On Interpolation of Variable Exponent Besov and Triebel-Lizorkin Spaces

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[1] based on joint work with P. Hästö

Variable integrability

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Function spaces with variable smoothness and integrability

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Interpolation

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Define:

$$\varphi_p(t) = \begin{cases} t^p & \text{if } p \in (0, \infty), \\ 0 & \text{if } p = \infty \text{ and } t \leqslant 1, \\ \infty & \text{if } p = \infty \text{ and } t > 1. \end{cases}$$

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$$\|f\|_{p(\cdot)} := \inf \left\{ \lambda > 0 \colon \varrho_{p(\cdot)}\left(\frac{1}{\lambda} f\right) \leqslant 1 \right\}$$

defines a quasi-norm (norm if $p \in \mathcal{P}$).

"Optimal" assumptions on the exponents

Local log-Hölder continuity: there exists $c_{\log} > 0$ such that

$$|g(x) - g(y)| \leqslant \frac{c_{\log}}{\log(e + 1/|x-y|)}, \quad x, y \in \mathbb{R}^n \quad \left[g \in C^{\log}_{\mathrm{loc}}\right].$$

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Decay (at infinity): there exists $g_\infty \in \mathbb{R}$ such that

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Class $\mathcal{P}_0^{\log}$**:** $p \in \mathcal{P}_0$ with $\frac{1}{p} \in C^{\log}_{\mathrm{loc}}$ and having a log-decay at infinity.

Class $\mathcal{P}^{\log}$**:** $p \in \mathcal{P}$ with $\frac{1}{p} \in C^{\log}_{\mathrm{loc}}$ and having a log-decay at infinity.

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Fourier analytical approach

Let $\varphi, \Phi \in \mathcal{S}$ (Schwartz class) satisfy

- $\operatorname{supp} \hat{\varphi} \subseteq \{\xi \in \mathbb{R}^n : \ \frac{1}{2} \leqslant |\xi| \leqslant 2\}$ and $|\hat{\varphi}(\xi)| \geqslant c > 0$ when $\frac{3}{5} \leqslant |\xi| \leqslant \frac{5}{3}$,
- $\operatorname{supp} \hat{\Phi} \subseteq \{\xi \in \mathbb{R}^n : |\xi| \leqslant 2\}$ and $|\hat{\Phi}(\xi)| \geqslant c > 0$ when $|\xi| \leqslant \frac{5}{3}$.

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Recall: Besov and Triebel-Lizorkin spaces...

$$\|f\|_{B^\alpha_{p,q}} := \Big\| \big\| 2^{\nu\alpha} \varphi_\nu * f \big\|_p \Big\|_{\ell^q} \quad \text{and} \quad \|f\|_{F^\alpha_{p,q}} := \Big\| \big\| 2^{\nu\alpha} \varphi_\nu * f \big\|_{\ell^q} \Big\|_p,$$

($\alpha \in \mathbb{R}$ and $0 < p, q \leqslant \infty$, with $p < \infty$ in the F-case).

Variable exponent Triebel-Lizorkin spaces [D-H-R'09]

$$F^{\alpha(\cdot)}_{p(\cdot),\,q(\cdot)} = \left\{ f \in \mathcal{S}' : \|f\|_{F^{\alpha(\cdot)}_{p(\cdot),\,q(\cdot)}} = \left\| \{2^{\nu\alpha(\cdot)}\,\varphi_\nu * f\} \right\|_{L^{p(\cdot)}(\ell^{q(\cdot)})} < \infty \right\}$$

$$\|f\|_{F^{\alpha(\cdot)}_{p(\cdot),\,q(\cdot)}} = \left\| \left(\sum_\nu |2^{\nu\alpha(x)}\,\varphi_\nu * f(x)|^{q(x)} \right)^{\frac{1}{q(x)}} \right\|_{L^{p(\cdot)}_x}.$$

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- $F^{\alpha(\cdot)}_{p(\cdot),\,q(\cdot)}$ well-defined if p, q, α are locally log-Hölder continuous and have log-decay at infinity ($0 < p^- \leqslant p^+ < \infty$, $0 < q^- \leqslant q^+ < \infty$).

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- $\|\cdot\|_{F^{\alpha(\cdot)}_{p(\cdot),\,q(\cdot)}}$ is a quasi-norm (norm if $\min\{p(x), q(x)\} \geqslant 1$).

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The **mixed Lebesgue-sequence space** $\ell^{q(\cdot)}(L^{p(\cdot)})$ is defined on sequences of $L^{p(\cdot)}$-functions by the (semi)modular

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\big((f_\nu)_\nu\big) := \sum_\nu \inf\left\{ \lambda_\nu > 0 \,\Big|\, \varrho_{p(\cdot)}\Big(f_\nu / \lambda_\nu^{\frac{1}{q(\cdot)}}\Big) \leqslant 1 \right\}.$$

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- If $q^+ < \infty$, then $\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\big((f_\nu)_\nu\big) = \sum_\nu \big\| |f_\nu|^{q(\cdot)} \big\|_{\frac{p(\cdot)}{q(\cdot)}}$.

Remarks on the space $\ell^{q(\cdot)}(L^{p(\cdot)})$

$$\|(f_\nu)_\nu\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} = \inf\left\{\mu > 0 \,\Big|\, \varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\Big(\tfrac{1}{\mu}(f_\nu)_\nu\Big) \leqslant 1\right\}.$$

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- $q \geqslant 1$ is constant and $p(x) \geqslant 1$, [A-H'10]
- or $\frac{1}{p(x)} + \frac{1}{q(x)} \leqslant 1$, [A-H'10]
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(3) The condition $\min\{p(x), q(x)\} \geqslant 1$ is not sufficient for $\|\cdot\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}$ to be a norm!

(4) The Hardy-Littlewood maximal operator is not necessarily bounded in $\ell^{q(\cdot)}(L^{p(\cdot)})$.

Additional tools to deal with the variable B-T-L spaces

- The η-**functions**:

$$\eta_{\nu,m}(x) := \frac{2^{n\nu}}{(1+2^{\nu}|x|)^m}, \quad \nu \in \mathbb{N},\ m > 0.$$

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- r-**trick**: for all $g \in \mathcal{S}'$ with $\operatorname{supp} \hat{g} \subset \{\xi : |\xi| \leqslant 2^{\nu+1}\}$,

$$|g(x)| \lesssim (\eta_{\nu,m} * |g|^r(x))^{1/r}, \quad r > 0, \quad \nu \geqslant 0, \quad m > n.$$

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For instance, for log-Hölder continuous exponents,

$$\|(\eta_{\nu,2m} * f_\nu)_\nu\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \leqslant c\,\|(f_\nu)_\nu\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}, \quad m > n.$$

Why variable smoothness and integrability?

1. **Unification**:
   - $F^{\alpha}_{p(\cdot),\,2} = \mathcal{L}^{\alpha,p(\cdot)}, \quad 1 < p^- \leqslant p^+ < \infty$ [D-H-R'09]
     (variable exponent Bessel potential spaces);
     $\mathcal{L}^{\alpha,p(\cdot)} = W^{\alpha,p(\cdot)}, \quad \alpha \in \mathbb{N}_0$ [A-S'06]
   - $B^{\alpha(\cdot)}_{\infty,\infty} = C^{\alpha(\cdot)},\ 0 < \alpha^- \leqslant \alpha^+ < 1$ [A-H'10]
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2. **Traces**: if $\left(\alpha - \frac{1}{p} - (n-1)\max\left\{0, \frac{1}{p} - 1\right\}\right)^- > 0$,

$$\mathrm{Tr} F^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) = F^{\alpha(\cdot)-\frac{1}{p(\cdot)}}_{p(\cdot),p(\cdot)}(\mathbb{R}^{n-1})$$ [D-H-R'09]

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3. **Embeddings**: for $\alpha_0(x) - \frac{n}{p_0(x)} = \alpha_1(x) - \frac{n}{p_1(x)}$,

$$B^{\alpha_0(\cdot)}_{p_0(\cdot),q(\cdot)} \hookrightarrow B^{\alpha_1(\cdot)}_{p_1(\cdot),q(\cdot)}$$ [A-H'10]

Other useful results

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- Basic embeddings:

(i) If $q_0 \leqslant q_1$, then

$$B^{\alpha(\cdot)}_{p(\cdot),\, q_0(\cdot)} \hookrightarrow B^{\alpha(\cdot)}_{p(\cdot),\, q_1(\cdot)}.$$

(ii) If $(\alpha_0 - \alpha_1)^- > 0$, then

$$B^{\alpha_0(\cdot)}_{p(\cdot),\, q_0(\cdot)} \hookrightarrow B^{\alpha_1(\cdot)}_{p(\cdot),\, q_1(\cdot)}.$$

(iii) If $p^+, q^+ < \infty$, then

$$B^{\alpha(\cdot)}_{p(\cdot),\, \min\{p(\cdot),q(\cdot)\}} \hookrightarrow F^{\alpha(\cdot)}_{p(\cdot),\, q(\cdot)} \hookrightarrow B^{\alpha(\cdot)}_{p(\cdot),\, \max\{p(\cdot),q(\cdot)\}}.$$

Other contributions...

Variable smoothness: $B^{\alpha(\cdot)}_{p,q}$

- H.-G. Leopold (1989): Besov spaces of variable order of differentiation which include $B^{\alpha(\cdot)}_{p,p}$ as a particular case.
- O. Besov (1997)

Variable integrability: $B^{\alpha}_{p(\cdot),q}$ and $F^{\alpha}_{p(\cdot),q}$

- J.-S. Xu (2008)

Variable smoothness and integrability:

- H. Kempka (2009) : $B^{\mathbf{w}}_{p(\cdot),q}$ and $F^{\mathbf{w}}_{p(\cdot),q(\cdot)}$

Real interpolation of variable Besov spaces

RECALL:

$(A_0, A_1)_{\theta,q}$, $0 < \theta < 1$, $0 < q \leqslant \infty$, denotes the interpolation space (of compatible quasi-Banach spaces) obtained by the real method, i.e., consisting of all $a \in A_0 + A_1$ s.t.

$$\|a\|_{\theta,q} = \left(\int_0^\infty t^{-\theta q} K(t,a)^q \frac{dt}{t} \right)^{1/q} < \infty \quad \text{(modification if } q = \infty\text{)}$$

where K denotes the well-known Peetre functional.

Real interpolation of variable Besov spaces

If $0 < \alpha_0 - \alpha_1$ is constant, then

$$\left(B^{\alpha_0(\cdot)}_{p(\cdot),\infty}, B^{\alpha_1(\cdot)}_{p(\cdot),\infty}\right)_{\theta,q} \hookrightarrow B^{\alpha(\cdot)}_{p(\cdot),q} \qquad (3.1)$$

with $\alpha(x) = (1-\theta)\alpha_0(x) + \theta\alpha_1(x)$.

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with $\alpha(x) = (1-\theta)\alpha_0(x) + \theta\alpha_1(x)$.

Moreover, we can show that

$$B^{\alpha(\cdot)}_{p(\cdot),q} \hookrightarrow \left(B^{\alpha_0(\cdot)}_{p(\cdot),s}, B^{\alpha_1(\cdot)}_{p(\cdot),s}\right)_{\theta,q} \tag{3.2}$$

for $0 < s < q$.

Real interpolation of variable Besov spaces

From (3.1) and (3.2) (with $0 < s < \min\{q_0^-, q_1^-\}$), we get

Theorem

Let $0 < \theta < 1$ *and* $q \in (0, \infty]$. *Moreover, let* $p, q_0, q_1 \in \mathcal{P}_0^{\log}$ *and* $\alpha_0, \alpha_1 \in C_{\mathrm{loc}}^{\log} \cap L^\infty$. *If* $0 \neq \alpha_0 - \alpha_1$ *is a constant, then*

$$\left(B^{\alpha_0(\cdot)}_{p(\cdot),q_0(\cdot)}, B^{\alpha_1(\cdot)}_{p(\cdot),q_1(\cdot)}\right)_{\theta,q} = B^{\alpha(\cdot)}_{p(\cdot),q}$$

with $\alpha(x) = (1-\theta)\alpha_0(x) + \theta\alpha_1(x)$.

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$$\left(B^{\alpha_0(\cdot)}_{p(\cdot),\,q_0(\cdot)}, B^{\alpha_1(\cdot)}_{p(\cdot),\,q_1(\cdot)}\right)_{\theta,q} = B^{\alpha(\cdot)}_{p(\cdot),\,q}$$

with $\alpha(x) = (1-\theta)\alpha_0(x) + \theta\alpha_1(x)$.

In particular,

$$\left(C^{\alpha_0(\cdot)}, C^{\alpha_1(\cdot)}\right)_{\theta,\infty} = C^{\alpha(\cdot)}, \qquad \alpha(x) = (1-\theta)\alpha_0(x) + \theta\alpha_1(x),$$

for $\alpha_0 - \alpha_1$ constant and $0 < \alpha_0^- \leqslant \alpha_0(x) < \alpha_1(x) \leqslant \alpha_1^+ < 1$.

Some consequences

Corollary

Let $0 < \theta < 1$, $q_0, q_1 \in (0, \infty]$, $p \in \mathcal{P}_0^{\log}$ *and* $\alpha \in C_{\text{loc}}^{\log} \cap L^\infty$. *Then*

$$\left(B^{\alpha(\cdot)}_{p(\cdot),\,q_0}, B^{\alpha(\cdot)}_{p(\cdot),\,q_1}\right)_{\theta,q} = B^{\alpha(\cdot)}_{p(\cdot),\,q} \quad \textit{with} \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Some consequences

Corollary

Let $0 < \theta < 1$, $q_0, q_1 \in (0, \infty]$, $p \in \mathcal{P}_0^{\log}$ *and* $\alpha \in C_{\rm loc}^{\log} \cap L^\infty$. *Then*

$$\left(B^{\alpha(\cdot)}_{p(\cdot),\,q_0}, B^{\alpha(\cdot)}_{p(\cdot),\,q_1}\right)_{\theta,q} = B^{\alpha(\cdot)}_{p(\cdot),\,q} \quad \text{with} \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Corollary

Let $0 < \theta < 1$ *and* $q \in (0, \infty]$. *Moreover, let* $\alpha_0, \alpha_1 \in C_{\rm loc}^{\log} \cap L^\infty$, $q_0, q_1 \in \mathcal{P}_0$ *and* $p \in \mathcal{P}_0^{\log}$ *with* $p^+ < \infty$. *If* $0 \neq \alpha_0 - \alpha_1$ *is a constant, then*

$$\left(B^{\alpha_0(\cdot)}_{p(\cdot),\,q_0(\cdot)}, F^{\alpha_1(\cdot)}_{p(\cdot),\,q_1(\cdot)}\right)_{\theta,q} = \left(F^{\alpha_0(\cdot)}_{p(\cdot),\,q_0(\cdot)}, F^{\alpha_1(\cdot)}_{p(\cdot),\,q_1(\cdot)}\right)_{\theta,q} = B^{\alpha(\cdot)}_{p(\cdot),\,q}$$

with $\alpha(x) = (1-\theta)\alpha_0(x) + \theta\alpha_1(x)$.

Special interpolation formulas

- **Besov | Bessel potential spaces:**

$$\left(B^{\alpha_0}_{p(\cdot),\,q_0(\cdot)}, \mathcal{L}^{\alpha_1,p(\cdot)}\right)_{\theta,q} = \left(\mathcal{L}^{\alpha_0,p(\cdot)}, \mathcal{L}^{\alpha_1,p(\cdot)}\right)_{\theta,q} = B^{\alpha}_{p(\cdot),\,q}$$

with $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$, $\alpha_0 \neq \alpha_1$, $1 < p^- \leqslant p^+ < \infty$.

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- **Besov | Sobolev spaces:**

$$\left(B^{k_0}_{p(\cdot),\,q_0(\cdot)}, W^{k_1,p(\cdot)}\right)_{\theta,q} = \left(W^{k_0,p(\cdot)}, W^{k_1,p(\cdot)}\right)_{\theta,q} = B^{\alpha}_{p(\cdot),\,q}$$

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- **Besov | Lebesgue spaces:**

$$\left(B^{\alpha}_{p(\cdot),\,q_0(\cdot)},L^{p(\cdot)}\right)_{\theta,q}=B^{(1-\theta)\alpha}_{p(\cdot),\,q},\qquad \alpha\neq 0.$$

Application: trace operator

Theorem

Let $p \in \mathcal{P}_0^{\log}$ *with* $p^+ < \infty$, $q \in (0, \infty]$ *and* $\alpha \in C_{\mathrm{loc}}^{\log}$ *having a limit at infinity. If* $\left(\alpha - \frac{1}{p} - (n-1)\max\left\{0, \frac{1}{p} - 1\right\}\right)^- > 0$, *then*

$$\mathrm{Tr} \;:\; B^{\alpha(\cdot)}_{p(\cdot),\,q}(\mathbb{R}^n) \hookrightarrow B^{\alpha(\cdot)-\frac{1}{p(\cdot)}}_{p(\cdot),\,q}(\mathbb{R}^{n-1}).$$

Complex interpolation

Theorem

Let $0 < \theta < 1$, $p_0, p_1 \in \mathcal{P}^{\log}$, $1 < p_i^- \leqslant p_i^+ < \infty$, $q_0, q_1 \in [1, \infty)$ *and* $\alpha_0, \alpha_1 \in C^{\log}_{\mathrm{loc}} \cap L^\infty \cap L^\infty$. *Then*

$$\left[B^{\alpha_0(\cdot)}_{p_0(\cdot),\, q_0}, B^{\alpha_1(\cdot)}_{p_1(\cdot),\, q_1}\right]_\theta = B^{\alpha(\cdot)}_{p(\cdot),\, q},$$

where $\alpha(x) = (1-\theta)\alpha_0(x) + \theta\alpha_1(x)$,

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(retraction technique + complex interpolation between appropriate weighted variable Lebesgue spaces)

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Thank You !

Applications

- Image restoration [Chen, Levine, Rao (2004)]

$$\min \int_\Omega \Phi(x, \nabla u) + \frac{\lambda}{2}(u - I)^2,$$

where

$$\Phi(x, r) := \begin{cases} \frac{1}{p(x)}|r|^{p(x)} & |r| \leqslant \beta \\ |r| - \frac{\beta p(x) - \beta^{p(x)}}{p(x)} & |r| > \beta \end{cases}$$

($\beta > 0$ fixed, $1 \leqslant p(x) \leqslant 2$).

Applications

- Variational problems with non-standard growth [Zhikov, Marcellini, Acerbi, Mingione (1997-)]

$$\min \int \varphi(x, \nabla u)dx, \qquad |\xi|^{p(x)} \leqslant \varphi(x, \xi) \leqslant c\,(1 + |\xi|^{p(x)}).$$

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- Fluids dynamics [Růžička (2000)]

$$-\mathrm{div}\left[\left(1 + |Du(x)|^2\right)^{\frac{p(x)-2}{2}} Du(x)\right].$$