SOME SMALL PROBLEMS ON FUNCTION SPACES

AND LINEAR OPERATORS

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1. In my recent paper (MIA 2008) I proved a theorem on interpolation of compact operators in arbitrary Banach function spaces (lattices). It turned out that this theorem comprises a larger class of spaces which I called *pseudo-lattices*. Namely, a space of measurable functions $X(\Omega)$ is said to be a pseudo-lattice if $\|f\chi_D\|_X \leq C_X\|f\|_X$ for any measurable set $D \subset \Omega$ with a constant C_X independent of f and D.

The class of pseudo-lattices looks as a very natural generalization of usual lattices and seems perspective for getting other properties of lattices. Nevertheless, it is almost not investigated. For instance, I even do not know any example of pseudo-lattice which is non-equivalent to some standard lattice. Maybe, the following idea could be useful for obtaining such an example: to take a non-positive linear operator $T : L_p(0,1) \to L_q(0,1)$ with $p, q > 1$ and to define a space X as completion of L_p in the norm

$$\|f\|_X = \|f\|_{L_1} + \sup_D \|T(f\chi_D)\|_{L_q},$$

where the supremum is taken over all measurable subsets $D \subset \Omega$.

2. The second problem is about properties of functions $f(t) \in BMO$. As one of consequences from general results, described in my talk on this Workshop, we have that the Zygmund space *exp L* is the *smallest* r.i. invariant space containing BMO. The similar question may be posed about the *largest* r.i. space, contained in BMO. The conjecture is that the space BMO does not contain any r.i. subspace larger than L_∞. In other words, it should be shown that, for any unbounded function $f(t) \in BMO$, there exists an equimeasurable function $g(t)$ (that is, $g^*(t) = f^*(t)$) which does not belong to BMO.

3. The last problem concerns spectral properties of a linear operator T acting in a Hilbert space H. Suppose that $T^n x \to 0$ as $n \to \infty$ for any $x \in H$, but the spectral radius

$$r(T) := \lim_{n\to\infty} \sqrt[n]{\|T^n\|} = 1.$$

In this case there are elements x, for which the sequence $\|T^n x\|$ turns to 0 arbitrarily slow; moreover, the set of such "unpleasant" elements is rather large (residual).

It is convenient to describe the rate of convergence $T^n x \to 0$ via the *local spectral radius*

$$r_x(T) := \limsup_{n\to\infty} \sqrt[n]{\|T^n x\|}.$$

If $r_x(T) < 1$ then the convergence is as fast as a geometric progression, which is usually required in iterations. Otherwise the convergence is slow and not suitable for iterative approximation methods.

There are many assertions showing how large is the set of elements with $r_x(T) = 1$. At the same time there is no example of an operator T as above, for which $r_x(T) = 1$ for all $x \in H$. Hence we may conjecture that *an element with* $r_x(T) < 1$ *exists for any operator* T *with supposed properties.* Of course, this immediately implies infinitely many such x.

As a principal example of operators T defined above we can consider the product of two orthogonal projection operators $T = P_{S_1} P_{S_2}$, where the closed linear subspaces $S_1, S_2 \subset H$ are such that $S_1 \cap S_2 = \{0\}$ and the space $S_1 + S_2$ is not closed in H. The last property implies that the angle between subspaces S_1 and S_2 is zero and the spectral radius of $P_{S_1} P_{S_2}$ is 1. At the same time, the famous theorem of von Neumann states that $\lim_{n\to\infty} \|(P_{S_1} P_{S_2})^n x\| = 0$ for any $x \in H$.

This situation is the worst one in the so-called *alternating projection methods.* The confirmation of conjecture could give a rule for the choice of initial approximation point x_0 such that the convergence $(P_{S_1} P_{S_2})^n x_0 \to 0$ becomes sufficiently fast even when the problem in general is ill-posed.