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Abstract

Let $k \in \mathbb{N}$ and let $E$ be a Banach space such that every $k$-homogeneous polynomial defined on a subspace of $E$ has an extension to $E$. We prove that every norm one $k$-homogeneous polynomial, defined on a subspace, has an extension with a uniformly bounded norm. The analogous result for holomorphic functions of bounded type is obtained. We also prove that given an arbitrary subspace $F \subset E$, there exists a continuous morphism $\phi_{k,F} : \mathcal{P}^k(F) \to \mathcal{P}^k(E)$ satisfying $\phi_{k,F}(P)_{|F} = P$, if and only if $E$ is isomorphic to a Hilbert space.

Introduction and preliminaries

Consider the non-weakly sequentially continuous polynomial $P$ on $\ell_2$, given by $P((a_i)) = \sum_{i=1}^{\infty} a_i^2$. Since every polynomial defined on a Banach space having the Dunford–Pettis property is weakly sequentially continuous (see [23, corollary 3]), $P$ is not the restriction of a continuous polynomial defined on $l_\infty$, no matter how $\ell_2$ is linearly embedded in $l_\infty$.

This shows that we can not expect, in general, that a given continuous scalar polynomial $P \in \mathcal{P}(^nF)$ of degree $n > 1$, defined on a closed subspace $F$ of a Banach space $E$, has an extension $\hat{P} \in \mathcal{P}(^nE)$ defined on $E$.

There are, however, specific cases where it is known that the continuous polynomials defined on subspaces have extensions to the entire space: for an arbitrary Banach space $E$, every polynomial of degree one defined on a closed subspace extends to a degree one polynomial defined on $E$ (this is, of course, the Hahn–Banach Theorem). On the other hand, if a Banach space $E$ is isomorphic to a Hilbert space, its closed subspaces $F \subset E$ are complemented. Hence, by composition with a projection, one sees that any polynomial on $F$ can be extended to a polynomial on $E$. The well-known result of B. Maurey, [21, corollaire 3], provides another family of examples: If a Banach space has type 2, then any scalar polynomial of degree 2 defined on a subspace has an extension (this can be found also in
The lack of other examples in the literature leads us to the study of the extension of polynomials through the following problems:

Problem 1: Does there exist a Banach space $E$, other than the type 2 spaces, such that every quadratic polynomial $P \in \mathcal{P}(2F)$, defined on a closed subspace $F$, is the restriction of a quadratic polynomial $\hat{P} \in \mathcal{P}(2E)$?

Problem 2: Does there exist a Banach space $E$, other than those isomorphic to a Hilbert space, such that for some $n > 2$, every polynomial $P \in \mathcal{P}(nF)$, defined on a closed subspace $F$, is the restriction of a polynomial $\hat{P} \in \mathcal{P}(nE)$?

We introduce in Definition 1.1 the $n$-Extension Property as the property stemming from these two problems: we say that a Banach space $E$ has the $n$-Extension Property if every $n$-homogeneous polynomial $P \in \mathcal{P}(nF)$ defined on an arbitrary closed subspace $F \subset E$ is the restriction of a polynomial $\hat{P} \in \mathcal{P}(nE)$.

Undoubtedly, among the most studied extension problems in the Geometry of Banach spaces are the problems concerning the extension of linear operators. In that context, we find the problems related to the existence of non-complemented subspaces (as in [4, 5] and [25]); those related to the so called extension property (see [24, 4.9.2]); those related to the compact extension property (see [24, 6.1.4]) or, more recently, those related to the extension properties $M_p$ introduced in [7]. A common important feature of these problems is that they are local, that is, that they have a finite dimensional reformulation.

The main contributions of this paper are the development of appropriate local techniques in the context of polynomial maps and a proof of the fact that that the extendibility of polynomials is a local property. Since we focus our study on Problems 1 and 2, we will deal here only with scalar valued polynomials, which provide the first non trivial case of extension of polynomial maps. Of course, it is also possible to consider the vector valued case.

Observe that the $n$-Extension Property just introduced does not ask for any a priori estimate of the norm of the extension $\hat{P}$. We prove in Theorem 1.3, that once there exist extensions for all polynomials of degree $n$ defined on subspaces, then there exist extensions with an upper estimate of the norm, controlled by an absolute constant. With the Hahn–Banach Theorem in mind, it is natural to ask if this absolute constant is equal to one. A finite dimensional example suffices to show that even a Banach space isomorphic to a Hilbert space may fail to have the constant equal to one. Such a question would give rise to an isometric property which is out of the scope of Problems 1 and 2 and, hence, will not be treated here.

We will study, however, two more extension properties that will help us to better understand the context where Problems 1 and 2 take place. On the one hand, we will introduce the $\mathcal{H}_b$-Extension Property, which involves the holomorphic functions of bounded type (Definition 1.5). This property implies the $n$-Extension Property for all $n$ (see Proposition 1.6). On the other hand, we will prove that if the $n$-Extension Property is fulfilled through an extension morphism (that is, if there is a linear and continuous selection map), then the space is necessarily isomorphic to a Hilbert space. The case $n = 1$ of this result (already proved by H. Fakhoury in [13]) is Theorem 2.1 and Theorem 2.3 covers the general case $n \in \mathbb{N}$.

In the following diagram we summarize the relations among the extension properties just introduced that we prove in this paper. Later on, we will see that each of these properties has an associated extension constant. In the diagram we allude to these constants with the
When $m > 2$, proving every missing left arrow in the diagram is the same as giving a negative answer to Problem 2. A counter example of some of these missing implications will solve Problem 2 in the affirmative. We will prove that all the notions appearing in the diagram are local, in the sense that they are fully determined by the respective behavior of the extensions from the finite dimensional subspaces. Theorem 1·3 covers the case of the $n$-Extension Property; Theorem 1·7 covers the case of the extension of holomorphic functions of bounded type ($\mathcal{H}_b$-Extension Property) and Theorems 2·1 and 2·3 cover the case of the extension through morphisms. Each of these theorems requires a specific proof. We have structured them with the use of two lemmas, explaining in this way that all three proofs share a common pattern (surely shared also by other extension properties).

It is worth noticing that Theorem 1·3 delimits the context where Problem 1 takes place to the setting of the local theory of Banach spaces. Indeed, as we will prove in a forthcoming paper, a Banach space with the $n$-Extension Property for some $n > 1$ necessarily has type $p$ for all $p < 2$.

We must mention that the problem of whether a continuous polynomial does admit an extension has been treated from a wide variety of approaches. The former discussion by S. Dineen in [11] and the well-known paper [1], by R. Aron and P.D. Berner, appeared in the seventies. Since then, quite a number of related results concerning specific polynomials or specific spaces, have been published (see, for instance, [2, 6, 15, 17 or 18]). An account of results and references on the matter can be found in [12] and [26].

We fix the notation that will be used in the following: given a Banach space $E$ and $n \in \mathbb{N}$, a map $P: E \to \mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$) is an $n$-homogeneous polynomial if it is the restriction to the subset $\{x \times \cdots \times x; x \in E\}$ of $E \times \cdots \times E$, of a continuous $n$-linear map, which is unique if it is required to be symmetric. We shall denote by $\mathcal{P}^{(n)}(E)$ the Banach space of $n$-homogeneous polynomials on $E$ with the norm $\|P\| = \sup\{\|P(x)\|: x \in E, \|x\| = 1\}$. Recall that if $\hat{\otimes}^n_E$ denotes the symmetric projective tensor product of $E$, then $\mathcal{P}^{(n)}(E)$ is isometrically isomorphic to the dual space $(\hat{\otimes}^n_E)^*$, where the duality is given by $\langle P, \hat{\otimes}^n_E x \rangle = P(x)$. By subspace we will always understand closed subspace. We refer the reader to [12] for the general background.

1. **Extendibility of polynomials is a local property**

The main results in this section are Theorems 1·3 and 1·7, where we prove that the property of extension for homogeneous polynomials of degree $n$, as well as the property of extension for holomorphic functions of bounded type, are both local properties. The proofs of these results make use of the fact that the $n$-Extension Property (a property just involving $n$-homogeneous polynomials) and the analogous extension property for polynomials of degree less or equal to $n$, are equivalent. This is proved in Proposition 1·2. This Proposition gives also an estimate between the norms of the extensions in both cases that will be a
key factor in the proof of the case of holomorphic functions of bounded type (that is, in Theorem 1.7).

The Hahn–Banach Theorem does not only assert the existence of a continuous linear extension of any functional defined on a subspace, but also the existence of an extension with the same norm. Before proving the main results in this Section, let us show that requiring the extensions to preserve the norm gives rise to a property that two isomorphic spaces may not share.

In [1] it is proved, with an example due to M. Schottenloher, that there exists a subspace \( F \) of \( \ell_\infty^2 \) and a norm one polynomial \( P \in \mathcal{P}(\ell_\infty^2) \), such that every extension \( \tilde{P} \) to \( \ell_\infty^2(\mathbb{C}) \) has \( \| \tilde{P} \| > 1 \). Of course, \( \ell_\infty^2(\mathbb{C}) \) is isomorphic to the Hilbert space \( \ell_2^2 \), where, on the contrary, the composition with the orthogonal projection onto each subspace gives the existence of extensions preserving the norm. In [12] an extension property for polynomials (for a fixed pair \( F \subset E \)) is introduced, where the extension is required to preserve the norm. We will not deal here with this “isometric” point of view. The extension property that we will study is the following:

**Definition 1.1.** Let \( E \) be a Banach space and \( n \in \mathbb{N} \). We say that \( E \) has the \( n \)-Extension Property if for every polynomial \( P \in \mathcal{P}(nE) \) defined on a closed subspace \( F \subset E \) there exists a polynomial \( \tilde{P} \in \mathcal{P}(nE) \) such that \( \tilde{P}_{|F} = P \).

Clearly, every closed subspace of a Banach space with the \( n \)-Extension Property has the \( n \)-Extension Property. Now, we have:

**Proposition 1.2.** Let \( n \in \mathbb{N} \) be fixed and let \( F \) be a subspace of a Banach space \( E \) such that every \( P \in \mathcal{P}(nE) \) has an extension \( \tilde{P} \in \mathcal{P}(nE) \). Then, for all \( k = 1, 2, \ldots, n - 1 \), every \( Q \in \mathcal{P}(n-k\cdot E) \) has an extension \( \tilde{Q} \in \mathcal{P}(n-k\cdot E) \). Moreover, if there is a constant \( M > 0 \) such that every \( P \in \mathcal{P}(nE) \) has an extension with \( \| \tilde{P} \| \leq M \| P \| \), then for \( k = 1, 2, \ldots, n - 1 \), every \( Q \in \mathcal{P}(n-k\cdot E) \) admits an extension \( \tilde{Q} \in \mathcal{P}(n-k\cdot E) \) such that \( \| \tilde{Q} \| \leq M(n^n/n!) \| Q \| \). In particular, a Banach space has the \( m \)-Extension Property for some \( m \in \mathbb{N} \) if and only if it has the \( n \)-Extension Property for every \( n \leq m \).

**Proof.** For a fixed \( k \), consider \( Q \in \mathcal{P}(n-k\cdot E) \). Let \( F = [x_0] \oplus G \) be a decomposition of \( F \) satisfying that the projection onto \( [x_0] \) has norm one and that \( \| x_0 \| = 1 \). Consider the linear map

\[
T_k : \mathcal{P}(n-k\cdot E) \rightarrow \mathcal{P}(n-k\cdot E)
\]

\[
(\lambda x_0 + g) \otimes (\lambda x_0 + g) \mapsto Q
\]

where \( \lambda x_0 + g \in [x_0] \oplus G = F \). We can construct a polynomial \( P \in \mathcal{P}(nE) \) using \( T_k \) as follows: \( P(\lambda x_0 + g) := T_k((\lambda x_0 + g) \otimes (\lambda x_0 + g)) = \lambda^k Q(\lambda x_0 + g) \). If the space \( E \) has the \( n \)-Extension Property, there exists a polynomial \( \tilde{P} \in \mathcal{P}(nE) \) such that \( \tilde{P}_{|F} = P \). This polynomial determines a linear map \( T_k^{\tilde{P}} \in \mathcal{L}(\mathcal{P}(n-k\cdot E)) \) by the relation \( T_k^{\tilde{P}}((x \otimes \cdots \otimes x))(x) := \tilde{P}(x) \) for every \( x \in E \). Let us define now \( \tilde{Q}(x) := T_k^{\tilde{P}}((x \otimes \cdots \otimes x_0))(x) \) for all \( x \in E \). Observe that \( \tilde{Q} \in \mathcal{P}(n-k\cdot E) \) is an extension of \( Q \), so the first assertion in Proposition 1.2 is already proved. Now, if we assume also the existence of \( M > 0 \), and an extension \( \tilde{P} \) such that \( \| \tilde{P} \| \leq M \| P \| \), we have the following estimate:

\[
\| \tilde{Q} \| = \| T_k^{\tilde{P}}(x_0 \otimes \cdots \otimes x_0)(\cdot) \| \leq \| T_k^{\tilde{P}} \| \frac{n^n}{n!} \| \tilde{P} \| \leq \frac{n^n}{n!} M \| P \| \leq \frac{n^n}{n!} M \| T_k \| \leq \frac{n^n}{n!} M \| Q \|
\]
where the estimate $\|T_k^{\tilde{P}}\| \leq (n^n/n!)\|\tilde{P}\|$ can be directly derived from the well-known relation between the norm of a polynomial and that of its associated symmetric multilinear map (see [12, proposition 1-8]). This finishes the proof.

A similar argument was used in [3] to prove the case $k = 1$. Obviously, it is possible to derive the general case $n - k$ from the case $k = 1$ by an induction argument; however, such an induction argument does not provide the bound on the norms given in Proposition 1-2. This bound will be crucial in the proof of Theorem 1-7.

In the introduction of this section we explain why we have not considered any a priori estimate on the norm of the extensions in the definition of the $n$-Extension Property. The next theorem states that the existence of the extensions suffices for having extensions with a norm controlled by some constant which depends on $n$ and on the space. The theorem also exhibits the local nature of the $n$-Extension Property:

**Theorem 1-3.** Let $E$ be a Banach space and $n \in \mathbb{N}$. The following are equivalent:

(i) $E$ has the $n$-Extension Property;

(ii) there is a constant $M > 0$ such that whenever $F \subset E$ is a finite dimensional subspace, every $P \in \mathcal{P}(n, F)$ has an extension $\tilde{P} \in \mathcal{P}(n, E)$ with $\|\tilde{P}\| \leq M\|P\|$;

(iii) there is a constant $M > 0$ such that whenever $F \subset E$ is a subspace, every polynomial $P \in \mathcal{P}(n, F)$ has an extension $\tilde{P} \in \mathcal{P}(n, E)$ with $\|\tilde{P}\| \leq M\|P\|$.

**Proof.** Implications (iii)$\Rightarrow$(ii) and (iii)$\Rightarrow$(i) are obvious. Let us prove that (ii) implies (iii): Assume $E$ satisfies (ii) with constant $M$. Let $F \subset E$ be any infinite dimensional subspace and let $P \in \mathcal{P}(n, F)$. For each finite dimensional subspace $G \subset F$, consider a polynomial $\tilde{P}_G \in \mathcal{P}(n, G)$ which extends $P|_G \in \mathcal{P}(n, G)$ with $\|\tilde{P}_G\| \leq M\|P|_G\| \leq M\|P\|$. The family:

$$\mathcal{F} = \{\tilde{P}_G; \ G \subset F, G \text{ of finite dimension}\} \subset \mathcal{P}(n, E)$$

determines a bounded net in the dual space $\mathcal{P}(n, E) = (\hat{\mathcal{S}}, n)^*$, thus there exists a $w^*$-convergent subnet $\mathcal{F}' = \{\tilde{P}_G\}$. Let $P \in \mathcal{P}(n, E)$ be the $w^*$-limit of $\mathcal{F}'$. Clearly $P$ extends $P$ and

$$\|\tilde{P}\| = \sup_{x \in B_g} |\tilde{P}(x)| = \sup_{x \in B_g} \lim_{G \in \mathcal{F}'} |\tilde{P}_G(x)| \leq \sup_{x \in B_g} \sup_{G \in \mathcal{F}'} \lim_{G \in \mathcal{F}} \left(\|\tilde{P}_G\| \cdot \|x\|^n\right) \leq M\|P\|.$$ 

This finishes the proof of (ii) implies (iii). Note that the arguments used in the proof of the equivalence (ii)$\Leftrightarrow$(iii) show that one can choose the same constant $M$ in both cases. We will refer to this constant as an extension constant associated to the $n$-Extension Property.

**Remark 1-4.** Proposition 1-2 implies that if $E$ is a Banach space satisfying (ii) in Theorem 1-3 for degree $n$ with constant $M$, then it also verifies the condition (ii) for homogeneous polynomials of degree $n - k$, with $M' \leq (n^n/n!)M \leq e^nM$ as an extension constant.

To complete the proof of Theorem 1-3 let us verify that (i) implies (ii). We will prove this implication with the use of two lemmas.

**Lemma A1.** Let $X$ be a Banach space, and $Y$ a subspace of $X$ satisfying (ii) in Theorem 1-3 with constant $M$. If $G \subset X$ is a finite dimensional subspace, and $G \cap Y = \{0\}$, then $G + Y$ also satisfies (ii) in Theorem 1-3 with constant $M' \leq \rho^nM$, and $\rho$ depending only on the dimension of $G$.

**Proof of Lemma A1.** Let $k = \text{dim} \ G$. Since the subspace $Y$ has codimension $k$ in $G + Y$, there is a projection map $\Pi$, that we fix for the moment, defined on $G + Y$ onto $Y$ with
any polynomial \( P \) defined as follows: a normalized basis of \( \text{Ker} \ P \). In order to prove that \( G + Y \) also satisfies (ii) in Theorem 1.3, we need to construct an extension defined on the entire \( G + Y \), of any polynomial \( P \in \mathcal{P}(n) \) defined on a finite dimensional subspace \( F \subset (\text{Ker} \ P) \oplus Y = G + Y \). We will do it by dividing the proof into three different cases, depending on the position of \( F \) inside \( (\text{Ker} \ P) \oplus Y \): 

Case \( F \subset Y \). Since \( Y \) satisfies (ii) with constant \( M \), there exists an extension \( \tilde{P} \in \mathcal{P}(n) \), with \( \|\tilde{P}\| \leq M\|P\| \). The composition \( \tilde{P} \circ \Pi \in \mathcal{P}(n) \) defines an extension of \( P \) with \( \|\tilde{P} \circ \Pi\| \leq \|\tilde{P}\| \cdot \|\Pi\|\|P\| \cdot \|\Pi\|^n \leq M_1 \|P\| \) for a constant \( M_1 \leq M\|P\|^n \) independent of \( F \) and \( P \).

Case \( F = (\text{Ker} \ P) \oplus H \), for some \( H \subset Y \) of finite dimension. Let us choose \( \{z_1, \ldots, z_k\} \) a normalized basis of \( \text{Ker} \ P \) for which \( \|g\|_{\infty} \leq \|g\| \) for all \( g \in \text{Ker} \ P \). Consider the symmetric multilinear map generated by \( P \), \( \tilde{P} \in \mathcal{L}(F \times \cdots \times F; \mathbb{K}) \). For \( g = \sum_{i=1}^k a_i z_i \in (\text{Ker} \ P) \), \( h \in H \) and \( 0 \leq m < n \), we can write 

\[
P(g + h) = P(g) + \sum_{m=0}^{n-1} \binom{n}{m} \tilde{P}(g, \ldots, g, h, n-m, h)
\]

Every set of indices \( \{j_1, \ldots, j_m\} \) appearing above determines a polynomial \( P_{j_1,\ldots,j_m} \in \mathcal{P}(n) \) by the relation \( P_{j_1,\ldots,j_m}(h) := \tilde{P}(z_{j_1}, \ldots, z_{j_m}, h, n-m, h) \). Since \( Y \) verifies (ii) with constant \( M \) for degree \( n \), by Remark 1.4 we know that there exists a polynomial \( Q_{j_1,\ldots,j_m} \in \mathcal{P}(n) \) extending \( P_{j_1,\ldots,j_m} \) such that \( \|Q_{j_1,\ldots,j_m}\| \leq M(n^n/n!) \|P_{j_1,\ldots,j_m}\| \leq M(n^n/n!) \|\tilde{P}\| \leq M(n^n/n!^2) \|P\| \). Consider now the polynomial \( \tilde{P} \in \mathcal{P}(n) \) defined as follows:

\[
\tilde{P}(g + y) = P(g) + \sum_{m=0}^{n-1} \binom{n}{m} \left( \sum_{j_1=1}^n a_{j_1} \cdots a_{j_m} Q_{j_1,\ldots,j_m}(y) \right).
\]

It is easy to check that \( \tilde{P} \) is an \( n \)-homogeneous polynomial extending \( P \). Let us estimate now its norm: if \( g + y \in (\text{Ker} \ P) \oplus Y \) is any norm one vector and \( g = \sum_{i=1}^k a_i z_i \) is the expression of \( g \) in the fixed basis, we have that for every \( i = 1, \ldots, k \), \( |a_i| \leq \|g\| \leq \|I - \Pi\| \). We will use the estimates: \( \|Q_{j_1,\ldots,j_m}(y)\| \leq M(n^n/n!)^2 \|P\| \cdot \|\Pi(g + y)\|^{n-m} \leq M(n^n/n!)^2 \|P\| \cdot \|\Pi\|^n \) and \( |a_{j_1} \cdots a_{j_m}| \leq \|I - \Pi\|^m \leq \|I - \Pi\|^n \). With them, we have

\[
\|\tilde{P}(g + y)\| \leq \|P\| \cdot \|\Pi\|^n + \sum_{m=0}^{n-1} \binom{n}{m} \left| a_{j_1} \cdots a_{j_m} \right| \cdot \|Q_{j_1,\ldots,j_m}(y)\|
\]

\[
\leq \|P\| \cdot \|\Pi\|^n + M \left( \frac{n^n}{n!} \right)^2 \|P\| \cdot \|\Pi\|^n \|I - \Pi\|^m \sum_{m=0}^{n-1} \binom{n}{m} k^m
\]

\[
\leq \|P\| \cdot \|I - \Pi\|^n (1 + M \left( \frac{n^n}{n!} \right)^2 \|\Pi\|^n (1 + k)^n).
\]
Bounding some of the terms of the form \((1 + a), a \geq 1\), by \(2a\) and using that \(\|\Pi\| \leq 2\sqrt{k}\), we obtain that \(\|\tilde{P}\| \leq M\|P\| (4e k)^2 n\).

Thus, we have that \(\tilde{P}\) determines an extension of \(P\) to \(G + Y = (\text{Ker } \Pi) \oplus Y\) with \(\|\tilde{P}\| \leq M_2\|P\|\), where \(M_2 \leq M\eta^n\) is a constant independent of \(P\) and \(F\).

The general case \(F \subset (\text{Ker } \Pi) \oplus Y\). Recall that \(\Pi\) is a fixed projection corresponding to the decomposition \((\text{Ker } \Pi) \oplus Y\). Thus, \(F \subset (1 - \Pi)(F) \oplus \Pi(F) \subset (\text{Ker } \Pi) \oplus \Pi(F)\). We will prove that there exists a constant \(M_3\) (independent of \(P\) and \(F\)) and an extension \(\tilde{P} \in \mathcal{P}(n(\text{Ker } \Pi) \oplus \Pi(F))\) such that \(\|\tilde{P}\| \leq M_3\|P\|\). This is enough, since we can then use the preceding case to find an extension, \(\tilde{P} \in \mathcal{P}(n(\text{Ker } \Pi) \oplus Y)\), of \(P\) (thus, of \(P\)) with \(\|\tilde{P}\| \leq M_2\|P\|\).

Observe that \(F\) has codimension at most \(k = \text{dim}(\text{Ker } \Pi)\) in \((\text{Ker } \Pi) \oplus \Pi(F)\). Then, we can find a projection \(T \in \mathcal{L}((\text{Ker } \Pi) \oplus \Pi(F), (\text{Ker } \Pi) \oplus \Pi(F))\) onto \(F\) with \(\|T\| \leq 2\sqrt{k}\) (again, see [16, theorem 8]). Let \(\tilde{P} = P \circ T \in \mathcal{P}(n(\text{Ker } \Pi) \oplus \Pi(F))\). \(\tilde{P}\) extends \(P\) and \(\|\tilde{P}\| \leq M_3\|P\|\), where \(M_3 = (2\sqrt{k})^n\), which does not depend on \(F\) or \(P\).

This finishes the proof of Lemma A1. If we track the extension constants appearing above, we see that \(M\eta^n\) is valid as an extension constant for the space \(G + Y\), where \(\rho := 2^7 e^2 (\text{dim } G)^{5/2}\) is independent of the degree \(n\).

**Lemma A2.** If a Banach space \(E\) fails (ii), then there exist a pair of sequences of subspaces \((F_k)_k\), \((Y_k)_k\) and a sequence of polynomials \((P_k)_k\) with the following conditions: \(Y_0 = E\) and

1. \(F_k\) is finite dimensional and \(F_k \subset Y_{k-1}\);
2. \(P_k \in \mathcal{P}(n F_k), \|P_k\| = 1\) and any extension \(\tilde{P}_k \in \mathcal{P}(n E)\) of \(P_k\) has \(\|\tilde{P}_k\| \geq k^3\);
3. \(Y_k \subset Y_{k-1}\) and \(Y_k\) is of finite codimension in \(Y_{k-1}\) (thus in \(E\));
4. \((F_1 \oplus \cdots \oplus F_k) \cap Y_k = \{0\}\) and there exists a projection \(\pi_k : F_1 \oplus \cdots \oplus F_k \oplus Y_k \to F_1 \oplus \cdots \oplus F_k\) with \(\|\pi_k\| \leq 2\).

**Proof of Lemma A2.** A Banach space \(X\) fails condition (ii) in Theorem 1.3 if and only if it satisfies:

\[
\text{for every } \Lambda \geq 1 \text{ there exist a finite dimensional subspace } F \subset X \text{ and a polynomial } P \in \mathcal{P}(n F), \|P\| \leq 1 \text{ such that every extension } \tilde{P} \in \mathcal{P}(n X) \text{ of } P \text{ satisfies } \|\tilde{P}\| \geq \Lambda.
\]

Thus, if \(E\) fails (ii), then there exist a finite dimensional subspace \(F_1 \subset E\) and \(P_1 \in \mathcal{P}(n F_1)\) satisfying (\(\ast\)) with \(\Lambda = 1\). Now, given any finite dimensional subspace, say \(F_1\), there exists \(Y_1\) satisfying (3) and (4) (this fact can be found in [10, remark 1]). Assume now that we have already constructed \((F_j)_{j=1}^{k-1}, (Y_j)_{j=1}^{k-1}, (P_j)_{j=1}^{k-1}\) satisfying (1) to (4). Let us construct \(F_k, Y_k\) and \(P_k\). By (3), the subspace \(Y_{k-1}\) has finite codimension in \(E\). This, joined with Lemma A1, implies that \(Y_{k-1}\) does not satisfy (ii). Consequently, (\(\ast\)) holds with \(X = Y_{k-1}\). Observe that, in principle, (\(\ast\)) would give the existence of some \(P\) and a lower bound of the norm of its extensions to \(Y_{k-1}\), but being \(Y_{k-1}\) a subspace of \(E\), the same bound works for the extensions of \(P\) to \(E\). Thus, we can assert that if we take \(\Lambda = (k)^3\), there exist \(F_k \subset Y_{k-1}\) and \(P_k \in \mathcal{P}(n F_k)\) verifying (1) and (2). Using [10, remark 1] again, we find a subspace \(Z_k\) of finite codimension in \(E\), such that \((F_1 \oplus \cdots \oplus F_k) \cap Z_k = 0\), and a projection \(Q_k : F_1 \oplus \cdots \oplus F_k \oplus Z_k \to F_1 \oplus \cdots \oplus F_k\) with \(\|Q_k\| \leq 2\). The subspace \(Y_k := Z_k \cap Y_{k-1}\) and the projection \(\pi_k := Q_k_{|F_1 \oplus \cdots \oplus F_k \oplus Z_k}\) satisfy the conditions (3) and (4).
Now we can finish the proof of Theorem 1.3. It only remains to prove the implication (i)⇒(ii). Assume that $E$ does not verify (ii). Let us consider two sequences of subspaces $(F_k)_k$, $(Y_k)_k$ and a sequence of polynomials $(P_k)_k$ as in Lemma A2. Consider the closed subspace $F = \bigcup_k (F_1 \oplus \cdots \oplus F_k) \subset E$. Since $\pi_{iF} : F \to F_1 \oplus \cdots \oplus F_k$ are uniformly bounded projection maps, $F$ has the following Schauder decomposition $F = \sum_{k=1}^{\infty} F_k$ (see [20, Chapter 1, g]). We now construct a polynomial $P$ defined on $F$ as follows:

$$P(x) = \sum_{k=1}^{\infty} \frac{P_k(x_k)}{k^2}, \quad \text{where} \quad x = \sum_{k=1}^{\infty} x_k \in F.$$ 

$P \in \mathcal{P}(\mathbb{F})$ is well defined by the Uniform Boundedness Principle for polynomials (see [22]). Let us check that $P$ has no extension to $E$: Assume, on the contrary, that there exists an extension $\tilde{P} \in \mathcal{P}(\mathbb{E})$ of $P$. It is easy to check that for every $k \in \mathbb{N}$, the polynomial $k^2 \tilde{P} \in \mathcal{P}(\mathbb{E})$ is an extension of $P_k \in \mathcal{P}(\mathbb{F}_k)$. We have constructed $P_k$ satisfying condition (2) in Lemma A2, so, it must satisfy $k^2 \| \tilde{P} \| = \| k^2 \tilde{P} \| \geq k^3$. However, this estimate is clearly false for all $k > \| \tilde{P} \|$. This contradiction proves that $P$ has no extension to $E$; hence, that $E$ does not verify (i) in Theorem 1.3.

We finish this section with the study of an extension property which involves holomorphic functions of bounded type. For a complex Banach space $E$, recall that an entire function $f : E \to \mathbb{C}$ is said to be of bounded type if $f$ is bounded on $B$, for any bounded set $B \subset E$. The space of entire functions of bounded type on $E$ is denoted by $\mathcal{H}_b(E)$. It is well-known that the Taylor series of $f \in \mathcal{H}_b(E)$ at 0, say $\sum_{n=0}^{\infty} P_n$, is uniformly convergent on all bounded sets. Furthermore, an entire function $\tilde{f} = \sum_{n=0}^{\infty} P_n$ belongs to $\mathcal{H}_b(E)$ if and only if $\limsup_{n \to \infty} \| P_n \|^{1/n} = 0$ (see [12, page 164]). In this context, we introduce the following extension property:

**Definition 1.5.** Let $E$ be a Banach space. We say that $E$ has $\mathcal{H}_b$-Extension Property, if for every $f \in \mathcal{H}_b(F)$ defined on a subspace $F \subset E$ there exists a function $\tilde{f} \in \mathcal{H}_b(E)$ such that $\tilde{f}|_F = f$.

**Proposition 1.6.** If $E$ is a complex Banach space with the $\mathcal{H}_b$-Extension Property, then $E$ has the $n$-Extension Property for every $n \in \mathbb{N}$.

**Proof.** Let $F \subset E$ be a subspace and consider $P \in \mathcal{P}(\mathbb{F})$. Since $P$ is also a holomorphic function of bounded type, the $\mathcal{H}_b$-Extension Property implies that there exists some $\tilde{f} = \sum_{n=0}^{\infty} Q_n \in \mathcal{H}_b(E)$ such that $\tilde{f}|_F = P$. The uniqueness of the Taylor series expansion implies that the $n$-homogeneous polynomial $Q_n \in \mathcal{P}(\mathbb{F})$ verifies $Q_n|_F = P$ and, consequently, that $P$ has an $n$-homogeneous polynomial as an extension.

We do not know if the converse statement is true; namely, if a Banach space which has the $n$-Extension Property for all $n \in \mathbb{N}$ has the $\mathcal{H}_b$-Extension Property. However, if we require the extension constant associated to each $n$-Extension Property to grow like $K^n$ for some $K > 0$, then the converse is true. This result is part of the following theorem:

**Theorem 1.7.** Let $E$ be a Banach space. The following are equivalent:

(i) $E$ has the $\mathcal{H}_b$-Extension Property;

(ii) there is a constant $K > 0$ such that for every $n \in \mathbb{N}$, whenever $F \subset E$ is a finite dimensional subspace, every $P \in \mathcal{P}(\mathbb{F})$ admits an extension $\tilde{P} \in \mathcal{P}(\mathbb{E})$ with $\| \tilde{P} \| \leq K^n \| P \|$;

$$\| \tilde{P} \| \leq K^n \| P \|;$$
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(iii) there is a constant \( K > 0 \) such that for every \( n \in \mathbb{N} \), whenever \( F \subset E \) is a subspace, every polynomial \( P \in \mathcal{P}(nF) \) admits an extension \( \tilde{P} \in \mathcal{P}(nE) \) with \( \| \tilde{P} \| \leq K^n \| P \| \).

**Proof.** The implication (iii)⇒(ii) is obvious. To prove (iii)⇒(i), let us assume that \( E \) satisfies (iii) and let us consider a function \( f \in \mathcal{H}_b(F) \) where \( F \) is a closed subspace of \( E \). By the previously mentioned characterization of holomorphic functions of bounded type, we know that \( f = \sum_{n=0}^{\infty} P_n \) with \( P_n \in \mathcal{P}(nF) \) for every \( n \) and \( \limsup_{n \to \infty} \| P_n \|^{1/n} = 0 \).

Let \( K > 0 \) and, for every \( n \), let \( P_n \) be an extension of \( P_n \), as in (iii). We can use them to construct the function \( \tilde{f} := \sum_{n=0}^{\infty} P_n \) which extends \( f \). Since \( \limsup_{n \to \infty} \| P_n \|^{1/n} \leq \limsup_{n \to \infty} K^n \| P_n \|^{1/n} = 0 \), we have that \( \tilde{f} \) is well defined and it is a holomorphic function of bounded type. This proves that \( E \) satisfies (i).

To prove the implication (ii)⇒(iii) we use, for every fixed \( n \in \mathbb{N} \), the implication (ii)⇒(iii) in Theorem 1.3. Recall that we have pointed out, at the end of the proof of (ii)⇒(iii) in Theorem 1.3, that the constants appearing there can be taken to be equal; so, assuming (ii), the constant \( K^n \) works in (iii) for each \( n \).

Once we have checked the implication (i)⇒(ii), the proof of Theorem 1.7 will be done. We now prove (i)⇒(ii) again using two lemmas:

**Lemma B1.** Let \( X \) be a Banach space, and \( Y \) a subspace of \( X \) satisfying (ii) in Theorem 1.7 with constant \( K > 0 \). If \( G \subset X \) is a finite dimensional subspace, and \( G \cap Y = \{0\} \), then \( G + Y \) satisfies (ii) in Theorem 1.7 with constant \( K_1 = \rho K \), where \( \rho \) is a constant only depending on the dimension of \( G \).

**Proof of Lemma B1.** For each \( n \in \mathbb{N} \), we can use Lemma A1 taking the constant \( M \) exactly equal to \( K^n \). We obtain, in this way, that \( G + Y \) satisfies (ii) in Theorem 1.3 for every \( n \). It is important now to use the fact that for each \( n \), (ii) in Theorem 1.3 is fulfilled with a constant \( \rho^n M \), where \( \rho \) depends on the dimension on \( G \) but it is independent on the degree \( n \). So, if we choose \( K_1 = \rho K \), we have that for every \( n \), \( G + Y \) satisfies (ii) in Theorem 1.3 with constant \( \rho^n K^n = (K_1)^n \) and consequently, it satisfies (ii) in Theorem 1.7 with constant \( K_1 \).

**Lemma B2.** If the Banach space \( E \) fails (ii) in Theorem 1.7, then there exist a pair of sequences of subspaces \((F_k)_k \), \((Y_k)_k \), a sequence of natural numbers \((n_k)_k \), and a sequence of polynomials \((P_k)_k \in \mathcal{P}(n_k F_k) \) with the following conditions: \( Y_0 = E \) and

1. \( F_k \) is finite dimensional and \( F_k \subset Y_{k-1} \);
2. \( P_k \in \mathcal{P}(n_k F_k) \), \( \| P_k \| = 1 \) and any extension of \( P_k \), \( \tilde{P}_k \in \mathcal{P}(n_k E) \) has \( \| \tilde{P}_k \| \geq (2k^2)^{n_k} \);
3. \( Y_k \subset Y_{k-1} \) and \( Y_k \) is of finite codimension in \( Y_{k-1} \) (thus in \( E \));
4. \( (F_1 \oplus \cdots \oplus F_k) \cap Y_k = \{0\} \) and there exists a projection \( \pi_k : F_1 \oplus \cdots \oplus F_k \oplus Y_k \to F_1 \oplus \cdots \oplus F_k \) with \( \| \pi_k \| \leq 2 \).

**Proof of Lemma B2.** The construction of the sequences can be done following word for word the proof of Lemma A2, taking into account that in this case, the condition equivalent to the failure of (ii) is:

\[
\text{for every } \Lambda \geq 1 \text{ there exist } n \in \mathbb{N}, \text{ a finite dimensional subspace } F \subset X \text{ and a polynomial } P \in \mathcal{P}(nF), \| P \| \leq 1, \text{ such that every extension } \tilde{P} \in \mathcal{P}(nX) \text{ of } P \text{ satisfies } \| \tilde{P} \| \geq \Lambda^n. \tag{**} \]

For each \( k \in \mathbb{N} \) we choose, inductively, the elements \( n_k, F_k, P_k \in \mathcal{P}(n_k F_k) \) and \( Y_k \) provided by (**) with \( \Lambda = 2k^2 \), analogously as we did in Lemma A2.
Now we can finish the proof of the remaining equivalence in Theorem 1.7, (i) ⇒ (ii). Let us assume that $E$ does not verify (ii), that is, that $E$ satisfies $(**)$ above and let us consider the sequences $(n_k)_k$, $(F_k)_k$, $(P_k)_k$ and $(Y_k)_k$ as in Lemma B2. We will construct a holomorphic function of bounded type defined on a closed subspace that does not have an extension to $E$, proving in this way that (i) fails. We do the construction in two cases, depending on the sequence of degrees $(n_k)_k$.

If the sequence $(n_k)_k$ is bounded, there exists a subsequence $(n_k)_j$ which is constant. In this case, let $d = n_k$ for all $j$, $d > 1$. We have a sequence of polynomials $P_k \in \mathcal{P}(d F_k)$ of the same degree, such that for every extension $\tilde{P}_k$, $\|\tilde{P}_k\| \geq (2k_j^2)^d$. We have similar circumstances to those in the proof of Theorem 1.3. Therefore, we can construct a closed subspace $F = \sum_{j=1}^{\infty} F_{k_j} \subset E$ and a polynomial $P \in \mathcal{P}(d F)$ defined on it as follows:

$$P(x) = \sum_{j=1}^{\infty} \frac{P_k(x_{k_j})}{k_j^2}, \quad \text{where} \quad x = \sum_{j=1}^{\infty} x_{k_j} \in F.$$ 

Being a polynomial, clearly $P \in \mathcal{H}_b(F)$. As we argue in Proposition 1.6, if $P$ is the restriction of a holomorphic function of bounded type defined on $E$, then necessarily this function is an $n$-homogeneous polynomial of the same degree $\tilde{P} \in \mathcal{P}(d E)$. But if such an extension exists, for every $j \in \mathbb{N}$ the polynomial $k_j^2 \tilde{P} \in \mathcal{P}(d E)$ is an extension of $P_{k_j} \in \mathcal{P}(d F_{k_j})$. Condition (2) in Lemma B2 implies that $\|k_j^2 \tilde{P}\| \geq (2k_j^2)^d$. This inequality is false for infinite many $j$, since $d \geq 2$.

In the case where the sequence $(n_k)_k$ is unbounded, we choose an increasing subsequence $(n_k)_j$ and we take the associated sequences $(F_k)_j$, $(P_k)_j$ and $(Y_k)_j$ provided by Lemma B2. We define the Banach space $F = \sum_{j=1}^{\infty} F_{k_j} \subset E$ as before, and the entire function of bounded type $f$ defined on $F$ as follows:

$$f(x) = \sum_{j=1}^{\infty} \frac{P_k(x_{j})}{(k_j)^{2n_j}}, \quad \text{for all} \quad x = \sum_{j=1}^{\infty} x_{j} \in F.$$ 

If we assume that $f$ has an extension $\tilde{f} \in \mathcal{H}_b(E)$, then $\tilde{f} = \sum_{n=0}^{\infty} Q_n$ for some $Q_n \in \mathcal{P}(\mathbb{C}^n E)$, $n \in \mathbb{N}$. Given any $x_j \in F_j$ we would have $f(x_j) = \tilde{f}(x_j)$ and, therefore,

$$\sum_{n=0}^{\infty} Q_n(x_j) = \frac{P_k(x_j)}{(k_j)^{2n_j}}.$$ 

The uniqueness of the Taylor series representation gives that $Q_{n_{k_j}}$ extends $P_{k_j}/(k_j)^{2n_j}$. Thus, $(k_j)^{2n_j} Q_{n_{k_j}}$ extends $P_{k_j}$ from $F_j$ to $E$. Condition (2) in Lemma B2 gives the estimation $(2k_j^2)^{n_j} \leq \|(k_j)^{2n_j} Q_{n_{k_j}}\|$. This implies $2 \leq \limsup \|Q_{n_{k_j}}\|^{1/n_{k_j}} = 0$, which is a contradiction.

2. Extensions through morphisms

When $F$ is a complemented subspace of a Banach space $E$, that is, when there exists a continuous projection map $\Pi_F : E \to E$ onto $F$, we can define the operator

$$\Phi_F : \mathcal{P}(\mathbb{C}^n F) \longrightarrow \mathcal{P}(\mathbb{C}^n E) \quad P \mapsto P \circ \Pi_F,$$

which is a linear and continuous extension morphism (also called a selection map): For every $P$, $\Phi_F(P)$ is an extension of $P$ and $\Phi_F$ is linear with $\|\Phi_F\| \leq \|\Pi_F\|^n$. 

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If a Banach space $E$ is isomorphic to a Hilbert space, every subspace $F$ is complemented. Consequently, we can construct the extension morphism $\Phi_F$ for any subspace $F$ and, hence, the space has the $n$-Extension Property for every $n \in \mathbb{N}$. Even more, in this case, since there exists a constant $K > 0$ such that every subspace is complemented by a projection with norm bounded by $K$, we can see that every Banach space isomorphic to a Hilbert space satisfies the $n$-Extension Property with constant $K^n$ for each $n \in \mathbb{N}$. By Theorem 1-7, it is equivalent to have the $\mathcal{H}_b$-Extension Property with constant $K$.

This section is devoted to the study of the converse statement, namely, to determine the spaces where the $n$-Extension Property, for some $n$ (or the $\mathcal{H}_b$-Extension Property), is fulfilled through an extension morphism. We prove that these are, precisely, the spaces which are isomorphic to a Hilbert space (Theorem 2-3). It turns out that this extension property is independent of the degree of the polynomials considered. We will prove this fact in two steps. In the first step we prove the particularly relevant linear case (Theorem 2-1). This result was first established in [13, théorème 3-7]. Recently, another proof has appeared in [8]. We prove it here in a different way, with a proof that follows the same pattern as the proofs of Theorems 1-3 and 1-7. In the second step (Theorem 2-3) we broaden the result to the general case of degree $n$ and the case of holomorphic functions of bounded type.

**Theorem 2-1.** Let $E$ be a Banach space. The following conditions are equivalent:

(i) for any $F \subset E$ there exists a continuous extension morphism $\psi \in \mathcal{L}(F^*, E^*)$;

(ii) there is a constant $M > 0$ such that whenever $F \subset E$ is a finite dimensional subspace, there exists an extension morphism $\psi_f : F^* \to E^*$ with $\|\psi_f\| \leq M$;

(iii) there exists $M > 0$ such that given any pair of finite dimensional subspaces $F \subset G \subset E$, there exists a projection map $\Pi$ from $G$ onto $F$ with $\|\Pi\| \leq M$;

(iv) $E$ is isomorphic to a Hilbert space.

**Proof.** (ii)⇒(iii) Consider a pair of finite dimensional subspaces $F \subset G \subset E$ and let $\psi : F^* \to E^*$ be a continuous extension morphism of norm $\leq M$. The map $\psi|_G : G \to F^{**}$ is a projection onto $F$ of norm $\leq M$. This proves (iii).

(iii)⇒(iv) This follows from the beautiful local version of the complemented subspace problem stated in [14, theorem 6-7]. Indeed, consider a Banach space $E$ satisfying (iii) and let $G$ be a finite dimensional subspace $G \subset E$. We have that every subspace of $G$ is $M$-complemented in $G$, which says precisely that $G$ satisfies the hypotheses of [14, theorem 6-7]. Recall that this theorem ensures the existence of a constant $f(M) > 0$ such that $d(G, l_2^{dim G}) \leq f(M)$, where $d$ denotes the Banach-Mazur distance between Banach spaces. In this way it is proved that every finite dimensional subspace of $E$ is uniformly close to a finite dimensional subspace of $l_2$. In [19, proposition 3] it is proved that this is equivalent to the existence of an isomorphism between $E$ and a Hilbert space.

(iv)⇒(i) This implication was already stated at the beginning of the section, namely: If $T \in \mathcal{L}(E, H)$ is an isomorphism, for each subspace $F \subset E$ the map $((T|_{(F^*)^*})^{-1} \circ \Pi_{T(F)} \circ T)^* \in \mathcal{L}(F^*, E^*)$ is an extension morphism, where $\Pi_{T(F)}$ is the orthogonal projection onto $T(F)$ $((T|_{(F^*)^*})^{-1} \circ \Pi_{T(F)} \circ T$ is the projection map that we called $\Pi_F$ in the introduction of this Section).

(i)⇒(ii) We will follow the same scheme used in Theorems 1-3 and 1-7 to prove the corresponding implication.

**Lemma C1.** Let $X$ be a Banach space and $Y$ a subspace of $X$ satisfying (ii) in Theorem 2-1. If $G \subset X$ is a finite dimensional subspace and $G \cap Y = \{0\}$, then $G+Y$ also satisfies (ii) in Theorem 2-1.
Proof of Lemma C1. Consider a projection map \( \Pi \) defined on \( G + Y \) onto \( Y \) with norm bounded by \( 2\sqrt{k} \) where \( k = \text{dim } G \). Its existence was justified in Lemma A1. Observe that \( \Pi^* : Y^* \to ((\text{Ker } \Pi) \oplus Y)^* \) is a continuous extension morphism. Consider \( F \subset (\text{Ker } \Pi) \oplus Y \) a finite dimensional subspace. We will prove the existence of an extension morphism from \( F^* \) to \( ((\text{Ker } \Pi) \oplus Y)^* \) by cases, depending on the position of \( F \) inside \( (\text{Ker } \Pi) \oplus Y \).

Case \( F \subset Y \). Since \( Y \) satisfies (ii), there exists an extension morphism \( \psi : F^* \to Y^* \) with \( \| \psi \| \leq M \). The map \( \tilde{\psi} = \Pi^* \circ \psi \) defines an extension morphism from \( F^* \) to \( ((\text{Ker } \Pi) \oplus Y)^* \) with \( \| \tilde{\psi} \| \leq M \| \Pi \| \).

Case \( F = (\text{Ker } \Pi) \oplus H \), for some \( H \subset Y \) of finite dimension. In this case we may identify \( F^* \) with \( (\text{Ker } \Pi)^* \oplus H^* \). Since \( Y \) satisfies (ii), there exists an extension morphism \( \psi : H^* \to Y^* \) with \( \| \psi \| \leq M \). Defining \( \tilde{\psi}(g^* + h^*) = g^* + \psi(h^*) \) for every \( g^* + h^* \in F^* \), we get an extension morphism from \( F^* \) to \( (\text{Ker } \Pi)^* \oplus Y^* \simeq ((\text{Ker } \Pi) \oplus Y)^* \) of norm bounded by some \( KM \), where \( K \) is a constant depending only on \( \Pi \).

The general case \( F \subset (\text{Ker } \Pi) \oplus Y \). If \( T : (\text{Ker } \Pi) \oplus \Pi(F) \to F \) denotes, as in Lemma A1, a projection onto \( F \) of norm \( \leq 2\sqrt{k} \) and \( \Psi : ((\text{Ker } \Pi) \oplus \Pi(F))^* \to ((\text{Ker } \Pi) \oplus Y)^* \) is an extension morphism constructed following the previous case, then \( \tilde{\Psi} = \psi \circ T^* : F^* \to ((\text{Ker } \Pi) \oplus Y)^* \) is an extension morphism with \( \| \tilde{\Psi} \| \leq 2\sqrt{k}(M + 2)\| \Pi \| \), a constant which does not depend on \( F \).

Lemma C2. If a Banach space \( E \) does not satisfy (ii), we can construct two sequences of subspaces \( (F_k)_k \) and \( (Y_k)_k \), with the following conditions: \( Y_0 = E \) and

\[
\begin{align*}
(1) & \quad F_k \text{ is finite dimensional and } F_k \subset Y_{k-1}; \\
(2) & \quad \text{every extension morphism } \psi \in \mathcal{L}(F_k^*, E^*) \text{ has } \| \psi \| \geq k; \\
(3) & \quad Y_k \subset Y_{k-1} \text{ and } Y_k \text{ is of finite codimension in } Y_{k-1} \text{ (thus in } E); \\
(4) & \quad (F_1 \oplus \cdots \oplus F_k) \cap Y_k = \{0\} \text{ and there exists a projection } \pi_k : F_1 \oplus \cdots \oplus F_k \oplus Y_k \to F_1 \oplus \cdots \oplus F_k \text{ with } \| \pi_k \| \leq 2.
\end{align*}
\]

Proof of Lemma C2. This proof can be done following the proof of Lemma A2. In this case, a Banach space which does not verify (ii) must verify:

\[
\text{for every } \Lambda \geq 1 \text{ there exists a finite dimensional subspace } F \subset X \text{ such that every extension morphism } \psi \in \mathcal{L}(F^*, E^*) \text{ has } \| \psi \| \geq \Lambda. \}
\]

(***)

Considering \( \Lambda = k \) for each \( k \in \mathbb{N} \), we find inductively a finite dimensional subspace \( F_k \) of a finite codimensional subspace \( Y_{k-1} \), such that any linear extension morphism \( \psi : F_k^* \to Y_{k-1}^* \) has norm \( \| \psi \| \geq k \). Here, it is necessary to use the fact that \( Y_{k-1} \), having finite codimension in \( E \), neither verifies (***), this precisely Lemma C1.

Now, we have all the elements to justify that (i) \( \Rightarrow \) (ii). If (ii) fails, let us check that there is no continuous linear extension morphism from the space \( F^* \) to the space \( E^* \), where \( F = \sum_{j=1}^{\infty} F_j \subset E \), contradicting (i). If \( \psi \) is an extension morphism and \( \Pi_k := \pi_k|_{F} : F \to F = \sum_{j=1}^{k} F_j \) are the projection maps associated to that decomposition of \( F \), then \( \psi \circ (\Pi_k - \Pi_{k-1})^* \) is an extension morphism from \( F_k^* \) to \( E^* \) of norm bounded by \( 2\| \psi \| \sup_k \| \Pi_k \| < \infty \). This contradicts step (2) in the construction of the \( F_k \), and the proof is complete.

In [19, theorem 1], Lindenstrauss and Tzafriri characterized the Banach spaces isomorphic to a Hilbert space as the spaces where all the closed subspaces are complemented (solving the so called Complemented Subspace problem). We deduce the following strengthened version of this result from Theorem 2.1.
COROLLARY 2.2. Let $E$ be a Banach space. If for every subspace $F \subset E$, the bidual space $F^{**}$ is complemented in $E^{**}$, then $E$ is isomorphic to a Hilbert space.

Proof. It is well known that given $F \subset E$, there exists a continuous linear extension morphism from $F^*$ to $E^*$ if and only if $F^{**}$ is complemented in $E^{**}$ (see [26]). Applying Theorem 2.1 we finish the proof.

We are now in position to extend Theorem 2.1 to the case of homogeneous polynomials of arbitrary degree and of holomorphic maps of bounded type.

THEOREM 2.3. Let $E$ be a Banach space and $n \in \mathbb{N}$. Assume that for every subspace $F \subset E$ there exists an extension morphism $\phi \in \mathcal{L}(\mathcal{P}(n F), \mathcal{P}(n E))$ that is, a continuous linear operator such that $\phi(P)_{|F} = P$. Then $E$ is isomorphic to a Hilbert space.

Proof. Let $n \in \mathbb{N}$ be fixed and consider a Banach space $E$ as in Theorem 2.3. It is proved in [26, theorem 2.2] that for a fixed subspace $F \subset E$, there is an extension morphism from $\mathcal{P}(n F)$ to $\mathcal{P}(n E)$ if and only if there is an extension morphism from $F^*$ to $E^*$. Then there is no loss of generality in assuming that $n = 1$. This is condition (i) in Theorem 2.1. Thus $E$ has to be isomorphic to a Hilbert space.

Remark 2.4. Among other equivalences, Aron and Berner proved in [1, theorem 1.1] that given $F \subset E$, there exists a continuous linear extension mapping $T : \mathcal{H}_b(F) \to \mathcal{H}_b(E)$ if and only if there exists a continuous linear extension mapping $\psi : F^* \to E^*$. This result follows immediately from Theorem 2.3, using again the uniqueness of the Taylor series. They also state in [1, corollary 1.2] that if a Banach space $E$ is reflexive and verifies that there exists an extension morphism $\psi : F^* \to E^*$ for every $F$, then the space is isomorphic to a Hilbert space. Recall that Theorem 2.1 states this result without the assumption on the reflexivity of $E$.

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