METRIC ENTROPY AND SMALL DEVIATIONS OF GAUSSIAN PROCESSES

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Abstract. It is well known that a Gaussian measure μ on a separable Banach space E is uniquely determined by its reproducing kernel Hilbert space H_μ, which is always compactly embedded into E. In many problems of probability theory and statistics the logarithmic small ball probabilities of μ,

$$\varphi_\mu(\varepsilon) := \log \mu\left(\{x \in E : \|x\| \leq \varepsilon\}\right) ,$$

play an important role, e.g. in the case when μ is the distribution of a Gaussian process $X = (X_t)_{t\in[0,1]}$ with paths almost surely in $E = L_2[0,1]$ or $C[0,1]$. Then the small deviation probabilities of the process X,

$$\mathbb{P}\left(\int_0^1 |X_t(\omega)|^2 \, dt \leq \varepsilon^2\right) \quad \text{resp.} \quad \mathbb{P}\left(\sup_{0\leq t\leq 1} |X_t(\omega)| \leq \varepsilon\right) ,$$

coincide with the small ball probabilities $\mu\left(\{x \in E : \|x\| \leq \varepsilon\}\right)$ of the measure μ.

Kuelbs and Li discovered a tight connection between the logarithmic small ball probabilities $\varphi_\mu(\varepsilon)$ and the ε-entropy of the unit ball K of the RKHS H_μ, considered as a subset of E, that means the quantities

$$\mathcal{H}_\varepsilon(K) := \log_2 \mathcal{N}_\varepsilon(K) ,$$

where $\mathcal{N}_\varepsilon(K)$ is the minimal number of ε-balls in E that are required to cover K.

In the talk we illustrate this connection and determine the exact asymptotic rate (as $\varepsilon \to 0$) of the logarithmic small deviation probabilities with respect to L_2-norm and sup-norm for a family of smooth Gaussian processes. The results are based on joint work with Frank Aurzada, Fuchang Gao, Wenbo Li and Qi-Man Shao.

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