Manifolds which are complex and symplectic but not Kähler

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- (M, J) a complex manifold, that is $J \in \mathrm{End}(TM)$ satisfies $J^2 = -\mathrm{id}$, and the Nijenhius tensor $N_J = 0$,

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Hermitian metric: $h = g + i\,\omega$,
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Kodaira (1954)

Smooth algebraic variety $S \subset \mathbb{CP}^N \iff S$ is Kähler and $[\omega] \in H^2(S, \mathbb{Z}) \subset H^2(S)$.

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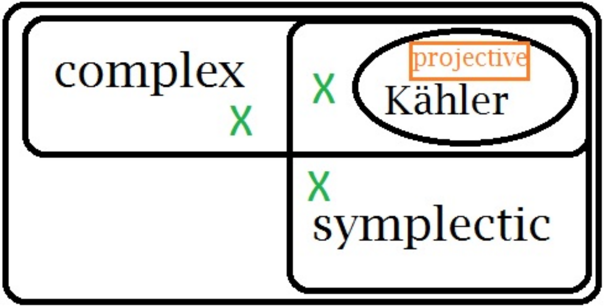

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Do these properties determine when a smooth manifold admit a Kähler structure?

Question

Does it exist a (compact) manifold M satisfying some/several/all topological properties admitting a complex/symplectic structure but not admitting a Kähler structure?

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Kodaira (1964)

Complex manifold with $b_1 = 3$.

$$H = \left\{ A = \begin{pmatrix} 1 & z & w \\ 0 & 1 & \bar{z} \\ 0 & 0 & 1 \end{pmatrix} \,|\, z, w \in \mathbb{C} \right\}$$

$\Gamma = \{A \in H \,|\, z, w \in \Lambda\}$, where $\Lambda \subset \mathbb{C}$ is a lattice closed by muliplication.
$M = H/\Gamma$.

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For complex surfaces, $b_1(M)$ even $\iff$ M admits a Kähler structure (Enriques-Kodaira classification).

Symplectic $\implies$ Kähler?

Thurston (1976)

Symplectic manifold with $b_1 = 3$. Take the Heisenberg manifold

$$H = \left\{ A = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\}$$

$\Gamma = \{A \in H \mid a, b, c \in \mathbb{Z}\}$

Then $S^1 \to H/\Gamma \to S^1 \times S^1$, $(a, b, c) \mapsto (a, b)$,

$\alpha = da, \beta = db$.

Connection 1-form $\eta = dc - b\,da \in \Omega^1(H/\Gamma)$, $d\eta = \alpha \wedge \beta$.

$M = (H/\Gamma) \times S^1$, $\gamma = d\theta$.

The symplectic form is: $\omega = \alpha \wedge \gamma + \beta \wedge \eta$.

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This is diffeomorphic to Kodaira's manifold (with $\Lambda = \mathbb{Z} + i\mathbb{Z}$).

$\rightsquigarrow$ Kodaira-Thurston manifold.

Constructions of symplectic manifolds

Fiber connected sum, Gompf (1995)

Given M_1, M_2 symplectic manifolds,
$S_j \subset M_j$ symplectic submanifolds of codimension 2,
$S_1 \cong S_2$, with normal bundles $\nu_{S_1} \cong \nu^*_{S_2}$.
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For any finitely presented group Γ and $2n \geq 4$, there is a symplectic $2n$-manifold M with $\pi_1(M) \cong \Gamma$.
If Γ is not a Kähler group, then M does not admit a Kähler structure.

Constructions of symplectic manifolds

For constructing simply-connected symplectic manifolds, we have:

Symplectic blow-up, McDuff (1984)

If $S \subset M$ is a symplectic submanifold,
then there is a symplectic manifold $\widetilde{M}$, $\pi : \widetilde{M} \to M$, such that
$\pi : E = \pi^{-1}(S) \to S$ is the complex projectivization of the normal bundle $E = \mathbb{P}(\nu_S) \to S$,
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So $\widetilde{\mathbb{CP}^n}$ does not admit a Kähler structure.

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A space M is formal if $(\Omega^*(M), d)$ is formal.

Deligne-Griffiths-Morgan-Sullivan (1975)

A Kähler manifold is formal.

Massey products

Let $a_1, a_2, a_3 \in H^*(A)$ be cohomology classes such that $a_1 \cup a_2 = 0$ and $a_2 \cup a_3 = 0$. Take forms $\alpha_i \in A$ with $a_i = [\alpha_i]$ and write $\alpha_1 \wedge \alpha_2 = d\xi,\ \alpha_2 \wedge \alpha_3 = d\zeta$.
Then
$d(\alpha_1 \wedge \zeta - (-1)^{|a_1|}\xi \wedge \alpha_3) = (-1)^{|a_1|}(\alpha_1 \wedge \alpha_2 \wedge \alpha_3 - \alpha_1 \wedge \alpha_2 \wedge \alpha_3) = 0.$

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The Massey product of the classes a_i is defined as:

$$\langle a_1, a_2, a_3 \rangle = [\alpha_1 \wedge \zeta - (-1)^{|a_1|}\xi \wedge \alpha_3] \in \frac{H^*(A)}{a_1 \cup H^*(A) + H^*(A) \cup a_3}.$$

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Proposition

If $\langle a_1, a_2, a_3 \rangle$ is non-zero, then X is non-formal.

Massey products can be transferred through quasi-isomorphisms.
Massey products vanish (obviously) on $(H^*(A), 0)$.

Non-formal symplectic manifolds

The Kodaira-Thurston manifold is non-formal.

$$H = \left\{ A = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\}, \Gamma = \{A \in H \mid a, b, c \in \mathbb{Z}\},$$

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The Massey product

$$\langle \alpha, \beta, \beta \rangle = \eta \wedge \beta \neq 0 \in \frac{H^2(KT)}{\alpha \cup H^1(KT) + H^1(KT) \cup \beta}.$$

Hence KT is non-formal.

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Salamon (2001)

There are non-formal 6-nilmanifolds which admit both complex and symplectic structures.
They cannot be Kähler.

Non-formal simply-connected symplectic manifolds

Babenko-Taimanov (2000)

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Yet ... what happens in dimension 8?

Symplectic resolution of singularities

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$\Lambda = \mathbb{Z} + \xi\mathbb{Z}$, $\xi = e^{2\pi i/3}$

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The minimal model is $(\bigwedge(\alpha, \bar{\alpha}, \beta, \bar{\beta}, \eta, \bar{\eta}, \gamma, \bar{\gamma}), d)$, $d\eta = \alpha \wedge \beta$.

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Let $\mathbb{Z}_3$ act on M by $(a, b, c, z) \mapsto (\xi a, \xi b, \xi^2 c, \xi z)$,
so $(\alpha, \beta, \eta, \gamma) \mapsto (\xi\alpha, \xi\beta, \xi^2\eta, \xi\gamma)$.
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Local model around a singular point: $B \cong (B(0,1)/\langle(\xi, \xi, \xi^2, \xi)\rangle, \hat{\omega})$.
With change of variables $(a', b', c', z') = (a, b - i\bar{c}, \bar{b} - ic, z)$,
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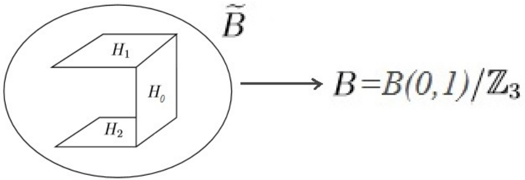

Symplectic resolution of singularities

Definition

The symplectic resolution is $\widetilde{M}_s = (\widehat{M} - \{0\}) \cup_\psi \widetilde{B}$.

Glue the symplectic forms on $\widehat{M}$ and $\widetilde{B}$ to get $(\widetilde{M}_s, \tilde{\omega})$.

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Complex, symplectic $\implies$ Kähler?

Guan (1995)

Construction of simply-connected, holomorphic symplectic and not Kähler manifolds of dimension $4n \geq 8$.

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$M = H/\Gamma \times \mathbb{C}/\Lambda$ has a complex structure J.
$(\widehat{M} = M/\mathbb{Z}_3, \hat{J})$ is a complex orbifold.

Complex, symplectic $\Longrightarrow$ Kähler?

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With coordinates (a, b, c, z),
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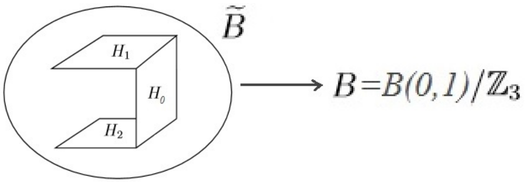

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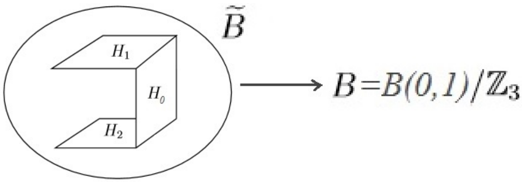

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Locally, the symplectic and complex resolutions coincide, but using different charts!

Complex, symplectic $\implies$ Kähler?

Comparing the charts, there is a diffeomorphism

$$\begin{array}{ccc} B & \xrightarrow{F} & B \\ \downarrow \psi & & \downarrow \varphi \\ B(0,1)/\mathbb{Z}_3 & = & B(0,1)/\mathbb{Z}_3 \end{array}$$

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F is isotopic to the identity through $\{F_t\}$, with $F_0 = \mathrm{id}$, $F_1 = F$.
Define $\hat{F} : \widehat{M} \to \widehat{M}$ as

$$\hat{F} = \begin{cases} F & \text{on } \widehat{M} - B(0,1)/\mathbb{Z}_3 \\ F_{2t-1} & \text{on } \partial B(0,t)/\mathbb{Z}_3,\ \frac{1}{2} \le t \le 1 \\ \mathrm{id} & \text{on } B(0,\frac{1}{2})/\mathbb{Z}_3 \end{cases}$$

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Now $\hat{F}$ extends to a diffeomorphism $\tilde{F} : \widetilde{M}_s \to \widetilde{M}_c$.

Construction in dimension 6

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What happens in dimension 6?

- Simply-connected 6-manifolds are formal.
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Bazzoni-Fernández-M, arxiv:1410.6045

There is a simply-connected 6-dimensional manifold, which is complex and symplectic but can not be hard-Lefschetz.
Hence it does not admit Kähler structures.

Proof of theorem

Take the complex Heisenberg group

$$H = \left\{ A = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \,\Big|\, a, b, c \in \mathbb{C} \right\},$$

$\Gamma = \{A \in H \,|\, a, b, c \in \Lambda\}$,

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$\alpha = \alpha_1 + i\alpha_2$, $d\alpha = 0$,
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Let $\omega = -i\alpha \wedge \bar{\alpha} + \beta \wedge \eta + \bar{\beta} \wedge \bar{\eta}$,

$d\omega = 0$ and $\omega^3 \neq 0$. So ω is symplectic.

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Hence $\widetilde{N}_c = \widetilde{N}_s$ admits complex and symplectic structures (but not Kähler ones!).

Proof of theorem

Singular points:

- 6 isolated orbifold points given by $(\frac{1}{3}k(1+\xi), \frac{1}{3}l(1+\xi), \frac{1}{3}m(1+\xi) + \frac{2}{9}kl(1+\xi)^2)$, $k, l, m = 0, 1, 2$, $(l, m) \neq (0, 0)$.
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- 5 tori given by $S_{p,q} = \{(w, p, pw + q) \mid w \in \mathbb{C}/\Lambda\}$, $p, q = 0, \frac{1}{2}, \frac{\xi}{2}, \frac{1+\xi}{2}$, $(p, q) \neq (0, 0)$.
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- One torus $S_0 = \{(w, 0, 0) \,|\, w \in \mathbb{C}/\Lambda\}$ with local model $\left((\mathbb{C}/\Lambda) \times \mathbb{C}^2/\langle \xi^3 \rangle\right)/\langle \xi \rangle$.

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  Change of coordinates: $a' = a$, $b' = b - i\,\bar{c}$, $c' = \bar{b} - i\,c$ to get
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  Symplectic/complex resolution of singularities transversally.
- For S_0, we need to do the transversal resolution of singularities to be also $\langle \xi \rangle$-equivariant.
  The diffeomrophism between complex and symplectic resolutions allows to put a Kähler form on a neighbourhood of the orbifold locus.

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  Then $(\beta \wedge \bar{\beta}) \wedge x \wedge y = 0$, for any $x, y \in H^2(\widetilde{N}_s)$.
  Therefore $L_\omega : H^2(\widetilde{N}_s) \to H^4(\widetilde{N}_s)$ has $\beta \wedge \bar{\beta}$ in its kernel for any possible ω.

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Hence $\tilde{N}_c = \widetilde{N}_s$ is a simply-connected 6-manifold which admits complex and symplectic structures but does not admit Kähler structures.

□

THANKS!

HAPPY BIRTHDAY SIMON