ERGODIC SOLENOIDAL GEOMETRY

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Abstract. We present a survey of recent results in the geometry of ergodic solenoids. We discuss the ideas behind the theory and its perspectives.

Dedicated to Emilio Elizalde in his 60th anniversary.

1. Introduction.

Geometry is at the origin of numerous applications of Mathematics to other fields, and to Mathematics itself. Classical Differential Geometry is nowadays a fundamental tool in Theoretical Physics. Needless to say that it is one of the most successful modern interaction and has marked the development of both fields. The objects of classical Differential Geometry are manifolds. One can envision many other geometric theories whose fundamental objects can be very different from classical manifolds. This idea is already present in Riemann's fundamental Memoir on the Foundations of Geometry.

We present in this article recent results on *Ergodic Solenoidal Geometry*. We do an informal presentation of the theory and we refer to our recent articles [2][3][4][5][6][7] for precise definitions, theorems in full generality, and complete proofs.

Ergodic Solenoidal Geometry is a generalization Differential Geometry where the central objects that extend manifolds are *uniquely ergodic solenoids*. Roughly speaking a *solenoid* is an abstract foliated space by finite dimensional leafs with transverse structure embedding into a finite dimensional space. Thus, contrary to other theories whose objects are foliated spaces, here we request some sort of finite dimensional transverse structure. This hypothesis is natural considering that our aim is to generalize finite dimensional Differential Geometry and not, for example, Banachian Differential Geometry. Example of solenoids are manifold with a foliation, but we have many more, as the dyadic solenoid $\hat{\mathbb{T}} = \varprojlim \{\mathbb{T} \to \mathbb{T};\ x \mapsto 2x\}$, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.

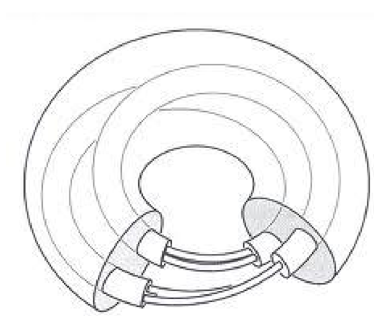

Figure 1. The dyadic solenoid

2000 *Mathematics Subject Classification.* Primary: 37A99. Secondary: 58A25, 57R95, 55N45.

Key words and phrases. Solenoids, real homology, Ruelle-Sullivan current, Schwartzman current, solenoid, ergodic theory, harmonic forms, cohomology, Hodge theory.

Partially supported through grant MEC (Spain) MTM2007-63582 and CNRS (UMR 7539).

The objects corresponding to compact and connected manifolds (we shall restrict to them from now on) are compact connected solenoids, but we need to restrict furthermore the structure in order to be able to generalize basic properties. Note that a compact manifold is a solenoid with trivial atomic transverse structure and a unique leaf. We consider solenoids that are topologically minimal, that is, all leaves are dense. We also consider solenoids possessing *daval* measures (the name comes from "measures that *decompose as volume along leaves*"). These are probability measures that locally disintegrate along the leaves as a product of a measure on a local transversal and a riemannian measure on the local leafs. It is easy to see that the existence of daval measure is equivalent to the existence of *transverse measure* in the sense of the theory of foliations. We recall that such a transverse measure is a collection of measures supported in each local transversal that are transported into each other by holonomy maps. Transverse measures are considered up to equivalence by multiplication by a positive scalar. Obviously any riemannian metric defines a daval measure on a manifold, the transverse measures being trivial atomic masses. But, in general, as is well known from foliation theory, transverse measures do not need to exist.

Thus, for the moment, our generalized objects are compact minimal solenoids admitting a transverse measure. A transverse measure is *ergodic* if for any local transversal, any Borel set that is invariant by the holonomy pseudogroup has zero or full measure. This is obviously the case for the trivial atomic measure associated to connected manifolds, since the holonomy is transitive on the points of the transversal. Thus we realize that it is natural to request to the transverse measure to be ergodic. And this is not enough to have a proper generalization. For manifolds the transverse measure is unique by the precedent argument. Thus we require *unique ergodicity* of the transverse measure: The transverse measure is unique (up to multiplication by a positive scalar as usual). It turns out that unique ergodicity implies topological minimality. Also unique ergodicity is determined by the geometry, but ergodicity is not.

We then arrive to the basic object of our geometry.

The objects in Ergodic Solenoidal Geometry that generalize compact connected manifolds in classical Differential Geometry are compact uniquely ergodic solenoids.

It is important to understand why we do not require just ergodicity. We could do this, but then only the underlying topological space is not sufficient to fully determine the object, more precisely its measurable transverse structure as happens for classical manifolds. Indeed the correct intuition is that the same solenoid endowed with distinct transverse ergodic measures should be considered as two objects of the geometry. By classical ergodic theory, ergodic measures generate all transversal measures. Note that two distinct ergodic measures are mutually singular.

In the following we explain how uniquely ergodic solenoids arise naturally from classical differential topology when we desire to represent geometrically real homology classes by a geometric class of currents on a manifold à la Ruelle-Sullivan, or à la Schwarzman, both being equivalent by Birkhoff ergodic theorem ([3]). Some important problems in geometry are about representing geometrically homology clasees, as the famous Hodge conjecture. Indeed solenoidal geometry allows to put in a new context the Hodge conjeture, and allows to isolate de geometric aspects from the algebraic ones. We formulating a natural solenoidal Hodge conjecture in this context (see [4] and section 2). We present then homological and cohomological theories for solenoids (section 3), and how Hodge theory extends to ergodic solenoidal geometry (section 4).

Acknowledgements. As a student, the second author first learned Riemannian geometry from Emilio Elizalde notes in the class "Metodos Matemáticos" at the University of Barcelona. Thus it is a double pleasure to congratulate him for his birthday with this geometric article, and to thank him for his excellent notes.

2. Geometric realization of the real homology.

We describe in this section our original motivation for introducing uniquely ergodic solenoids.

We consider a smooth compact connected oriented manifold M of dimension $n \geq 1$. Any closed oriented submanifold $N \subset M$ of dimension $0 \leq k \leq n$ determines a homology class in $H_k(M, \mathbb{Z})$. This homology class in $H_k(M, \mathbb{R})$, as dual of De Rham cohomology, is explicitly given by integration of the restriction to N of differential k-forms on M. Also, any immersion $f : N \to M$ defines an integer homology class in a similar way by integration of pull-backs of k-forms. Unfortunately, because of topological reasons dating back to Thom [11] [12], not all integer homology classes in $H_k(M, \mathbb{Z})$ can be realized in such a way. Geometrically, we can realize any class in $H_k(M, \mathbb{Z})$ by topological k-chains. The real homology $H_k(M, \mathbb{R})$ classes are only realized by formal combinations with real coefficients of k-cells. This is not satisfactory for various reasons. In particular, for diverse purposes it is important to have an explicit realization, as geometric as possible, of real homology classes.

The first contribution in this direction came in 1957 from the work of S. Schwartzman [9]. Schwartzman showed how, by a limiting procedure, one-dimensional curves embedded in M can define a real homology class in $H_1(M, \mathbb{R})$. More precisely, he proved that this happens for almost all curves solutions to a differential equation admitting an invariant ergodic probability measure. Schwartzman's idea is very natural. It consists on integrating 1-forms over large pieces of the parametrized curve and normalizing this integral by the length of the parametrization. Under suitable conditions, the limit exists and defines an element of the dual of $H^1(M, \mathbb{R})$, i.e. an element of $H_1(M, \mathbb{R})$. This procedure is equivalent to the more geometric one of closing large pieces of the curve by relatively short closing paths. The closed curve obtained defines an integer homology class. The normalization by the length of the parameter range provides a class in $H_k(M, \mathbb{R})$. Under suitable hypothesis, there exists a unique limit in real homology when the pieces exhaust the parametrized curve, and this limit is independent of the closing procedure. In the article [3], we study the different aspects of the Schwartzman procedure, that we extend to higher dimension.

Later in 1975, D. Ruelle and D. Sullivan [8] defined, for arbitrary dimension $0 \leq k \leq n$, geometric currents by using oriented k-laminations embedded in M and endowed with a transversal measure. They applied their results to stable and unstable laminations of Axiom A diffeomorphisms. In a later article Sullivan [10] extended further these results and their applications. The point of view of Ruelle and Sullivan is also based on duality. The observation is that k-forms can be integrated on each leaf of the lamination and then all over the lamination using the transversal measure. This makes sense locally in each flow-box, and then it can be extended globally by using a partition of unity. The result only depends on the cohomology class of the k-form. In [3] we review and extend Ruelle-Sullivan theory.

It is natural to ask whether it is possible to realize every real homology class using a topologically minimal Ruelle-Sullivan current. In order to achieve this goal we must enlarge the class of Ruelle-Sullivan currents by considering immersions of abstract oriented solenoids. For these oriented solenoids we can consider k-forms that we can integrate provided that we are given a transversal measure invariant by the holonomy group. We define an immersion of a solenoid S into M to be a regular map $f : S \to M$ that is an immersion in each leaf. If the solenoid S is endowed with a transversal measure μ, then any smooth k-form in M can be pulled back to S by f and integrated. The resulting numerical value only depends on the cohomology class of the k-form. Therefore we have defined a closed current that we denote by (f, S_μ) and that we call a *generalized current*. This gives a homology class $[f, S_\mu] \in H_k(M, \mathbb{R})$. The main result from [4] is the following:

Theorem 2.1. (Realization Theorem) *Every real homology class in $H_k(M,\mathbb{R})$ can be realized by a generalized current (f, S_μ) where S_μ is an oriented, minimal, uniquely ergodic solenoid.*

This result strengthens De Rham's realization theorem of homology classes by abstract currents, i.e. forms with coefficients distributions. It is a geometric De Rham's Theorem where the abstract currents are replaced by generalized currents that are geometric objects. Moreover, we prove in [6] that such geometric currents are dense in the space of currents.

We can ask why we do need to enlarge the class of Ruelle-Sullivan currents. The result does not hold for minimal Ruelle-Sullivan currents due to the observation that homology classes with non-zero self-intersection cannot be represented by Ruelle-Sullivan currents with no compact leaves ([5]). Therefore it is not possible to represent a real homology class in $H_k(M,\mathbb{R})$ with non-zero self-intersection by a minimal Ruelle-Sullivan current that is not a submanifold. Note that this obstruction only exists when $n-k$ is even. This may be the historical reason behind the lack of results on the representation of an arbitrary homology class by minimal Ruelle-Sullivan currents.

The space of solenoids is large, and we would like to realize the real homology classes by a minimal class of solenoids enjoying good properties. We are first naturally led to topological minimality. As we prove in [2], the spaces of k-solenoids is inductive and therefore there are always minimal k-solenoids. However, the transversal structure and the holonomy group of minimal solenoids can have a rich structure. In particular, such a solenoid may have many distinct transversal measures, each one yielding a different generalized current for the same immersion f. Also when we push Schwartzman ideas beyond 1-homology for some nice classes of solenoids, we see that in general, even when the immersion is an embedding, the generalized current does not necessarily coincide with the Schwartzman homology class of the immersion of each leaf (actually not even this Schwartzman class needs to be well defined). Indeed the classical literature lacks of information about the precise relation between Ruelle-Sullivan and Schwartzman currents, in particular in higher dimension. One would naturally expect that there is some relation between the generalized currents and the Schwartzman current (if defined) of the leaves of the lamination. We study this problem in [3].

The main result in [4] is that there is such relation for the class of minimal, ergodic solenoids with a trapping region. A solenoid with a trapping region has holonomy group generated by a single map. Then the bridge between generalized currents and Schwartzman currents of the leaves is provided by Birkhoff's ergodic theorem. The main result of [4] is the following:

Theorem 2.2. *Let S_μ be a minimal solenoid endowed with an ergodic transversal measure μ and possessing a trapping region W. Let $f : S_\mu \to M$ be an immersion of S_μ into M such that $f(W)$ is contained in a ball of M. Then for μ-almost all leaves $l \subset S_\mu$, the Schwartzman homology class of $f(l) \subset M$ is well defined and coincides with the homology class $[f, S_\mu]$.*

If moreover S is uniquely ergodic, then this happens for all leaves.

The solenoids constructed for the proof of the Realization Theorem do satisfy the hypothesis of this theorem and the transversal measure is unique, that is, the solenoids are uniquely ergodic.

Solenoidal Hodge Conjecture.

The Hodge Conjecture is an statement about the geometric realization of an integral class of pure type (p,p) in a complex (projective) manifold. If we drop the condition of the class being integral, then theorem 2.1 suggests a natural conjecture for *real* homology classes of pure type as follows.

For a compact Kähler manifold M of complex dimension n, a complex immersed solenoid $f : S_\mu \to M$ (that is, a solenoid where the images $f(l)$ of the leaves $l \subset S_\mu$ are complex immersed submanifolds), of dimension $k = 2(n-p)$, defines a class in $H_{n-p,n-p}(M) = H^{p,p}(M)^* \subset H_k(M, \mathbb{R})$. It is natural to formulate the following conjecture (see [4]):

Conjecture 2.3. ***(Solenoidal Hodge Conjecture)*** Let M be a compact Kähler manifold. Then any class in $H^{p,p}(M)$ is represented by a complex immersed solenoid of dimension $k = 2(n-p)$.

Note that the standard Hodge Conjecture is stated for projective complex manifolds, since it fails for Kähler manifolds [14]. The counterexamples of [14] are non-algebraic complex tori. It is easy to see that conjecture 2.3 holds for complex tori (using non-minimal complex solenoids).

3. Differential geometry of solenoids.

We describe in this section how theories and tools of differential topology do extend to Ergodic Solenoidal Geometry.

3.1. **De Rham cohomology.** Let S be a solenoid. The space of p-forms $\Omega^p(S)$ consist of p-forms on leaves with function coefficients that are smooth on leaves and partial derivatives of all orders continuous transversally. Using the differential d in the leaf-wise directions, we obtain the De Rham differential complex $(\Omega^*(S), d)$. The De Rham cohomology groups of the solenoid are defined as the quotients

$$H^p_{DR}(S) := \frac{\ker(d : \Omega^p(S) \to \Omega^{p+1}(S))}{\operatorname{im}(d : \Omega^{p-1}(S) \to \Omega^p(S))} . \tag{3.1}$$

We can also consider the spaces $\Omega^p_m(S)$ differential forms with function coefficients that are smooth on leaves and measurable transversally. Then define in the same way the De Rham measurable cohomology groups $H^p_{DRm}(S)$ using the complex $(\Omega^*(S), d)$. Note the natural map $H^p_{DR}(S) \to H^p_{DRm}(S)$.

Proposition 3.1. *Let $\underline{\mathbb{R}}_c$ and $\underline{\mathbb{R}}_m$ be respectively the sheaf of functions which are locally constant on leaves and transversally continuous, resp. measurable. Then we have isomorphisms*

$$H^p_{DR}(S) \cong H^p(S, \underline{\mathbb{R}}_c) ,$$

and

$$H^p_{DRm}(S) \cong H^p(S, \underline{\mathbb{R}}_m) .$$

Remark 3.2. The spaces $\Omega^p(S)$ are topological vector spaces. Therefore the De Rham cohomology (3.1) inherits a natural topology. In general, these spaces are infinite dimensional (even for compact solenoids). In some references, it is customary to take the closure of the spaces $\operatorname{im} d$ in definition (3.1), obtaining the *reduced De Rham cohomology groups*

$$\bar{H}^p_{DR}(S) = \frac{\ker d|_{\Omega^p}}{\overline{\operatorname{im} d|_{\Omega^{p1}}}} .$$

This is equivalent to quotienting $H^p_{DR}(S)$ by $\overline{\{0\}}$, obtaining thus Hausdorff vector spaces.

We shall list some basic properties of the De Rham cohomology:

(1) **Functoriality**. Let S_1, S_2 be two solenoids. A smooth map $f : S_1 \to S_2$ is a map sending leaves to leaves and transversally continuous. f defines a map on De Rham cohomologies, $f^* : H^p_{DR}(S_2) \to H^p_{DR}(S_1)$, by $f^*[\omega] = [f^*\omega]$. This applies in particular to an immersion of a solenoid into a smooth manifold $f : S \to M$, or to the inclusion of a leaf $i : l \to S$.

(2) **Mayer-Vietoris sequence**. Let U, V be two open subsets of a solenoid S. There is a short exact sequence of complexes: $\Omega^\bullet(U\cup V)\to \Omega^\bullet(U)\oplus\Omega^\bullet(V)\to\Omega^\bullet(U\cap V)$.
(3) **Homotopy**. A homotopy between two maps $f_0, f_1 : S_1 \to S_2$ is a map $F : S_1 \times [0,1] \to S_2$ (where $S_1\times[0,1]$ is given the solenoid structure with leaves $l\times[0,1]$, for $l\subset S_1$ a leaf of S_1) such that $F(x,0)=f_0(x)$ and $F(x,1)=f_1(x)$. We say that the maps f_0, f_1 are homotopic, written $f_0\sim f_1$. In this case $f_0^*=f_1^* : H^p_{DR}(S_2)\to H^p_{DR}(S_1)$.
(4) **Homotopy type**. We say that two solenoids S_1, S_2 are of the same homotopy type if there are maps $f: S_1\to S_2$, $g: S_2\to S_1$, such that $f\circ g\sim Id_{S_2}$, $g\circ f\sim Id_{S_1}$. Then the cohomology groups of S_1 and S_2 are isomorphic.

3.2. **Fundamental classes.** Let S be an *oriented* compact k-solenoid (dimension of leaves is k). The De Rham cohomology groups do not depend on any measure of S. If $\mu=(\mu_T)$ is a transversal measure, then the integral $\int_{S_\mu}$ descends to cohomology giving a map

$$\int_{S_\mu} : H^k_{DR}(S)\to\mathbb{R}\ . \tag{3.2}$$

We define the solenoidal homology as

$$H_p(S,\underline{\mathbb{R}}_c) := H^p(S,\underline{\mathbb{R}}_c)^* = H^p_{DR}(S)^*.$$

Then the map (3.2) defines a homology class $[S_\mu]\in H^k(S,\underline{\mathbb{R}}_c)^* = H_k(S,\underline{\mathbb{R}}_c)$. We shall call this element the *fundamental class* of S_μ.

Any map $f: S_1\to S_2$ defines a map $f^*: H^p_{DR}(S_2)\to H^p_{DR}(S_1)$ and hence, by dualizing, a map $f_*: H_p(S_1,\underline{\mathbb{R}}_c)\to H_p(S_2,\underline{\mathbb{R}}_c)$. Applying this to an immersion $f: S_\mu\to M$ of an oriented, measured, compact solenoid into a smooth manifold, then we have the equality

$$f_*[S_\mu]=[S_\mu, f]\ ,$$

with the generalized Ruelle-Sullivan class defined in the previous section.

Note that if S has a dense leaf (in particular when S it is minimal, i.e. all leaves are dense), then $H_0(S,\underline{\mathbb{R}}_c)=\mathbb{R}$. On the other hand, the dimension of the top degree homology counts the number of mutually singular tranverse measures on S.

Theorem 3.3. *Let S be a compact, oriented k-solenoid. Then $H_k(S,\underline{\mathbb{R}}_c)$ is isomorphic to the real vector space generated by all transversal measures.*

Remark 3.4. There is no Poincaré duality for $H^*_{DR}(S)$ in general. Moreover these spaces may be infinite dimensional (even for uniquely ergodic solenoids): if S is a two-torus foliated by irrational lines, then $H^1_{DR}(S)$ can be infinite-dimensional.

This result supports the intuition that the natural objects of Ergodic Solenoidal Geometry are uniquely ergodic solenoids with only one transversal measure.

3.3. **Singular cohomology.** We consider the space $\mathrm{Map}(I^n,S)$ of continuous maps $T: I^n\to S$ mapping into a leaf, and endow it with the uniform convergence topology. The degenerate maps (see [1]) form a closed subspace, therefore the quotient, $\mathrm{Map}'(I^n,S)$, has a natural quotient topology. The space of singular chains $C_n(S)$ is the free abelian group generated by $\mathrm{Map}'(I^n,S)$. There is a natural boundary map $\mathbf{d}: C_n(S)\to C_{n-1}(S)$.

Let G be any topological abelian group. Define the cochains $C^n(S,G)=\mathrm{Hom}_{cont}(C_n(S),G)$ as the continous homomorphisms. That is, $\varphi: C_n(S)\to G$ such that if $T_k: I^n\to S$ are maps which converge to $T_o: I^n\to S$ in the uniform topology, then $\varphi(T_k)\to\varphi(T_o)$. Define the differential $\delta: C^n(S)\to C^{n+1}(S)$ by $\delta\varphi(T)=\varphi(\mathbf{d}T)$. The solenoid singular cohomology of S with coefficients in G is defined as:

$$H^n(S,G) := \frac{\ker(\delta: C^n(S,G)\to C^{n+1}(S,G))}{\mathrm{im}(\delta: C^{n-1}(S,G)\to C^n(S,G))}\ .$$

We have some basic properties:

(1) **Functoriality**. Let $f : S_1 \to S_2$ be a solenoid map. Then there is a map $f_* : C_n(S_1) \to C_n(S_2)$, $f_*(T) = f \circ T$, and a map $f^* : C^n(S_2, G) \to C^n(S_2, G)$, $f^*(\varphi) = \varphi \circ f$. Clearly $f^*\delta = \delta f^*$, so the map descends to cohomology: $f^* : H^n(S_2, G) \to H^n(S_2, G)$.
(2) **Homotopy**. Suppose that $f, g : S_1 \to S_2$ are two homotopic solenoid maps. The usual construction yields a chain homotopy H between f^* and g^* (one only have to check that this map sends continuous cochains into continuous cochains). Therefore $f^* = g^* : H^n(S_2, G) \to H^n(S_2, G)$.
(3) **Homotopy type**. If S_1, S_2 are of the same homotopy type, then $H^n(S_2, G) \cong H^n(S_2, G)$.
(4) If $U = D^k \times K(U)$ is a flow-box, then U is of the same homotopy type than $\{*\} \times K(U)$. Therefore $H^n(U) = 0$ for $n > 0$, and $H^0(U) = \mathrm{Map}_{cont}(K(U), G)$. In particular, this implies that
$$\underline{\mathbb{R}}_c \to C^0(-, \mathbb{R}) \xrightarrow{\delta} C^1(-, \mathbb{R}) \xrightarrow{\delta} \dots$$
is a resolution. Therefore there is an isomorphism $H^n(S, \mathbb{R}) \cong H^n(S, \underline{\mathbb{R}}_c)$.
(5) **Mayer-Vietoris sequence**. For two open sets U, V with $S = U \cup V$, define $C_n(S; U, V)$ as the subcomplex generated by those singular chains completely contained in either U or V. Define accordingly $C^n(S; U, V)$. It is not difficult to see that the restriction $C^n(S, G) \to C^n(S, G; U, V)$ is chain homotopy equivalence. Therefore the exact sequence $0 \to C^n(S, G; U, V) \to C^n(U, G) \oplus C^n(V, G) \to C^n(U \cap V, G) \to 0$ gives rise to a long exact sequence:
$$\dots \to H^p(U \cup V, G) \to H^p(U, G) \oplus H^p(V, G) \to H^p(U \cap V, G) \to H^{p+1}(U \cup V, G) \to \dots$$

3.4. **De Rham L^2-cohomology.** Now consider a k-solenoid S with a transversal measure μ. There is a notion of cohomology which takes into account the transversal measure structure. For this, we work with forms which are L^2-transversal relative to μ.

Definition 3.5. A function f is $L^2(\mu)$-*transversally smooth* if in any (good) flow-box $U = D^k \times K(U)$ all partial derivatives on the first variable exist and are in $L^2(\mu_{K(U)})$, i.e. if we write f as $f(x, y)$ then for all $r \geq 0$,
$$\int_{K(U)} ||f(-, y)||^2_{C^r} d\mu_{K(U)}(y) < \infty \ .$$

We consider the space of forms
$$\Omega^p_{L^2(\mu)}(S)$$
which are $L^2(\mu)$-transversally smooth, i.e. locally these are forms $\alpha = \sum f_I(x, y) dx_I$, where f_I are $L^2(\mu)$-transversally smooth functions. There is a well-defined differential along leaves $d : \Omega^p_{L^2(\mu)}(S) \to \Omega^{p+1}_{L^2(\mu)}(S)$ which defines the complex $(\Omega^*_{L^2(\mu)}(S), d)$. We define the *De Rham L^2-cohomology* vector space as the quotients
$$H^p_{DR}(S_\mu) := \frac{\ker(d : \Omega^p_{L^2(\mu)}(S) \to \Omega^{p+1}_{L^2(\mu)}(S))}{\mathrm{im}(d : \Omega^{p-1}_{L^2(\mu)}(S) \to \Omega^p_{L^2(\mu)}(S))} \ . \tag{3.3}$$
We also introduce the reduced De Rham L^2-cohomology:
$$\bar{H}^p_{DR}(S_\mu) := \frac{\ker d}{\overline{\mathrm{im}\, d}} \ . \tag{3.4}$$

Note that there are natural maps
$$H^p_{DR}(S) \to H^p_{DR}(S_\mu) \to H^p_{DRm}(S) \,,$$

since $C^{\infty,0}$-functions are $L^2(\mu)$-transversally smooth. The integration map $\int_{S_\mu}$ is well-defined for forms in $\Omega^k_{L^2(\mu)}$, since a $L^2(\mu)$-transversally smooth k-form is automatically $L^1(\mu)$-transversal (all measures are finite measures on compact transversals). So we have $\int_{S_\mu} : H^k_{DR}(S_\mu) \to \mathbb{R}$.

Let $\underline{\mathbb{R}}_\mu$ be the sheaf of measurable functions which are locally constant on leaves and $L^2(\mu)$-transversally. A standard Poincaré lemma shows that there is a resolution of sheaves

$$\underline{\mathbb{R}}_\mu \to \Omega^0_{L^2(\mu)} \to \Omega^1_{L^2(\mu)} \to \ldots \to \Omega^k_{L^2(\mu)} .$$

So we get a natural isomorphism

$$H^p_{DR}(S_\mu) \cong H^p(S, \underline{\mathbb{R}}_\mu) .$$

We review basic properties of the De Rham L^2-cohomology:

(1) There is not cup product, and therefore the $H^*_{DR}(S_\mu)$ are just vector spaces (not rings).
(2) Functoriality. If $f : S_1 \to S_2$ is a solenoidal map, then we require that $\mu_2 = f_*\mu_1$. This means that for any local transversal T_1 of S_1, $f(T_1)$ is a local transversal of S_2 and the transported measure $f_*\mu_1$ is a constant multiple of μ_2 on the transversal. Note that this is automatic when the solenoids are uniquely ergodic. Then for any form ω which is $L^2(\mu_2)$-transversally smooth we have that $f^*\omega$ is $L^2(\mu_1)$-transversally smooth.
(3) Mayer-Vietoris. It holds exactly as in subsection 3.1.
(4) Poincaré duality. We shall see that it holds for the reduced L^2-cohomology for ergodic solenoids (see [7]).

3.5. **Bundles over solenoids.** Let S be a k-solenoid. A vector bundle of rank n over S consists of a $(k+n)$-solenoid E and a projection map $\pi : E \to S$ satisfying the following condition: there is an open covering U_α for S, and solenoid isomorphisms $\psi_\alpha : E_\alpha = \pi^{-1}(U_\alpha) \xrightarrow{\cong} U_\alpha \times \mathbb{R}^n = D^k \times K(U_\alpha) \times \mathbb{R}^n$, such that $\pi = pr_1 \circ \psi_\alpha$, where $pr_1 : U_\alpha \times \mathbb{R}^n \to U_\alpha$ denotes the projection, and the transition functions

$$\psi_\alpha \circ \psi_\beta^{-1} : (U_\beta \cap U_\alpha) \times \mathbb{R}^n \to (U_\beta \cap U_\alpha) \times \mathbb{R}^n$$

are of the form $(x, y, v) \mapsto (x, y, g_{\alpha\beta}(x, y)(v))$, where $g_{\alpha\beta}$ is a $C^{\infty,0}$-smooth function from $U_\alpha \cap U_\beta$ to $\mathrm{GL}(n)$.

Some points are easy to check:

(1) The usual constructions of vector bundles remain valid here: direct sums, tensor products, symmetric and anti-symmetric products. Also there are notions of sub-bundle and of quotient bundle.
(2) A section of a bundle $\pi : E \to S$ is a map $s : S \to E$ such that $\pi \circ s = Id$. We denote the space of sections as $\Gamma(E)$.
(3) If S_μ is a measured solenoid, and $E \to S$ is a vector bundle, then we have the notion of sections which are $L^2(\mu)$-transversally smooth. Locally, in a chart $E_\alpha = D^k \times K(U) \times \mathbb{R}^n \to U_\alpha = D^k \times K(U)$, the section is written $s(x, y) = (x, y, v(x, y))$. We require that v is C^∞ on x and $L^2(\mu)$ on y. This does not depend on the chosen trivialization.
(4) If $f : S_1 \to S_2$ is a solenoid map, and $\pi : E \to S_2$ is a vector bundle, then the pull-back $f^*E = \{(p, v) \in S_1 \times E \mid f(p) = \pi(v)\}$ is naturally a vector bundle over S_1.
(5) The tangent bundle TS of S is an example of vector bundle. We have bundles of (p, q)-tensors $TS^{\otimes p} \otimes (TS^*)^{\otimes q}$ on any solenoid S. In particular, we have bundles of p-forms (anti-symmetric contravariant tensors) $\bigwedge^p T^*S$. Its sections are the p-forms $\Omega^p(S)$.

(6) A metric on a bundle E is a section of $\mathrm{Sym}^2(E^*)$ which is positive definite at every point. A metric on S is a metric on the tangent bundle. An orientation of a bundle E is a continuous choice of orientation for each of the fibers of E. An orientation of S is an orientation of its tangent bundle.

We define $\Omega^p(E) = \Gamma(\bigwedge^p T^*S \otimes E)$. A connection on a vector bundle $E \to S$ is a map

$$\nabla : \Gamma(E) \to \Omega^1(E),$$

such that $\nabla(f \cdot s) = f\nabla s + df \wedge s$. Consider a local trivialization in a flow-box U_α with coordinates (x, y). Then $\nabla|_{U_\alpha} = d + a_\alpha$, where $a_\alpha \in \Omega^1(U_\alpha, \mathrm{End}\, E)$. Under a change of trivialization $g_{\alpha\beta}$, for two trivializing open subsets U_α, U_β, we have the usual formula $a_\beta = g_{\alpha\beta}^{-1} a_\alpha g_{\alpha\beta} + g_{\alpha\beta}^{-1} dg_{\alpha\beta}$.

A partition of unity argument proves that there are always connections on a vector bundle $E \to S$. The space of connections is an affine space over $\Omega^1(\mathrm{End}\, E)$.

Given a connection ∇ on E, there is a unique map $d_\nabla : \Omega^p(E) \to \Omega^{p+1}(E)$, $p \geq 0$, such that $d_\nabla s = \nabla s$ for $s \in \Gamma(E)$, and $d_\nabla(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d_\nabla \beta$, for $\alpha \in \Omega^p(S)$, $\beta \in \Omega^q(E)$. It is easy to see that $\hat{F}_\nabla : \Gamma(E) \to \Omega^2(E)$, given by $\hat{F}_\nabla(s) = d_\nabla d_\nabla s$, has a tensorial character (i.e., it is linear on functions). Therefore there is a $F_\nabla \in \Omega^2(\mathrm{End}\, E)$, called *curvature* of ∇, such that $\hat{F}_\nabla(s) = F_\nabla \cdot s$. Locally on a trivialization U_α, we have the formula $F_\nabla = da_\alpha + a_\alpha \wedge a_\alpha$.

Given connections on vector bundles, there are induced connections on associated bundles (dual bundle, tensor product, direct sum, symmetric product, pull-back under a solenoid map, etc.). This follows in a straightforward way from the standard theory. In particular, if $l \hookrightarrow S$ is a leaf of a solenoid S, then we can perform the pull-back of the bundle and connection to the leaf, which consists on restricting them to l. This gives a bundle and connection of a complete k-dimensional manifold. Also, if $f : S \to M$ is an immersion of a solenoid in a smooth n-manifold, and $E \to M$ is a bundle with connection, then the pull-back construction produces a bundle with connection on S.

Consider a vector bundle $E \to S$ endowed with a metric. We say that a connection ∇ is compatible with the metric if it satisfies

$$d\langle s, t\rangle = \langle \nabla s, t\rangle + \langle s, \nabla t\rangle\,.$$

In the particular case of the tangent bundle TS of a Riemannian solenoid S, we have the Levi-Civita connection ∇^{LC}, which is the unique connection compatible with the metric and with torsion $T_\nabla(X, Y) = \nabla_X Y - \nabla_Y X = 0$. This is the Levi-Civita connection on each leaf, and the transversal continuity follows easily.

3.6. **Chern classes.** We can also define a complex vector bundle over a solenoid, by using $\mathbb{C}^n$ as fiber, and taking the transition functions with values in $\mathrm{GL}(n, \mathbb{C})$. An hermitian metric on a complex vector bundle is a positive definite hermitian form in each fiber with smoothness of type $C^{\infty,0}$ on any local trivialization.

Let $E \to S$ be a complex vector bundle over a solenoid of rank n. Put a hermitian structure on E, and consider any hermitian connection ∇ on E. Then the curvature F_∇ is a 2-form with values in $\mathrm{End}\, E$, i.e. $F_\nabla \in \Omega^2(\mathrm{End}\, E)$. The Bianchi identity says

$$d_\nabla F_\nabla = 0\,.$$

This holds leaf-wise, so it holds on the solenoid.

Consider the elementary functions: $\mathrm{Tr}_i : M_{r\times r} \to \mathbb{C}$, given by $\mathrm{Tr}_i(A) = \mathrm{Tr}(\bigwedge^i A)$. Then the Chern classes are

$$c_i(E) = \Big[\mathrm{Tr}_i\Big(\frac{\sqrt{-1}}{2\pi} F_\nabla\Big)\Big] \in H^{2i}_{DR}(S)\,.$$

These classes are well defined (since the forms inside are closed, which again follows by working on leaves) and do not depend on the connection (different connections give forms differing by exact forms), see [13, Chapter III].

We have some facts:

(1) If M is a manifold, we recover the usual Chern classes.
(2) If $f : S_1 \to S_2$ is a solenoid map, then $f^* c_i(E) = c_i(f^* E)$. In particular,
  - If $f : S \to M$ is an immersion of a solenoid in a manifold and $E|_S = f^* E$, then $c_i(E|_S) = f^* c_i(E)$.
  - If $j : l \to S$ is the inclusion of a leaf, then $c_i(E|_l) = j^* c_i(E)$.

Question. Are the Chern classes defined as elements in $H^{2i}(S, \underline{\mathbb{Z}})$?
(for line bundles it is true).

4. Hodge Theory of solenoids

4.1. **Sobolev norms.** Let S_μ be a compact Riemannian k-solenoid which is oriented and endowed with a transversal measure. We denote the associated (finite) daval measure also by μ. Now consider a vector bundle $E \to S$ and endow it with a metric. The space of sections of class $C^{\infty,0}$ is denoted $\Gamma(S, E)$. The space of $L^2(\mu)$-transversally smooth sections (sections of class C^∞ along leaves and L^2 in the transversal directions) is denoted by $\Gamma_{L^2(\mu)}(S, E)$.

Now let us introduce suitable completions of these spaces of sections. Fix a connection ∇ for E and the Levi-Civita connection for TS. There is an L^2-norm on sections of E, given by

$$(s,t)_E = \int_S \langle s, t \rangle \, d\mu \, .$$

We can complete the spaces of sections to obtain spaces of L^2-sections $L^2(S, E)$. We consider also Sobolev norms $W^{l,2}$ as follows. Take s a section of E. Then we set

$$||s||^2_{W^{l,2}_\mu} = \int_S \sum_{i=0}^{l} |\nabla^i s|^2 \, d\mu \, .$$

Completing with respect to this norm gives a Hilbert space consisting of sections with regularity $W^{l,2}$ on leaves and $L^2(\mu)$-transversally, denoted $W^{l,2}_\mu(S, E)$. These spaces do not depend on the choice of metrics and connections.

For future use, we also introduce the norms C^r_μ, which give spaces of sections with C^r-regularity on leaves and $L^2(\mu)$-transversally. Take s a section of E. Assume it has support in a flow box $U = D^k \times K(U)$, and assume that E has been trivialized by an orthonormal frame. Then

$$||s||^2_{C^r_\mu} = \int_{K(U)} ||s(\cdot, y)||^2_{C^r} \, d\mu_{K(U)}(y) \, .$$

These norms are patched (via partitions of unity, in a non-canonical way) to get a norm on the spaces of sections on the whole solenoid. The topology defined by this norm is independent of the partition of unity. The spaces of sections are denoted $C^r_\mu(S, E)$. Note that $\bigcap_{r \geq 0} C^r_\mu(S, E) = L^2(\mu)(S, E)$.

We can define the norm $W^{l,2}_\mu$ by using Fourier transforms. For this we have to restrict to a flow-box $U = D^k \times K(U)$. We Fourier-transform the section $s(x, y)$ in the leaf-wise directions, to get $\hat{s}(\xi, y)$, and then take the integral

$$\int_{K(U)} \left(\int (1 + |\xi|^2)^l |\hat{s}(\xi, y)|^2 d\xi \right) d\mu_{K(U)}(y).$$

Proposition 4.1 (Sobolev). *$W^{s,2}_\mu(S, E) \subset C^p_\mu(S, E)$, for $s > [k/2] + p + 1$.*

This is similar to Proposition 1.1 in Chapter IV of [13]. The proof carries over to the solenoid situation verbatim. As a consequence,

$$\bigcap_{r\geq 0} W^{r,2}_\mu(S,E)=\Gamma_{L^2(\mu)}(S,E)\,.$$

4.2. **Pseudodifferential operators.** Let E,F be two vector bundles over S of ranks n,m respectively. A differential operator L of order l is an operator

$$L:\Gamma(S,E)\to\Gamma(S,F)$$

which locally on a flow-box $U=D^k\times K(U)$ is of the form

$$L(s)=\sum_{|\alpha|\leq l} A_\alpha(x,y)D^\alpha s\,,$$

where A_α are $(n\times m)$-matrices of functions (with regularity $C^{\infty,0}$) and $\alpha=(\alpha_1,\ldots,\alpha_k)$ is a multi-index, with $|\alpha|=\sum\alpha_i$, $D^\alpha=\frac{\mathbf{d}^{|\alpha|}}{\mathbf{d}^{\alpha_1}x_1\ldots\mathbf{d}^{\alpha_k}x_k}$. Note that a differential operator gives rise to differential operators on each leaf. Moreover, L extends to

$$L:W^{p,2}_\mu(S,E)\to W^{p-l,2}_\mu(S,F)\,.$$

The usual properties, like the existence of adjoints, extend to this setting.

The symbol of a differential operator on a solenoid is defined in the same fashion as for the case of manifolds, and coincides with the symbol of the differential operator on the leaves. We recall that the symbol $\sigma_l(L)\in\mathrm{Hom}(\pi^*E,\pi^*F)$, $\pi:TS\to S$, has the form

$$\sigma_l(L)(x,y,v)=\sum_{|\alpha|=l}A_\alpha(x,y)v_1^{\alpha_1}\ldots v_k^{\alpha_k}\,.$$

The properties of the symbol map, such as the rule of the symbol of the composition of differential operators, or the symbol of the adjoint, hold here. This is just the fact that they can be done leaf-wise, and the continuous transversality is easy to check.

Differential operators can be generalized to pseudodifferential operators as in the case of manifolds. A pseudodifferential operator of order l on a flow-box $U=D^k\times K(U)$ is an operator

$$L(p):\Gamma_c(U,E)\to\Gamma(U,F)$$

which sends a (compactly supported) section $s(x,y)$ to

$$L(p)s(x,y)=\int p(x,\xi,y)\hat{s}(\xi,y)e^{i\langle x,\xi\rangle}d\xi\,,$$

where $\hat{s}(\xi,y)$ is the (leaf-wise) Fourier transform, and $p(x,\xi,y)$ is a function defined in $D^k\times\mathbb{R}^k\times K(U)$, smooth on x and ξ, continuous on y, and satisfying:

- $|D^\beta_x D^\alpha_\xi p(x,\xi,y)|\leq C_{\alpha\beta\,l}(1+|\xi|)^{l-|\alpha|}$, for constants $C_{\alpha\beta\,l}$,
- the limit $\sigma_l(p)(x,\xi,y)=\lim_{\lambda\to\infty}\frac{p(x,\lambda\xi,y)}{\lambda^l}$ exists,
- $p(x,\xi,y)-\sigma_l(p)(x,\xi,y)$ should be of order $\leq l-1$ for $|\xi|\geq 1$.

A pseudodifferential operator of order l on S is an operator $L:\Gamma(S,E)\to\Gamma(S,F)$ which is locally of the form $L(p_U)$ for some p_U as above. The symbol of L is $\sigma_l(L)=\sigma_l(p_U)$ for a local representative $L|_U=L(p_U)$. This symbol is well-defined and independent of choices, which is a delicate point but it is analogous to the case of manifolds (see [13]). The usual properties of the symbol map (composition, adjoint) hold here.

A pseudodifferential operator of order l is an operator of order l, i.e., it extends as a continuous map to

$$L:W^{p,2}_\mu(S,E)\to W^{p-l,2}_\mu(S,E)\,.$$

This is done as in Theorem 3.4 of [13, Ch. IV], by noting that $||L(p)s(\cdot,y)||_{W^{p-l,2}}\leq C||s(\cdot,y)||_{W^{p,2}}$, where C is a constant depending on $C_{\alpha\beta\,l}$.

The key of the theory is the fact that we can construct a pseudodifferential operator given a symbol $\sigma_l(L)$.

Proposition 4.2. *Let S be a compact solenoid. Then there is an exact sequence $0 \to \mathrm{OP}_{l-1}(E,F) \to \mathrm{PDiff}_l(E,F) \to \mathrm{Symb}_l(E,F) \to 0$, where $\mathrm{OP}_{l-1}(E,F)$ is the space of operators of order $l-1$, $\mathrm{PDiff}_l(E,F)$ the space of pseudodifferential operators of order l, and $\mathrm{Symb}_l(E,F)$ the space of symbols of order l.*

4.3. **Elliptic operator theory for solenoids.** We say that a pseudodifferential operator $L : E \to F$ of order l is elliptic if the symbol $\sigma_l(L)$ satisfies that $\sigma_l(L)(x,v) : E_x \to F_x$ is an isomorphism for each $x \in S$, $v \in T_xS$, $v \neq 0$.

Theorem 4.3. *Let L be an elliptic pseudodifferential operator of order l. Then there exists a pseudo-inverse, a pseudodifferential operator $\tilde{L}$ of order $-l$ such that $L \circ \tilde{L} = Id + K_1$ and $\tilde{L} \circ L = Id + K_2$, where K_1, K_2 are operators of order -1.*

This is done as in Theorem 4.4 [13, Ch. IV]. The basic idea is to construct a pseudo-inverse by using Proposition 4.2. Note that K_1, K_2 are not ussually compact operators (this is due to the failure of the Rellich lemma in our situation), so we will not have finite-dimensionality of the kernel and cokernel of elliptic operators.

Corollary 4.4. *Let L be an elliptic pseudodifferential operator of order l, and let $\mathcal{K}_{L_s} = \ker(L : W^{s,2}_\mu(S,E) \to W^{s-l,2}_\mu(S,F))$. Then $\mathcal{K}_{L_s} \subset \Gamma_{L^2(\mu)}(S,E)$, and it is independent of s.*

An operator $L : \Gamma(E) \to \Gamma(E)$ is called self-adjoint if $L^* = L$. If L is an elliptic self-adjoint operator, then there is a pseudo-inverse G which is self-adjoint (just take the pseudo-inverse $\tilde{L}$ provided by Theorem 4.3 and let $G = (\tilde{L} + \tilde{L}^*)/2$). Then we have that $L \circ G = G \circ L$, because

$$\langle (L \circ G - G \circ L)s, s \rangle = \langle Gs, Ls \rangle - \langle Ls, Gs \rangle = 0\,.$$

In particular, $K_1 = K_2$ in Theorem 4.3.

For self-adjoint operators, we have the following result

Theorem 4.5. *Let L be an elliptic self-adjoint operator of order l. Then*

$$W^{s,2}_\mu(S,E) = \ker L \oplus \overline{\operatorname{im} L}\,.$$

and an analogous result for $\Gamma_{L^2(\mu)}(S,E)$.

A complex of differential operators is a sequence

$$\Gamma(E_0) \xrightarrow{L_0} \Gamma(E_1) \xrightarrow{L_1} \dots \xrightarrow{L_{m-1}} \Gamma(E_m)\,,$$

where E_i are vector bundles, and L_i are differential operators such that $L_i \circ L_{i-1} = 0$. The complex is called elliptic if the sequence of symbols

$$\pi^* E_0 \xrightarrow{\sigma(L_0)} \pi^* E_1 \xrightarrow{\sigma(L_1)} \dots \xrightarrow{\sigma(L_{m-1})} \pi^* E_m\,,$$

is exact for each $v \neq 0$. We define the cohomology of the complex as

$$H^q(S,E) = \frac{\ker(L_q : \Gamma(E_q) \to \Gamma(E_{q+1}))}{\operatorname{im}(L_{q-1} : \Gamma(E_{q-1}) \to \Gamma(E_q))}\,,$$

and the L^2-cohomology by

$$H^q(S_\mu,E) = \frac{\ker(L_q : \Gamma_{L^2(\mu)}(E_q) \to \Gamma_{L^2(\mu)}(E_{q+1}))}{\operatorname{im}(L_{q-1} : \Gamma_{L^2(\mu)}(E_{q-1}) \to \Gamma_{L^2(\mu)}(E_q))}\,.$$

The *reduced* L^2-cohomology is

$$\bar{H}^q(S_\mu,E) = \frac{\ker L_q}{\overline{\operatorname{im} L_{q-1}}}\,.$$

This is the group $H^q(S_\mu, E)$ quotiented by the closure of $\{0\}$, making it a Hausdorff space.

We construct the Laplacian operators of the elliptic complex as follows:

$$\Delta_j = L_j^* L_j + L_{j-1} L_{j-1}^* : \Gamma_{L^2(\mu)}(E_j) \to \Gamma_{L^2(\mu)}(E_j)\,.$$

These are self-adjoint elliptic operators. There is an associated operator G given by Theorem 4.5. Denote

$$\mathcal{H}^j(E) = \ker \Delta_j\,.$$

And note that $\Delta_j s = 0$ if and only if $L_j s = 0$ and $L_j^* s = 0$. We remove the subindex j from now on.

Theorem 4.6. *We have the following:*

1. $\overline{\operatorname{im}\Delta} = \overline{\operatorname{im} L} \oplus \overline{\operatorname{im} L^*}$*, and it is an orthogonal decomposition.*
2. $\Gamma_{L^2(\mu)}(S, E_j) = \mathcal{H}^j(E) \oplus \overline{\operatorname{im} L} \oplus \overline{\operatorname{im} L^*}$.
3. *There is a canonical isomorphism* $\mathcal{H}^j(E) \cong \bar{H}^j(S_\mu, E)$.

4.4. **Harmonic theory.** The Riemannian metric and the orientation give rise to a natural volume form along leaves $\mathrm{vol} \in \Omega^k(S)$. The usual Hodge-$*$ operator (see [13]) can be defined for forms on S, actually, it is the $*$ operator on leaves. This operator $* : \Omega^p(S) \to \Omega^{k-p}(S)$ is defined by

$$\alpha \wedge *\beta = (\alpha, \beta)\,\mathrm{vol}\,,$$

for $\alpha, \beta \in \Omega^p(S)$, where $(\cdot,\cdot)$ is the point-wise metric induced on forms. Note that $*$ extends to $* : \Omega^p_{L^2_\mu}(S) \to \Omega^{k-p}_{L^2_\mu}(S)$, since it is leaf-wise isometric. Note that $\mathrm{vol} = *1$.

Lemma 4.7. $d^* = \pm * d *$. □

The Laplacian is defined as $\Delta = dd^* + d^*d$. Note that if $\Delta s = 0$ then $(s, \Delta s) = (s, dd^*s) + (s, d^*ds) = (d^*s, d^*s) + (ds, ds) = ||d^*s||^2 + ||ds||^2$. So $d^*s = 0$ and $ds = 0$. We define the space of harmonic forms:

$$\mathcal{K}^j(S_\mu) = \mathcal{H}_\Delta(\wedge^j T^*S)\,.$$

Then the theory of elliptic operators says the following

Theorem 4.8. *We have*

- *The space of harmonic sections* $\mathcal{K}^j(S_\mu) \subset \Omega^j_{L^2(\mu)}(S)$.
- *There is a natural isomorphism* $\bar{H}^j_{DR}(S_\mu) \cong \mathcal{K}^j(S_\mu)$.

Corollary 4.9. *Poincaré duality:*

$$* : \mathcal{K}^p(S_\mu) \to \mathcal{K}^{k-p}(S_\mu)$$

is an isomorphism.

If S is ergodic, then $H^0(S_\mu) \cong H^k(S_\mu) \cong \mathbb{R}$ (with the isomorphism given by integration $\int_{S_\mu}$. Therefore

$$\int_{S_\mu} : \bar{H}^p_{DR}(S_\mu) \otimes \bar{H}^{k-p}_{DR}(S_\mu) \to \mathbb{R}$$

is a perfect pairing.

In general, the spaces $\mathcal{K}^p(S_\mu)$ are not finite dimensional. For instance, take a solenoid which is a fibration, i.e., S is a compact $(n+k)$-manifold such that there is a submersion $\pi : S \to B$ onto an n-dimensional manifold, and the transversal measure is induced by a measure μ on B. Then we have a fiber bundle $\mathbb{H}^p \to B$ such that $\mathbb{H}^p_y = H^j(F_y)$, $F_y = \pi^{-1}(y)$. Then $\mathcal{K}^p(S_\mu) \cong L^2(\mu)(\mathbb{H}^p)$.

Nonetheless, we propose the following conjecture, as is natural from the standpoint of Ergodic Solemoidal Geometry:

Conjecture 4.10. If S_μ is a uniquely ergodic solenoid then the spaces $\mathcal{K}^p(S_\mu)$ are of finite dimension.

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