formal. Babenko-Taimanov (2000): non-formal simply-connected symplectic manifolds for dimension ≥ 10. Fernández-Muñoz (2008): non-formal simply-connected symplectic manifolds for dimension 8.

Main results

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What happens in dimension 6?

Theorem [Bazzoni-Fernández-Muñoz, arxiv:1410.6045]

There is a simply-connected 6-dimensional manifold complex and symplectic which is not hard-Lefschetz. Hence it does not admit Kähler structures.

Proof of theorem

Take the complex Heisenberg group

$$H = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{C}/\Lambda \right\},$$

where $\Lambda = \mathbb{Z}\langle 1, \xi \rangle$, $\xi = e^{2\pi i/6}$.

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But $\wedge\omega : H^2(\hat{M}) \to H^4(\hat{M})$ is not an isomorphism.

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Hence $\widetilde{M}_c = \widetilde{M}_s$ admits complex and symplectic structures (but not Kähler ones!)

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  Do a complex resolution with coordinates (a', b', c') obtaining a Kähler form $\tilde{\omega}'$ which has to be glued to ω outside B/Γ via cut-off functions. This substitutes B'/Γ' by some $\tilde{B}'$ with a symplectic form.

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- Need to control the distortion on the radial direction to prevent that the Jacobian becomes non-invertible.

Finally, have to deal with non-isolated orbifold points.

Proof of theorem

To prove that $\tilde{M}_c$ and $\tilde{M}_s$ are diffeomorphic,
we need a diffeomorphism $\tilde{B} \to \tilde{B}'$.

This requires:

- That $\Gamma \subset U(3)$ is conjugated to $\Gamma' \subset U(3)$. In our case, by (**), they are equal.
- Need to isotop the identity to the change of coordinates (*).
- Use the isotopy to radially construct a diffeomorphism $B \to B'$ which is Γ-equivariant.
- Need to control the distortion on the radial direction to prevent that the Jacobian becomes non-invertible.

Finally, have to deal with non-isolated orbifold points. QED