Limiting real interpolation spaces for general couples

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For $0 < \theta < 1$ and $1 \leq q \leq \infty$

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Problems in image processing and denoising

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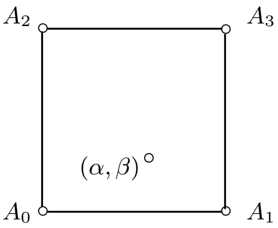

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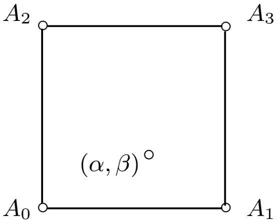

For $a \in \Sigma(\overline{\mathbb{A}}) = A_0 + A_1 + A_2 + A_3$ and $t > 0,\ s > 0$

$$\bar{K}(t,s;a) = \inf\left\{ \|a_0\|_{A_0} + t\|a_1\|_{A_1} + s\|a_2\|_{A_2} + ts\|a_3\|_{A_3} :\ a = \sum_{j=0}^{3} a_j,\ a_j \in A_j \right\},$$

$$\bar{J}(t,s;a) = \max\left\{ \|a\|_{A_0}, t\|a\|_{A_1}, s\|a\|_{A_2}, ts\|a\|_{A_3} \right\}, a \in \Delta(\overline{\mathbb{A}}) = A_0 \cap A_1 \cap A_2 \cap A_3.$$

$$\overline{\mathbb{A}}_{(\alpha,\beta),q;K} = \left\{ a \in \Sigma(\overline{\mathbb{A}}) : \|a\|_{\overline{\mathbb{A}}_{(\alpha,\beta),q;K}} = \left(\int_0^\infty \int_0^\infty \left(t^{-\alpha} s^{-\beta} \bar{K}(t,s;a) \right)^q \; \frac{dt}{t} \; \frac{ds}{s} \right)^{\frac{1}{q}} < \infty \right\}$$

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The norm on this space is

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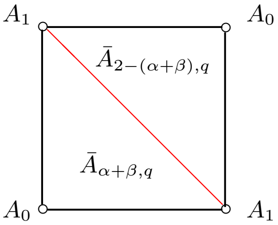

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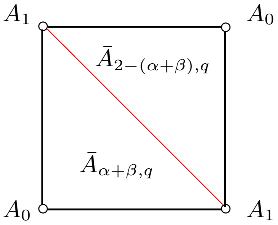

$$(A_0, A_1, A_1, A_0)_{(\alpha,\beta),q;K} = \begin{cases} (A_0, A_1)_{\alpha+\beta,q} & \text{if} \quad \beta < 1-\alpha, \\ (A_0, A_1)_{2-(\alpha+\beta),q} & \text{if} \quad 1-\alpha < \beta. \end{cases}$$

When $\beta = 1 - \alpha$ we obtain limiting real interpolation spaces

$$(A_0, A_1)_{1,q} = \left\{ a \in A_1 : \|a\|_{1,q} = \left(\int_1^\infty \left(t^{-1} K(t,a) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty \right\}$$

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The reason for cutting the integral is that

$$K(t,a) \sim t\|a\|_{A_1} \quad \text{for} \quad 0 < t \le 1.$$

Therefore,

$$\Big(\int_0^1 [t^{-1} K(t,a)]^q \frac{dt}{t} \Big)^{1/q} = (\int_0^1 \frac{dt}{t})^{1/q} \|a\|_{A_1} = \infty.$$

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Moreover, using that $A_0 \hookrightarrow A_1$ one can show that

$$\|a\|_{\theta,q} \sim \Big(\int_1^\infty [t^{-\theta} K(t,a)]^q \frac{dt}{t} \Big)^{1/q}$$

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$$(L_\infty, L_1)_{\theta,q} = L_{p,q} \quad if \quad \frac{1}{p} = \theta$$

and

$$L_{p,q} = \left\{ f : \|f\|_{L_{p,q}} = \left(\int_0^\infty (t^{\frac{1}{p}-1} \int_0^t f^*(s)ds)^q \, \frac{dt}{t} \right)^{1/q} < \infty \right\}.$$

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- Limiting J-spaces with $\theta = 0$.

$\triangleright$ F. Cobos, L.M. Fernández-Cabrera, T. Kühn and T. Ullrich, J. Funct. Anal. 256 (2009) 2321-2366.

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If $A_0 \hookrightarrow A_1$ and $1 \le q \le \infty$ then $(A_0, A_1)_{0,q}$ is formed by all those elements $a \in A_1$ for which there exists a strongly measurable function $u(t)$ with values in A_0 such that

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We set

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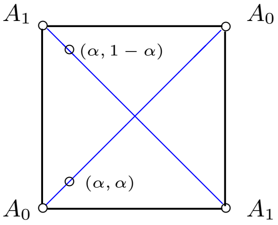
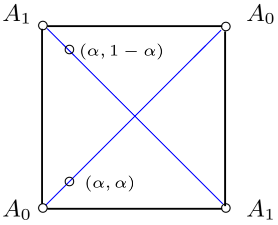
If $\mu(\Omega) < \infty$ then $(L_\infty, L_1)_{0,q} =$

$$L_{\infty,q}(\log L)_{-1} = \left\{f : \Big(\int_0^{\mu(\Omega)}\Big(\frac{1}{(1+|\log t|)t}\int_0^t f^*(s)ds\Big)^q \frac{dt}{t}\Big)^{1/q} < \infty\right\}.$$

In particular, we have $(L_\infty, L_1)_{0,\infty} = L_{exp}$



A_1
A_0
$(\alpha, 1-\alpha)$
(α, α)
A_0
A_1



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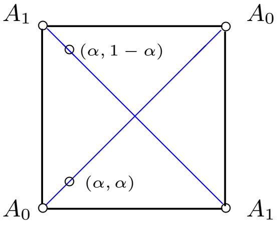

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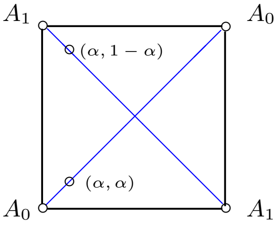

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- Limiting spaces for general Banach couples.
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with powers of the function $g(t) = 1 + |\log t|$

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• No weight

▷ F. Cobos, L.M. Fernández-Cabrera, P. Silvestre, Math. Nachr. (to appear).

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$$\begin{aligned} \sup_{0<t\le 1} t^{-1} K(t,a) &\le K(1,a) \\ &\lesssim K(1,a) \left(\int_1^\infty t^{-q} \frac{dt}{t} \right)^{1/q} \\ &\le \left(\int_1^\infty (t^{-1} K(t,a))^q \frac{dt}{t} \right)^{1/q} . \end{aligned}$$

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An integration by parts yields that

$$\begin{aligned} \int_0^1 \left(\int_0^x f^*(s)ds \right) \frac{dx}{x} &= \int_0^1 f^*(t)\log(\frac{1}{t})dt \\ &\backsim \int_0^1 f^*(t)(1 + |\log t|)dt. \end{aligned}$$

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- The result shows a symmetry which cannot be observed in the ordered case.

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$$a = \int_0^\infty v(t) \frac{dt}{t} \quad \text{(convergence in } A_0 + A_1) \,, \tag{1}$$

where $v(t)$ is a strongly measurable function with values in $A_0 \cap A_1$ such that

$$\int_0^1 J(t, v(t)) \frac{dt}{t} + \left(\int_1^\infty J(t, v(t))^q \frac{dt}{t} \right)^{1/q} < \infty \,. \tag{2}$$

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The space $\bar{A}_{1,q;J} = (A_0, A_1)_{1,q;J}$ is formed by all those $a \in A_0 + A_1$ for which there is a representation of the type (1) but satisfying now

$$\left(\int_0^1 (t^{-1} J(t, v(t)))^q \frac{dt}{t} \right)^{1/q} + \int_1^\infty t^{-1} J(t, v(t)) \frac{dt}{t} < \infty . \tag{3}$$

The norm in $\bar{A}_{1,q;J}$ is the infimum in (3) over all representations (1), (3).

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- If $A_0 \hookrightarrow A_1$ and $a \in \bar{A}_{0,q;J}$ with $a = \int_0^\infty v(t)\,\frac{dt}{t}$

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Writing $u(t) = v(t) + a_0\chi_{(1,e)}(t)$ for $1 \leq t < \infty$, we get that $a = \int_1^\infty u(t)dt/t$ with

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Therefore, $\bar{A}_{0,q;J} \hookrightarrow \bar{A}_{0,q}$. The reverse inclusion is clear.

THEOREM.- Let $\bar{A} = (A_0, A_1)$ be a Banach couple, let $0 < \alpha < 1$ and let $1 < q \leq \infty$. Put $\bar{\mathbb{A}} = (A_0, A_1, A_1, A_0)$. Then we have with equivalent norms

$$\bar{\mathbb{A}}_{(\alpha,\alpha),q;J} = \begin{cases} \bar{A}_{2\alpha,q} \cap \bar{A}_{0,q;J} & \text{if} \quad 0 < \alpha < 1/2\,, \\ \bar{A}_{1,q;J} \cap \bar{A}_{0,q;J} & \text{if} \quad \alpha = 1/2\,, \\ \bar{A}_{2-2\alpha,q} \cap \bar{A}_{0,q;J} & \text{if} \quad 1/2 < \alpha < 1\,, \end{cases}$$

and

$$\bar{\mathbb{A}}_{(\alpha,1-\alpha),q;J} = \begin{cases} \bar{A}_{1-2\alpha,q} \cap \bar{A}_{1,q;J} & \text{if} \quad 0 < \alpha < 1/2\,, \\ \bar{A}_{0,q;J} \cap \bar{A}_{1,q;J} & \text{if} \quad \alpha = 1/2\,, \\ \bar{A}_{2\alpha-1,q} \cap \bar{A}_{1,q;J} & \text{if} \quad 1/2 < \alpha < 1\,. \end{cases}$$

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- The result shows a symmetry which cannot be observed in the ordered case.

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- The result shows a symmetry which cannot be observed in the ordered case.
- For the proof, some other auxiliary interpolation spaces are required. Restriction $q \neq 1$ is due to one of the intermediate results.

COROLLARY.- Let (Ω, μ) be a σ-finite measure space. Then
$(L_\infty, L_1, L_1, L_\infty)_{(\alpha,\alpha),\infty;J}$

$$= \begin{cases} L_{1/2\alpha,\infty} \cap L_{\infty,\infty}(\log L)_{-1} & \text{if} \quad 0 < \alpha < 1/2\,, \\ L_{(1,\infty)}(\log L)_{-1} \cap L_{\infty,\infty}(\log L)_{-1} & \text{if} \quad \alpha = 1/2\,, \\ L_{1/(2-2\alpha),\infty} \cap L_{\infty,\infty}(\log L)_{-1} & \text{if} \quad 1/2 < \alpha < 1 \end{cases} ,$$

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$$= \begin{cases} L_{1/(1-2\alpha),\infty} \cap L_{(1,\infty)}(\log L)_{-1} & \text{if} \quad 0 < \alpha < 1/2\,, \\ L_{\infty,\infty}(\log L)_{-1} \cap L_{(1,\infty)}(\log L)_{-1} & \text{if} \quad \alpha = 1/2\,, \\ L_{1/(2\alpha-1),\infty} \cap L_{(1,\infty)}(\log L)_{-1} & \text{if} \quad 1/2 < \alpha < 1\,. \end{cases}$$

Here

$$L_{p,q}(\log L)_b$$
$$= \left\{ f : \|f\|_{L_{p,q}(\log L)_b} = \left(\int_0^\infty (t^{1/p}(1 + |\log t|)^b f^*(t))^q \frac{dt}{t} \right)^{1/q} < \infty \right\}$$

and we define $L_{(p,q)}(\log L)_b$ similarly but replacing f^* by f^{**}.