Local function spaces: Problems

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Wavelets $\psi_F \in C^u(\mathbb{R})$, $\psi_M \in C^u(\mathbb{R})$, $u \in \mathbb{N}$,

$$\int_{\mathbb{R}} \psi_M(x) x^v \, \mathrm{d}x = 0, \qquad v \in \mathbb{N}_0, \quad v < u.$$

$$\Psi^j_{G,m}(x) = 2^{jn/2} \prod_{w=1}^{n} \psi_{G_w}(2^j x_w - m_w), \qquad G \in G^j, \quad m \in \mathbb{Z}^n.$$

Q_{jm} cube in $\mathbb{R}^n$, $2^{-j}m$ left corner, side-length 2^{-j+1}, $j \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$,

$$\operatorname{supp} \Psi^j_{G,m} \subset Q_{jm}.$$

Wavelet expansion:

$$f = \sum_{j=0}^{\infty} \sum_{G \in G^j} \sum_{m \in \mathbb{Z}^n} \lambda^{j,G}_m \, 2^{-jn/2} \Psi^j_{G,m},$$

with

$$\lambda^{j,G}_m = \lambda^{j,G}_m(f) = 2^{jn/2} \int_{\mathbb{R}^n} f(x) \Psi^j_{G,m}(x) \, \mathrm{d}x.$$

Wavelets as above,

$$\Psi^u = \{\Psi^j_{G,m} : \ j \in \mathbb{N}_0, \ G \in G^j, \ m \in \mathbb{Z}^n\}, \quad \text{based on } \psi_{F,u},$$

$$\mathrm{supp}\ \Psi^j_{G,m} \subset Q_{jm}.$$

$$\mathbb{P}_{JM} = \{j \in \mathbb{N}_0, \ G \in G^j, \ m \in \mathbb{Z}^n : \ Q_{jm} \subset Q_{JM}\}, \qquad J \in \mathbb{N}_0, \quad M \in \mathbb{Z}^n.$$

Sequence spaces $\mathcal{L}^r b^s_{p,q}(\mathbb{R}^n)$:

$$\lambda = \{\lambda^{j,G}_m \in \mathbb{C} : \ j \in \mathbb{N}_0, \ G \in G^j, \ m \in \mathbb{Z}^n\}$$

quasi-normed by

$$\|\lambda \,|\mathcal{L}^r b^s_{p,q}(\mathbb{R}^n)\| = \sup_{J \in \mathbb{N}_0, M \in \mathbb{Z}^n} 2^{J(\frac{n}{p}+r)} \Big(\sum_{j=J}^\infty 2^{j(s-\frac{n}{p})q} \Big(\sum_{m,G:(j,G,m)\in \mathbb{P}_{JM}} |\lambda^{j,G}_m|^p \Big)^{q/p} \Big)^{1/q}.$$

Similarly $\mathcal{L}^r f^s_{p,q}(\mathbb{R}^n)$.

Definition 4.1. $0 < p, q \leq \infty$, $s \in \mathbb{R}$, $-n/p \leq r < \infty$,

$$u > \max(s + r^+, \sigma_p - s).$$

Then $\mathcal{L}^r B^s_{p,q}(\mathbb{R}^n)$ collects all $f \in S'(\mathbb{R}^n)$ for which

$$\|f \,|\mathcal{L}^r B^s_{p,q}(\mathbb{R}^n)\|_{\Psi^u} = \sup_{J,M} 2^{J(\frac{n}{p}+r)} \Big\| \sum_{(j,G,m)\in\mathbb{P}_{JM}} \lambda^{j,G}_m(f)\, 2^{-jn/2} \Psi^j_{G,m} \,|B^s_{p,q}(\mathbb{R}^n)\Big\|$$

$$\sim \|\lambda(f)\,|\mathcal{L}^r b^s_{p,q}(\mathbb{R}^n)\| < \infty.$$

Recall $\sigma_p = n\big(\max(1/p, 1) - 1\big)$.
Similarly $\mathcal{L}^r F^s_{p,q}(\mathbb{R}^n)$, both together denoted as $\mathcal{L}^r A^s_{p,q}(\mathbb{R}^n)$.

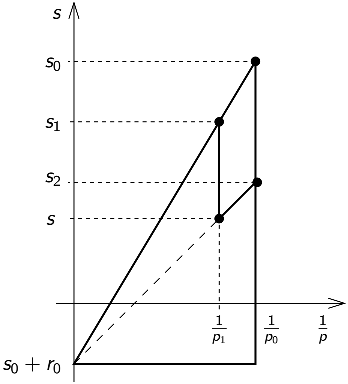

Figure: Limiting embeddings

Main problem: Necessary and sufficient conditions for

$$\mathcal{L}^{r_0} A^{s_0}_{p_0,q_0}(\mathbb{R}^n) \hookrightarrow \mathcal{L}^{r} A^{s}_{p,q}(\mathbb{R}^n)$$

Two decisive quantities:

$$\text{differential dimension}: \quad s + r,$$
$$\text{slope}: \quad |r|p.$$

Classical case: $r = -n/p$. Then differential dimension $s - \frac{n}{p}$, slope n. Now merging parameters: From (s, p, q) to (r, s, p, q) as in Newtonian Mechanics to Special Relativity.

Definition 4.2. $\Omega \subset \mathbb{R}^n$ domain, $0 < p, q \leq \infty$, $s \in \mathbb{R}$, $-n/p \leq r < 0$. Then

$$\mathcal{L}^r \widetilde{A}^s_{p,q}(\Omega) = \left\{ f \in \mathcal{L}^r A^s_{p,q}(\mathbb{R}^n) : \ \mathrm{supp}\ f \subset \overline{\Omega} \right\}.$$

Theorem 4.3. (i) Let $s_0 \in \mathbb{R}$, $0 < p_0 < \infty$, $-n/p_0 \leq r_0 < 0$. Let

$$p_0 \leq p < \infty, \qquad s + r = s_0 + r_0 < s \leq s_0 + r_0 \left(1 - \tfrac{p_0}{p}\right).$$

Then

$$\mathcal{L}^{r_0} \widetilde{B}^{s_0}_{p_0,\infty}(\Omega) \hookrightarrow \mathcal{L}^r \widetilde{B}^s_{p,\infty}(\Omega)$$

is continuous, but not compact.

(ii) Let Ω be a bounded domain and $s + r < s_0 + r_0$, otherwise as in (i), $0 < q_0, q \leq \infty$. Then

$$\mathcal{L}^{r_0} \widetilde{A}^{s_0}_{p_0,q_0}(\Omega) \hookrightarrow \mathcal{L}^r \widetilde{A}^s_{p,q}(\Omega)$$

is compact.

Dialectical method: Contradictory thesis and anti-thesis resolve at a higher level (Hegel, Marx).
Sobolev and Morrey as a dialectical couple:

Sobolev: Offer smoothness, ask for better integrability,
Morrey: Offer (refined) integrability, aks for better smoothness.

Morrey's refinement of the (uniform) Lebesgue spaces $\mathcal{L}_p(\mathbb{R}^n)$: $0 < p < \infty$, $-n/p \le r < 0$,

$$\|f \,|\mathcal{L}^r_p(\mathbb{R}^n)\| = \sup_{J \in \mathbb{N}_0, M \in \mathbb{Z}^n} 2^{J(r+\frac{n}{p})} \, \|f \,| L_p(Q_{JM})\|.$$

Recall that

$$\mathcal{L}^r_p(\mathbb{R}^n) = \mathcal{L}^r L_p(\mathbb{R}^n) = \mathcal{L}^r F^0_{p,2}(\mathbb{R}^n), \qquad \text{if } 1 < p < \infty.$$

Problem 4.4. Necessary and sufficient conditions for

$$\mathcal{L}^{r_0} A^{s_0}_{p_0,q_0}(\mathbb{R}^n) \hookrightarrow \mathcal{L}^r A^s_{p,q}(\mathbb{R}^n), \quad \mathcal{L}^{r_0} \widetilde{A}^{s_0}_{p_0,q_0}(\Omega) \hookrightarrow \mathcal{L}^r \widetilde{A}^s_{p,q}(\Omega)$$

So far, classical case, $r_0 = -n/p_0$, $r = -n/p$, Sickel-T (1995). Now Morrey's refinement of L_p-spaces should be included guided by:

$$s + r \le s_0 + r_0, \quad \text{decreasing differential dimensions},$$
$$|r|p \le |r_0|p_0, \quad \text{decreasing slopes}.$$

Problem 4.5. Ω bounded domain in $\mathbb{R}^n$. $s + r < s_0 + r_0$, otherwise in the above triangle. Entropy and approximation numbers for the compact embeddings

$$\mathcal{L}^{r_0}\widetilde{A}^{s_0}_{p_0,q_0}(\Omega) \hookrightarrow \mathcal{L}^{r}\widetilde{A}^{s}_{p,q}(\Omega).$$

Applications to spectral theory, elliptic PDE's, Schrödinger equations.