Weighted modular and norm inequalities for the Hardy operator in $L^{p(x)}$ spaces of decreasing functions

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Preliminaries

Let $p : \Omega \subset \mathbb{R} \longrightarrow [1, +\infty)$ be a measurable function on the open set Ω. Let us denote by $L^{p(\cdot)}(\Omega)$ the Banach space of measurable functions such that for $\lambda > 0$,

$$\int_\Omega |f(x)/\lambda|^{p(x)}\ dx < +\infty,$$

with norm

$$\|f\|_{L^{p(\cdot)}} = \inf\left\{\lambda > 0 : \int_\Omega |f(x)/\lambda|^{p(x)}\ dx \le 1\right\}.$$

Spaces $L^{p(\cdot)}(\Omega)$ are examples of the Musielak-Orlicz spaces.

Preliminaries

Recently, some possible extensions of results concerning classical operators in harmonic analysis have been studied in the context of variable exponents. Explicitly, inequalities in norm for an operator T

$$\|Tf\|_{L^{p(\cdot)}} \leq C\|f\|_{L^{p(\cdot)}},$$

or inequalities of modular type

$$\int_\Omega (Tf(x))^{p(x)}\, dx \leq C \int_\Omega (f(x))^{p(x)}\, dx.$$

In this sense, we can mention:

- A. Lerner (2005): modular inequalities for M, the Hardy-Littlewood maximal operator.
- L. Diening (2003): Cruz-Uribe, Fiorenza and Neugebauer (2004), norm inequalities for M.
- Cruz-Uribe, Fiorenza, Martell and Pérez (2006): norm inequalities for singular integral operators, commutators or fractional integrals.

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Preliminaries

Let us consider the Hardy operator

$$(Sf)(x) = \frac{1}{x}\int_0^x f(t)\,dt,\;\; x > 0.$$

- Weighted norm inequalities for the Hardy operator: Kokilashvili and Samko (powers weights) (2004) , Edmunds, Kokilashvili and Meshki (2005).
- Sinnamon (2000): shows that, for arbitrary functions, the only possibility for having a modular inequality for the Hardy operator is that p is essentially a constant function.
- A. Lerner (2005): concludes the same but, in this case, for the Hardy-Littlewood maximal operator or Calderón-Zygmund singular integrals.

Preliminaries

- I. Aguilar and P. Ortega (2006): weak type inequalities of modular type for operators of Hardy type:

$$\int_{\{x\in A:\ Tf(x)>\lambda\}} w(x)\ dx \leq \int_A \left(\frac{K\ |f(x)|}{\lambda}\right)^{p(x)}\ w(x)\ dx,$$

being T an operator of Hardy type, $A \subset \mathbb{R}$, $K > 0$ and f an arbitrary measurable function.

Modular strong type inequalities

In our work (JMAA (2008)), the main purpose was to characterize the weights w for which the following modular inequality holds for the Hardy operator restricted to decreasing functions,

$$\int_0^{+\infty} (Sf(x))^{p(x)}\ w(x)\ dx \leq C \int_0^{+\infty} (f(x))^{p(x)}\ w(x)\ dx, \qquad (1)$$

for some positive constant C and f decreasing.

If p is constant, the theory of Ariño and Muckenhoupt prove that (1) is equivalent to the B_p condition for the weight w; that is, the existence of some constant $C > 0$ such that, for $r > 0$,

$$\int_r^{+\infty} \left(\frac{r}{x}\right)^p\ w(x)\ dx \leq C \int_0^r w(x)\ dx. \quad B_p \text{ condition}$$

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Given $p : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that $0 < p^- \leq p^+ < +\infty$ and w a weight in $(0, +\infty)$, let us define the local oscillation of p as

$$\varphi_{p(\cdot),w}(\delta) = \sup_{x \in (0,\delta) \cap \text{supp } w} p(x) - \inf_{x \in (0,\delta) \cap \text{supp } w} p(x).$$

We observe that $\varphi_{p(\cdot),w}$ is an increasing and positive function such that

$$\lim_{\delta \to \infty} \varphi_{p(\cdot),w}(\delta) = p_w^+ - p_w^-,$$

where p_w^- and p_w^+ denote the infimum and essential supremum, respectively, of p on the support of w.

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Modular strong type inequalities

THEOREM.

Let w be a weight in $(0,\infty)$ and $p:\mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that $0<p^- \leq p^+ < +\infty$, let us suppose $\varphi_{p(\cdot),w}(0^+)=0$. The following facts are equivalent:

(a) There exists $C>0$ such that, for every f positive and decreasing:

$$\int_0^{+\infty} (Sf(x))^{p(x)}\ w(x)\ dx \leq C \int_0^{+\infty} (f(x))^{p(x)}\ w(x)\ dx.$$

(b) For every $r,s>0$

$$\int_r^{+\infty} \left(\frac{r}{sx}\right)^{p(x)}\ w(x)\ dx \leq C \int_0^{r} \frac{w(x)}{s^{p(x)}}\ dx. \tag{2}$$

(c) $p_{|\mathrm{supp\ w}} \equiv p_0$ a.e. and $w\in B_{p_0}$.

Modular strong type inequalities

REMARKS.

- The condition $\varphi_{p(\cdot),w}(0^+) = 0$ in the previous theorem is only necessary to prove (c) from (2).
- For w verifying (2), the behavior at the origin of $\varphi_{p(\cdot),w}$ is independent of w, since the support must contain a zero neighborhood.
- $\varphi_{p(\cdot),w}(0^+) = 0$ holds, if p belongs to Lip-α, $0 < \alpha \leq 1$, in a zero neighborhood.
- Due to the lack of homogeneity in the modular inequality, in condition $B_{p(\cdot)}$ the second parameter s must be introduced.

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Implicit in the proof of last theorem, we obtain the following.

COROLLARY.

Let w be a weight in the $B_{p(\cdot)}$ class, the function $\varphi_{p(\cdot),w}$ must be constant.

It is false that the modular inequality holds exclusively for constant exponents without the hypothesis $\varphi_{p(\cdot),w}(0^+) = 0$.

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EXAMPLE.

Let us consider the exponent $p(x)$ restricted to the interval $(0,1]$:

$$p(x) = \begin{cases} p^+ & \text{for } x \in A := \bigcup_{k=0}^{\infty} \left(\frac{1}{2^{2k+1}}, \frac{1}{2^{2k}} \right] \\ p^- & \text{for } x \in B := \bigcup_{k=1}^{\infty} \left(\frac{1}{2^{2k}}, \frac{1}{2^{2k-1}} \right], \end{cases}$$

where $1 < p^- < p^+ < +\infty$, and the weight $w(x) = \chi_{(0,1)}(x)$, for f a decreasing function, we have the corresponding strong modular inequality

$$\int_0^1 (Sf(x))^{p(x)}\, dx \leq C \int_0^1 (f(x))^{p(x)}\, dx.$$

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Modular strong type inequalities

As it happens in the case of constant exponents, we can prove that the modular inequality implies a $B_{p(\cdot)-\epsilon}$ condition for some $\epsilon > 0$.

PROPOSITION.

Let $p : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that $1 \leq p^- \leq p^+ < +\infty$ and w a weight for which the modular inequality holds. Then, for some $\epsilon > 0$, $w \in B_{p(\cdot)-\epsilon}$.

Modular strong type inequalities

Last proposition represents an improvement of the main theorem due to the following result that proves the following inclusion relations between weights in $B_{p(\cdot)}$:

PROPOSITION.

Given $p : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that $0 < p^- \leq p^+ < +\infty$ and $\delta > 0$, then:

(i) $B_{p(\cdot)} \subsetneqq B_{p(\cdot)+\delta}$.

(ii) $B_{\delta p(\cdot)} \subsetneqq B_{p(\cdot)}$, for $\delta \leq 1$.

REMARK.

In general, inclusion $B_{p(\cdot)} \subsetneqq B_{q(\cdot)}$ is false for $p(x) \leq q(x)$ a.e $x > 0$.
Let us consider $w \equiv 1 \in B_p$ if $p > 1$, and $q(x) \geq p$ such that the corresponding function $\varphi_{q(\cdot)}$ don't be constant in $(0, +\infty)$.
for example $q(x) = p\chi_{(0,1)} + 2p\chi_{(1,+\infty)}$

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REMARK.

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Modular weak type inequalities

We look for conditions on $p(\cdot)$ and in w to verify when it is true that

$$\sup_{r>0} \int_0^r (Sf(r))^{p(x)}\ w(x)\ dx \leq C \int_0^{+\infty} (f(x))^{p(x)}\ w(x)\ dx. \qquad (3)$$

for every non-increasing f in $[0, \infty)$.

For $p(\cdot)$ constant, inequality (3) becomes in

$$\sup_{r>0} (Sf(r))^p W(r)\ dx \leq C \int_0^{+\infty} (f(x))^p\ w(x)\ dx. \qquad (4)$$

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Modular weak type inequalities

- Neugebauer (1991) proved that the class of weights for which (4) holds is exactly B_p in the case $1 < p < \infty$.

- In the case $0 < p \leq 1$, Carro and Soria (1993) proved that the class of weights that characterize (4) is the class R_p, that is, those for which there exists $C > 0$ such that if $0 < r < t < +\infty$

$$\frac{1}{t^p}\int_0^t w(x)\ dx \leq C\frac{1}{r^p}\int_0^r w(x)\ dx. \quad R_p \text{ condition}$$

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THEOREM.(B., Soria) to appear in REMC

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(a) There exists $C>0$ such that, for each non-increasing and positive f:

$$\sup_{r>0}\int_0^r (Sf(r))^{p(x)}\ w(x)\ dx \leq C\int_0^{+\infty}(f(x))^{p(x)}\ w(x)\ dx. \quad W_{p(\cdot)}$$

(b) For all $0<t<r$, $s>0$

$$\int_0^r \left(\frac{st}{r}\right)^{p(x)}\ w(x)\ dx \leq C\int_0^t s^{p(x)}\ w(x)\ dx. \quad R_{p(\cdot)}$$

(c) $p_{|\mathrm{supp\ w}}\equiv p_0$ a.e. and then $w\in R_{p_0}$.

Modular weak type inequalities

Let us define the class $W_{p(\cdot)}$ consisting of all weights satisfying the weak modular inequality and $R_{p(\cdot)}$ the class consisting of all weights verifying the restricted weak modular inequality.

REMARK. Looking at the proof of the previous theorem, we conclude that condition $\varphi_{p(\cdot),w}(0^+) = 0$ implies, without any restriction in the exponent, that for w satisfying $R_{p(\cdot)}$, the exponent $p(\cdot)$ must be constant.

THEOREM.

If a weight is $w \in W_{p(\cdot)}$ and the exponent $p(\cdot)$ satisfies that $p^- > 1$ then $w \in B_{p(\cdot)}$.

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COROLLARY.

If $w \in W_{p(\cdot)}$, the exponent $p(\cdot)$ verifies that $p^- > 1$ and $\varphi_{p(\cdot)}(0^+) = 0$ then $p(\cdot) \equiv p_0$ and $w \in B_{p_0}$.

The previous corollary together with the previous remark led us to conclude that for exponents such that $\varphi_{p(\cdot)}(0^+) = 0$, if there weights satisfying $W_{p(\cdot)}$ then necessarily $p(\cdot) \equiv p_0$ and then,

- If $p_0 \leq 1$, $w \in R_{p_0}$
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It is false that the weak modular inequality holds exclusively for constant exponents without the hypothesis $\varphi_{p(\cdot),w}(0^+) = 0$.

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Modular weak type inequalities

The following proposition proves the connection between conditions $R_p(\cdot)$ and $B_{p(\cdot)}$:

PROPOSITION.

Let $w \in R_{p(\cdot)}$ then, for $\delta > 0$, $w \in B_{p(\cdot)+\delta}$.

Weighted norm inequalities

C.J. Neugebauer (2009), defines the $B_{p(\cdot)}$ class as

$$\int_r^{+\infty} \left(\frac{r}{x}\right)^{p(x)} \; w(x)\; dx \leq C \int_0^r w(x)\; dx. \tag{5}$$

Let $1 \leq p(x) \leq p^+ < \infty$ and $p(x)$ INCREASING, then, w belongs to the $B_{p(\cdot)}$ class given by (5) if and only if the modular inequality (1) holds for each f nondecreasing such that $f(0+) \leq 1$. Moreover, these two conditions are both equivalents to the following: for every $0 < \gamma \leq 1$ there exists $1 \leq c_\gamma < \infty$ such that

$$\|Sf\|_{p(x),\sigma w} \leq c_\gamma \|f\|_{p(x),\sigma w}$$

for every f decreasing with $f(0+) \leq 1$, and all $0 < \sigma < \infty$ for which $\|f\|_{p(x),\sigma w} \geq \gamma$.

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Weighted norm inequalities

In contrast with Neugebauer's result, we have proved, in the case of very simple non-constant exponents that can be INCREASING or DECREASING, a characterization of the weights for which a norm inequality holds for the Hardy operator with no restrictions on the class of non-decreasing functions.

PROPOSITION.

Let $p(x) = 2p_0\chi_{(0,1)}(x) + p_0\chi_{(1,+\infty)}(x)$, $1 < p_0 < +\infty$, the norm inequality

$$\|Sf\|_{p(x),w} \leq C\|f\|_{p(x),w}, \tag{6}$$

with $C > 0$, independent of f decreasing, holds if and only if:

i) $w\chi_{(0,1)} \in B_{2p_0}$.

ii) $\displaystyle\int_r^{+\infty} \left(\frac{r}{x}\right)^{p_0} w(x)\, dx \leq C \int_0^r w(x)\, dx,$
for some $C > 0$ and all $r > 1$.

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PROPOSITION.

Let $p(x) = 2p_0\chi_{(0,1)}(x) + p_0\chi_{(1,+\infty)}(x)$, $1 < p_0 < +\infty$, the norm inequality

$$\|Sf\|_{p(x),w} \leq C\|f\|_{p(x),w}, \tag{6}$$

with $C > 0$, independent of f decreasing, holds if and only if:

i) $w\chi_{(0,1)} \in B_{2p_0}$.

ii) $\displaystyle\int_r^{+\infty} \left(\frac{r}{x}\right)^{p_0} w(x)\,dx \leq C \int_0^r w(x)\,dx,$
for some $C > 0$ and all $r > 1$.

Weighted norm inequalities

$$\|f\|_{p(\cdot),w}^{p_0} \simeq$$

$$\int_1^{+\infty} f^{p_0}(x)w(x)dx + \sqrt{\left(\int_1^{+\infty} f^{p_0}(x)w(x)dx\right)^2 + \int_0^1 f^{2p_0}(x)w(x)dx}.$$

Weighted norm inequalities

We have to prove

$$\left(\int_1^{+\infty}(Sf)^{p_0}(x)w(x)dx\right)^2 \leq \left(\int_1^{+\infty} f^{p_0}(x)w(x)dx\right)^2 + \int_0^1 f^{2p_0}(x)w(x)dx. \tag{7}$$

and

$$\int_0^1 (Sf)^{2p_0}(x)w(x)dx \leq \left(\int_1^{+\infty} f^{p_0}(x)w(x)dx\right)^2 + \int_0^1 f^{2p_0}(x)w(x)dx. \tag{8}$$

Weighted norm inequalities

Since if f is decreasing, (8) is equivalent to

$$\int_0^1 (Sf)^{2p_0}(x)w(x)dx \leq \int_0^1 f^{2p_0}(x)w(x)dx$$

that it turns out to be equivalent to $w\chi_{(0,1)} \in B_{2p_0}$.
On the other hand, writing explicitly Sf in (7), that is equivalent to

$$\left(\int_0^1 f(x)dx\right)^{2p_0}\left(\int_1^{+\infty} \frac{w(x)}{x^{p_0}}dx\right)^2 \leq \left(\int_1^{+\infty} f^{p_0}(x)w(x)dx\right)^2 + \int_0^1 f^{2p_0}(x)w(x)dx, \tag{9}$$

and

$$\left(\int_1^{+\infty}\left(\int_1^x f(s)ds\right)^{p_0}\frac{w(x)}{x^{p_0}}dx\right)^2 \leq \left(\int_1^{+\infty} f^{p_0}(x)w(x)dx\right)^2 + \int_0^1 f^{2p_0}(x)w(x)dx. \tag{10}$$

Weighted norm inequalities

Since f is non-increasing, if we assume $\int_1^{+\infty} \frac{w(x)}{x^{p_0}} dx < +\infty$, inequality (9) can be expressed as an inclusion between Lorentz spaces $\Lambda^{2p_0}(w\chi_{(0,1)}) \hookrightarrow \Lambda^1(\chi_{(0,1)})$ which is satisfied if $w\chi_{(0,1)} \in B_{2p_0}$ (Sawyer (1990)).

Finally, to ensure that (10) holds it is equivalent to restrict the inequality to a decreasing function f with $f(x) \equiv 1$, $0 < x \leq 1$, i.e.

$$\left(\int_1^{+\infty} \left(\int_1^x f(s)ds\right)^{p_0} \frac{w(x)}{x^{p_0}} dx\right)^2 \leq \left(\int_1^{+\infty} f^{p_0}(x)w(x)dx\right)^2 + \int_0^1 w(x)dx. \tag{11}$$

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Weighted norm inequalities

Applying the inequality to $\chi_{(0,r)}$ with $r > 1$ the following necessary condition is obtained to ensure (11)

$$\int_r^{+\infty} \left(\frac{r}{x}\right)^{p_0} w(x)\, dx \leq \int_0^r w(x)\, dx, \text{ with } r > 1.$$

The proof ends checking that this condition is also enough.

Weighted modular and norm inequalities for the Hardy operator in $L^{p(x)}$ spaces of decreasing functions

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