# Problem Session

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**A. Hardy's inequality in $L^1(\mathbb{R}^n)$**

**Theorem** For all $f \in C_0^\infty(\mathbb{R}^n)$

$$\|f/|\cdot|\|_{L^{1,\infty}(\mathbb{R}^n)} \leq C(1,n)\|\nabla f\|_{L^1(\mathbb{R}^n)}, \tag{1}$$

where the optimal constant is $C(1,n) = \omega_n^{1-1/n}/n$, $\omega_n$ is the volume of the unit ball and

$$\|g\|_{L^{1,\infty}(\mathbb{R}^n)} := \sup_{t>0}\{t \operatorname{meas}\{\mathbf{x} : |g(\mathbf{x}| > t\}\}.$$

The result is sure to be well-known. It follows from the Sobolev inequality

$$\|f\|_{L^{n/(n-1)}} \leq C_n\|\nabla f\|_{L^1},$$

the fact that $|\cdot|^{-1} \in L^{n,\infty}$ and the "weak"-Holder inequality; the optimal constant $C_n = 1/n\omega_n^{1/n}$.

**B. Hardy's inequality in $L^1(\Omega)$**

Hardy's inequality in $L^p(\Omega)$, $1 < p < \infty$, is

$$\int_\Omega |\nabla f(\mathbf{x})|^p d\mathbf{x} \geq c(n,p,\Omega) \int_\Omega \frac{|f(\mathbf{x})|^p}{\delta(\mathbf{x})^p} d\mathbf{x}, \quad f \in C_0^\infty(\Omega),$$

where $\delta(\mathbf{x}) := \mathrm{dist}(\mathbf{x}, \mathbb{R}^{\mathrm{n}} \setminus \Omega)$ and if $\Omega$ is convex, the optimal constant is

$$c(n,p,\Omega) = \left(\frac{p-1}{p}\right)^p.$$

**Questions**

- When $p = 1$, is there an inequality of the form

$$\|f/|\cdot|\|_{L^{1,\infty}(\Omega)} \leq C(1,n,\Omega)\|\nabla f\|_{L^1(\Omega)}, \tag{2}$$

subject to some regularity condition on $\Omega$.

- If there is an inequality (2), what is the optimal constant $C(1,n)$ when $\Omega$ is convex. In this case, can the inequality be extended to one of the form

$$\int_\Omega |\nabla f(\mathbf{x})| d\mathbf{x} \geq C(1,n) \| \{1 + a(\delta, \partial\Omega)\} f / |\cdot| \|_{L^{1,\infty}(\Omega)} \tag{3}$$

where $a(\delta, \partial\Omega)$ depends on $\delta$ and geometric properties of the boundary $\partial\Omega$ of $\Omega$.

- Are there rearrangement-invariant spaces closer to $L^1$ than $L^{1,\infty}$ for which the inequalities (1), (2) and (3) hold.