Open problems involving approximation and entropy numbers

D. E. Edmunds

- **Approximation numbers**
- 1. Let Ω be a bounded open subset of $\mathbb{R}^n$, let $p \in (1, \infty)$

and denote by *id* the natural embedding of $\overset{0}{W}{}_p^1(\Omega)$ in $L_p(\Omega)$.Then

$$c_1 k^{-1/n} \leq a_k(id) \leq c_2 k^{-1/n} \ (k \in \mathbb{N})$$

- When $p = 2$,

$$a_k(id) = \lambda_k^{-1/2},$$

where λ_k is the k^{th} eigenvalue of the Dirichlet Laplacian.

- Asymptotic behaviour of λ_k is known:

$$\lim_{k\to\infty} k^{-2/n}\lambda_k = 4\pi^2 \left(\omega_n \left|\Omega\right|\right)^{-2/n},$$

where ω_n = volume of unit ball in $\mathbb{R}^n$. Hence when $p = 2$,

$$\lim_{k\to\infty} k^{1/n} a_k(id) = \frac{\left|\Omega\right|^{1/n}}{2\sqrt{\pi}\left(\Gamma\left(1+\frac{n}{2}\right)\right)^{1/n}}.$$

- Same holds for embedding of $W_2^1(\Omega)$ in $L_2(\Omega)$.

- When $n = 1$ and $1 < p < \infty$,

$$a_k(id) = \mu_k^{-1/p},$$

where μ_k is the k^{th} eigenvalue of the (Dirichlet) $p-$Laplacian.

- Suppose that $\Omega = (a, b)$. Then since

$$\mu_k = (p-1)\left\{\frac{k\pi_p}{b-a}\right\}^p, \ \pi_p = \frac{2\pi}{p\sin(\pi/p)},$$

it follows that

$$ka_k(id) = \frac{b-a}{\pi_p(p-1)^{1/p}} \ (k \in \mathbb{N})$$

- Question: does

$$\lim_{k\to\infty} k^{1/n} a_k(id) \text{ exist when } p \neq 2 \text{ and } n > 1?$$

  The same question arises for other embeddings of Sobolev type (see Evans-Harris) and for embeddings involving Besov and Lizorkin-Triebel spaces.
- Remainder terms?

- 2. Consider the Hardy operator $T : L_p(I) \to L_q(I)$, where $p, q \in (1, \infty)$, $I = (0, 1)$ and

$$Tf(x) = v(x) \int_0^x u(t) f(t) dt,$$

with

$$u \in L_{p'}(I),\ \ v \in L_q(I).$$

When $p = q$ it is known that

$$\lim_{k \to \infty} k a_k(T) = \gamma_p \int_0^1 |u(t) v(t)| \, dt,$$

where

$$\gamma_p = \frac{1}{2} \left(p'\right)^{1/p} p^{1/p'} \pi^{-1} \sin(\pi/p).$$

A corresponding formula is known when $p > q$.

If $p > q$, then

$$\lim_{k\to\infty} ka_k(T) = \gamma_{p,q}\left(\int_0^1 |u(t)v(t)|^{1/r}\,dt\right)^r,$$

where $r = 1/q + 1/p'$.

- Question: Is there an analogous result when $p < q$?

Entropy numbers

The same questions arise for the entropy numbers of the embeddings in 1).
Majority opinion: no such limits exist
Contrary views can be found: no-one seems to know even in the simple case $n = 1$, $p = 2$
For the Hardy operator $T : L_p(I) \to L_p(I)$ all that appears to be known is that

$$c_1 \|uv\|_{1,I} \leq \liminf_{k\to\infty} ke_k(T) \leq \limsup_{k\to\infty} ke_k(T) \leq c_2 \|uv\|_{1,I} .$$