Some problems

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Problem 1.
Do all uniformly non-octahedral (non-ℓ_1^3) Banach spaces have FPP?

Problem 1$'$
In the case this is not affirmative:
Is there any characterization of those uniformly non-octahedral spaces having FPP?

In the case this is affirmative:
Is the number 3 the smallest $n \in \mathbb{N}$ for which all uniformly non-ℓ_1^n spaces have FPP?

Problem 2 (cf. Kato-Saito-Tamura, Math. Anal. Appl. **7**, 2004)
Characterize the uniform non-squareness for the ψ-direct sum $(X_1 \oplus \cdots \oplus X_N)_\psi$ for N Banach spaces $X_1, \ldots, X_N$.

Recall that $X \oplus_\psi Y$: *UNS* $\iff$ X, Y: *UNS and* $\psi \neq \psi_1, \psi_\infty$.

In the N Banach spaces case ($N \geq 3$) it is not enough to remove the functions ψ_1 and ψ_∞ corresponding to the ℓ_1- and ℓ_∞-norms. We need to find out a couple of classes of convex functions in Ψ_N which should be removed: They would yield "ℓ_1-like" and "ℓ_∞-like" norms, respectively.

Remark P. Dowling and S. Saejung (JMAA **369** (2010), 53-59) derived a result for the case $N = 3$ from their Z-direct sum approach.

Problem 2$'$ More generally,
Characterize the uniform non-ℓ_1^n-ness for the ψ-direct sum $(X_1 \oplus \cdots \oplus X_N)_\psi$ for N Banach spaces $X_1, \ldots, X_N$.

(For the case $N = 2$ see Kato-Saito-Tamura, J. Nonlinear Convex Anal. **11** (2010), 13-33.)

Supplement for Problem 2

Correspondence between AN_N and Ψ_N

(Saito-Kato-Takahashi, JMAA **252** (2000), 879-975)

Let AN_N denote the family of all absolute normalized norms on $\mathbb{C}^N$.
Let

$$\Delta_N = \{(s_1, s_2, \cdots, s_{N-1}) \in \mathbb{R}^{N-1} :\ s_1+s_2+\cdots+s_{N-1} \leq 1,\ s_j \geq 0\ (\forall j)\}.$$

For any $\|\cdot\| \in AN_N$ define the function ψ on Δ_N by

$$\psi(s) := \|(1 - \sum_{J=1}^{N-1} s_j,\ s_1, \cdots, s_{N-1})\| \quad \text{for } s = (s_1, \cdots, s_{N-1}) \in \Delta_N.$$

Then $\underline{\psi \text{ is convex continuous on } \Delta_N}$ and we have the following.

(A_0) $\psi(0, \cdots, 0) = \psi(1, 0, \cdots, 0) = \cdots = \psi(0, \cdots, 0, 1) = 1,$

$$(A_1)\ \psi(s_1, \cdots, s_{N-1}) \geq (s_1 + \cdots + s_{N-1})\psi(\frac{s_1}{s_1 + \cdots + s_{N-1}}, \cdots, \frac{s_{N-1}}{s_1 + \cdots + s_{N-1}}),$$

$$(A_2)\ \psi(s_1, \cdots, s_{N-1}) \geq (1 - s_1)\psi(0, \frac{s_2}{1 - s_1}, \cdots, \frac{s_{N-1}}{1 - s_1}),$$

$\cdots\cdots\cdots$

$$(A_N)\ \psi(s_1, \cdots, s_{N-1}) \geq (1 - s_{N-1})\psi(\frac{s_1}{1 - s_{N-1}}, \cdots, \frac{s_{N-2}}{1 - s_{N-1}}, 0).$$

Let Ψ_N be the family of all convex continuous functions ψ on Δ_N satisfying these conditions $(A_0), (A_1), \cdots, (A_N)$.

Conversely for any $\psi \in \Psi_N$ define

$$\|(z_1, \cdots, z_N)\|_\psi = \begin{cases} (|z_1| + \cdots + |z_N|)\psi\left(\frac{|z_2|}{|z_1|+\cdots+|z_N|}, \cdots, \frac{|z_N|}{|z_1|+\cdots+|z_N|}\right) \\ \qquad\qquad\qquad \text{if } (z_1, \cdots, z_N) \neq (0, \cdots, 0), \\ 0 \qquad\qquad\qquad \text{if } (z_1, \cdots, z_N) = (0, \cdots, 0). \end{cases}$$

Then $\| \cdot \|_\psi \in AN_N$ and $\| \cdot \|_\psi$ satisfies

$$\psi(s) := \|(1 - \sum_{j=1}^{n-1} s_j,\ s_1, \cdots, s_{N-1})\|_\psi \quad \text{for } s \in \Delta_N. \tag{1}$$

Thus AN_N and Ψ_N correspond in a one-to-one way under the equation (1).

ψ-direct sum $(X_1 \oplus \cdots \oplus X_N)_\psi$

Let $X_1, ..., X_N$ be Banach spaces and let $\psi \in \Psi_N$.

The ψ-direct sum $(X_1 \oplus \cdots \oplus X_N)_\psi$ of $X_1, \ldots, X_N$ is their direct sum equipped with the norm

$$\|(x_1, \cdots, x_N)\|_\psi = \|(\|x_1\|, \cdots, \|x_N\|)\|_\psi$$

(Kato-Saito-Tamura, J. Aust. Math. Soc. **75** (2003), 413-422).

Problem 3

Describe the modulus of convexity $\delta_X(\varepsilon)$ of $X = X_1 \oplus_\psi X_2$ by means of the convex function ψ:

$$\delta_X(\epsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \; x, y \in S_X, \; \|x - y\| = \epsilon\right\} \quad (0 \le \epsilon \le 2)$$