Local Khintchine inequalitiy for symmetric spaces

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1. Khintchine inequality

Khintchine inequality

Theorem (Khintchine, 1923)

Given $1 \leq p < \infty$, there exists constants A_p, B_p such that

$$A_p \cdot \Big(\sum_{n=1}^{\infty} a_n^2 \Big)^{1/2} \leq \left(\int_0^1 \Big| \sum_{n=1}^{\infty} a_n r_n(t) \Big|^p dt \right)^{1/p} \leq B_p \cdot \Big(\sum_{n=1}^{\infty} a_n^2 \Big)^{1/2}$$

- $r_n(t) = (-1)^{k-1}, \quad t \in \Delta_n^k := \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right]$ $(1 \leq k \leq 2^n, n \in \mathbb{N})$

Rademacher functions

- Orthonormal system: $$\Big\| \sum_{n=1}^{\infty} a_n r_n \Big\|_2 = \Big(\sum_{n=1}^{\infty} a_n^2 \Big)^{1/2}$$
- $\displaystyle\sum_{n=1}^{\infty} a_n r_n(t)$ converges a.e. $\iff \displaystyle\sum_{n=1}^{\infty} a_n^2 < \infty.$
- $Rad(X) :=$ closed linear span of $\{r_n\}$ in X
  - $Rad(L^2) = \ell^2$
  - $Rad(L^p) \approx \ell^2$, for $1 \leq p < \infty$
  - $Rad(L^\infty) = \ell^1$ since

$$\sup_{t\in[0,1]} \sum_{n=1}^{N} a_n r_n(t) = \sum_{n=1}^{N} |a_n|$$

Beyond L^p

Any other function space X such that $Rad(X) \approx \ell^2$?

- Yes: the space $Exp\ L^2$ of all functions f such that

$$\int_0^1 \exp\left(\frac{|f(t)|}{\lambda}\right)^2 dt < \infty, \text{ for some } \lambda > 0.$$

- $Exp\ L^2$ is “close” to L^∞:

$$L^\infty \subsetneq Exp\ L^2 \subset L^p, \quad 1 \le p < \infty$$

Characterization of KI

Symmetric space: Banach function space where the norm of a function is determined its distribution function

$$Rad(X) \approx \ell^2 \iff \Big(\sum_{n=1}^{\infty} a_k^2\Big)^{1/2} \asymp \Big\| \sum_{n=1}^{\infty} a_k r_k(t) \Big\|_X, \quad (a_k) \in \ell^2$$

Theorem (Rodin & Semenov, 1975)

Let X be a symmetric space. Then,

$$Rad(X) \approx \ell^2 \iff \overline{L^\infty}^{Exp\ L^2} \subset X$$

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2. A local version of Khintchine inequality

Local version: idea

- If $E = \Delta$ a dyadic set or order N, e.g. $\Delta = \left[0, \frac{1}{2^N}\right]$
- Modulo the size of Δ, the Rademacher functions $\{r_{N+1}, r_{N+2}, \dots\}$ behave on Δ as $\{r_1, r_2, \dots\}$ behave on $[0, 1]$

  That is, for every $(a_k) \in \ell^2$,

$$\int_\Delta \Big| \sum_{n=N+1}^{\infty} a_k r_k(t) \Big|^2 dt = \frac{1}{2^N} \sum_{n=N+1}^{\infty} a_k^2$$

A result of Zygmund

Theorem (“Trigonometric Series”, 1935)

Let $\varepsilon > 0$*. For any measurable set* $E \subset [0,1]$ *with* $m(E) > 0$ *there exists* $N := N(E)$ *such that*

$$(1-\varepsilon)\sum_{n=N}^{\infty} a_k^2 \le \frac{1}{m(E)}\int_E \Big|\sum_{n=N}^{\infty} a_k r_k(t)\Big|^2 dt \le (1+\varepsilon)\sum_{n=N}^{\infty} a_k^2$$

for every $(a_k) \in \ell^2$*.*

The LKI for L^p

Theorem (Sagher & Zhou, 1990)

For any $1 \leq p < \infty$ *there exist constants* $A'_p, B'_p > 0$ *so that for any measurable set* $E \subset [0,1]$ *with* $m(E) > 0$ *there exists* $N := N(E)$ *such that*

$$A'_p \Big(\sum_{n=N}^{\infty} a_k^2 \Big)^{1/2} \leq \left(\frac{1}{m(E)} \int_E \Big| \sum_{n=N}^{\infty} a_k r_k(t) \Big|^p \, dt \right)^{1/p} \leq B'_p \Big(\sum_{n=N}^{\infty} a_k^2 \Big)^{1/2}$$

for every $(a_k) \in \ell^2$.

How to extend beyond L^p?

Two possibilities:

$$\left(\frac{1}{m(E)}\int_E \Big|\sum_{n=N}^{\infty} a_k r_k(t)\Big|^p dt\right)^{1/p} = \left(\int_E \Big|\sum_{n=N}^{\infty} a_k r_k(t)\Big|^p \frac{dt}{m(E)}\right)^{1/p}$$

$$\left(\frac{1}{m(E)}\int_E \Big|\sum_{n=N}^{\infty} a_k r_k(t)\Big|^p dt\right)^{1/p} = \frac{1}{m(E)^{1/p}}\Big\|\chi_E \cdot \sum_{n=N}^{\infty} a_k r_k(t)\Big\|_p$$

The LKI for $Exp\ L^1$

Theorem (Sagher & Zhou, 1996)

There exist $\alpha, \beta > 0$ so that for $E \subset [0,1]$ with $m(E) > 0$ there exists $N := N(E)$ such that

$$\alpha\Big(\sum_{n=N}^{\infty} a_k^2\Big)^{1/2} \le \Big\| \sum_{n\ge N} a_n r_n \Big\|_{Exp\ L^1\left(E, \frac{dt}{|E|}\right)} \le \beta\Big(\sum_{n=N}^{\infty} a_k^2\Big)^{1/2}$$

for every $(a_k) \in \ell^2$.

$$\|f\|_{Exp\ L^1\left(E, \frac{dt}{|E|}\right)} = \inf\left\{\lambda > 0 : \int_E \left(\exp\left(\frac{|f(t)|}{\lambda}\right) - 1\right)\frac{dt}{|E|} \le 1\right\}.$$

The LKI for *Exp* L^2

Theorem (Carrillo-Alanís, 2011)

There exist $A, B > 0$ *so that for* $E \subset [0,1]$ *with* $m(E) > 0$ *there exists* $N := N(E)$ *such that*

$$A\Big(\sum_{n=N}^{\infty} a_k^2\Big)^{1/2} \le \Big\|\sum_{n\ge N} a_n r_n\Big\|_{ExpL^2\left(E,\frac{dt}{|E|}\right)} \le B\Big(\sum_{n=N}^{\infty} a_k^2\Big)^{1/2}$$

for every $(a_k) \in \ell^2$.

$$\|f\|_{Exp\ L^2\left(E,\frac{dt}{|E|}\right)} = \inf\left\{\lambda > 0 : \int_E \left(\exp\left(\frac{|f(t)|}{\lambda}\right)^2 - 1\right)\frac{dt}{|E|} \le 1\right\}.$$

A local symmetric Khintchine inequality

- X symmetric satisfies a **Local Khintchine Inequality** if:

  there exist $\alpha, \beta > 0$ such that for $E \subset [0,1]$ with $m(E) > 0$ there exists $N := N(E)$ such that

$$\alpha \cdot \varphi_X(m(E)) \leq \frac{\left\| \chi_E \sum_{i=N}^{\infty} a_i r_i(t) \right\|_X}{\left\| \sum_{i=N}^{\infty} a_i r_i(t) \right\|_X} \leq \beta \cdot \varphi_X(m(E))$$

  for every $(a_i) \in \ell^2$ with $\sum_{i=N}^{\infty} a_i r_i(t) \in X$

- $\varphi_X := \|\chi_{[0,t]}\|_X$ is the fundamental function of X.

A local symmetric Khintchine inequality

- For L^p, the LKI was

$$A'_p\Big(\sum_{n=N}^{\infty} a_k^2\Big)^{1/2} \leq \left(\frac{1}{m(E)}\int_E \Big|\sum_{n=N}^{\infty} a_k r_k(t)\Big|^p dt\right)^{1/p} \leq B'_p\Big(\sum_{n=N}^{\infty} a_k^2\Big)^{1/2}$$

where

$$\varphi_{L^p}(t) = t^{1/p}$$

$$\Big(\sum_{n=N}^{\infty} a_k^2\Big)^{1/2} \asymp \Big\|\sum_{i=N}^{\infty} a_i r_i(t)\Big\|_{L^p}$$

A local symmetric Khintchine inequality

Theorem (Astashkin & C. (2011))

Let X be symmetric. Then

$$X \text{ satisfies a LKI} \iff \gamma_{\varphi_X} > 0$$

where $\gamma_{\varphi_X} > 0$ is the lower dilation index of the fundamental function φ_X.

$$\gamma_{\varphi_X} := \lim_{t \to 0^+} \frac{\log \mathcal{M}_{\varphi_X}(t)}{\log t}, \quad \text{for} \quad \mathcal{M}_{\varphi_X}(t) := \sup_{0<s\leq 1} \frac{\varphi_X(st)}{\varphi_X(s)}.$$

Example: for $X = L^p$ the dilation index is $\frac{1}{p}$.

The tail Rademacher multiplicator space

- For $f(t) = \sum_{j=1}^{2^n} c_j \chi_{\Delta_j^n}(t)$, step dyadic function of order n

$$\|f\|_{\Lambda_t(X)} := \sup \left\{ \Big\| f \cdot \sum_{k=n+1}^{\infty} a_k r_k \Big\|_X : \Big\| \sum_{k=n+1}^{\infty} a_k r_k \Big\|_X \leq 1 \right\}$$

- $\Lambda_t(X)$, the *tail Rademacher multiplicator space* of X, is the completion of the dyadic step functions with respect to the $\Lambda_t(X)$-norm

The tail Rademacher multiplicator space

A complicate norm in $\Lambda_t(X)$:

- $\left\| \sum_{j=1}^{2^n} c_j \chi_{\Delta_j^n}(t) \right\|_{\Lambda_t(X)} \asymp \left\| \sum_{j=1}^{2^n} c_j \log^{1/2}\left(\frac{2}{2^n t - k + 1} \right) \cdot \chi_{\Delta_j^n} \right\|_X$

- $\|\chi_E\|_{\Lambda_t(X)} \asymp \left\| \log^{1/2}\left(\frac{2m(E)}{t} \right) \chi_{[0,m(E)]} \right\|_X$

- $\left\| \sum_{j=1}^{m} \alpha_j \chi_{E_j} \right\|_{\Lambda_t(X)} \asymp \left\| \sum_{j=1}^{m} \alpha_j \log^{1/2}\left(\frac{2m(E_j)}{t - \beta_{j-1}} \right) \chi_{(\beta_{j-1},\beta_j]} \right\|_X$

for $\beta_j := \sum_1^j m(E_s)$, and $\beta_0 = 0$.

LKI and tail Rademacher multiplicator space

Theorem (Astashkin & C. (2011))

The following are equivalent:

- *The LKI holds in X,*

$$\varphi_X(m(E)) \cdot \Big(\sum_{k=N}^{\infty} a_k^2\Big)^{1/2} \asymp \Big\|\chi_E \sum_{k=N}^{\infty} a_k r_k(t)\Big\|_X$$

- $\log^{1/2}(e/t) \in \overline{L^\infty}^X$ *and* $\varphi_{\Lambda_t(X)} \asymp \varphi_X$,

$$\|\chi_{[0,a]}\|_X \asymp \Big\|\log^{1/2}\Big(\frac{2a}{t}\Big) \cdot \chi_{[0,a]}(t)\Big\|_X, \quad 0 < a \leq 1$$

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3. A weighted Khintchine inequality

L^p weighted Khintchine inequality

Theorem (Veraar, 2010)

Let $1 \leq p < \infty$. Let $w \in L^q([0,1])$, for some $q > p$, and $m(supp(w)) > 2/3$.

Then, there exists $C_1, C_2 > 0$ such that

$$C_1 \Big(\sum_{i=1}^{\infty} a_i^2 \Big)^{1/2} \leq \left(\int_0^1 \Big| \sum_{i=1}^{\infty} a_i r_i(t) \Big|^p |w(t)|^p \, dt \right)^{1/p} \leq C_2 \Big(\sum_{i=1}^{\infty} a_i^2 \Big)^{1/2},$$

for every $(a_i) \in \ell^2$.

Extension to symmetric spaces

- Restriction on the support of the weight: $m(\mathrm{supp}(w)) > 2/3$

  A result of Stechkin and Ul'yanov on zero sets of Rademacher series shows that they have measure $\leq 1/2$.

- The upper bound:

$$\sup_{a\in\ell^2} \frac{\left\| w \cdot \sum_{i=1}^{\infty} a_i r_i \right\|_X}{\left\| \sum_{i=1}^{\infty} a_i r_i \right\|_X} = \|w\|_{\Lambda(X)},$$

  where $\Lambda(X)$ is the Rademacher multiplicator space of X:

$$\Lambda(X) = \left\{ f : [0,1] \to \mathbb{R} : f \sum_{n=1}^{\infty} a_n r_n \in X \; : \; \text{for all } \sum_{n=1}^{\infty} a_n r_n \in X \right\}.$$

Symmetric weighted Khintchine inequality

Theorem (Astashkin & C., 2011)

Let X be a symmetric space with $Rad(X) \approx \ell^2$. Let $\Lambda(X)$ be the Rademacher multiplicator space of X. Consider a weight $w \in \Lambda(X)$ such that

$$\max\left\{m\big(M_\eta(w)\cap[0,\tfrac{1}{2}]\big), m\big(M_\eta(w)\cap[\tfrac{1}{2},1]\big)\right\} > \frac{1}{4}.$$

for some $\eta > 0$, where

$$M_\eta(w) := \left\{t \in [0,1] : |w(t)| \geq \eta\|w\|_{\Lambda(X)}\right\}.$$

Then, there exists constants $C_X, D_w > 0$ such that, for every $a = (a_i) \in \ell^2$,

$$D_w\|w\|_{\Lambda(X)}\|a\|_2 \leq \left\|w \cdot \sum_{i=1}^{\infty} a_i r_i\right\|_X \leq C_X\,\|w\|_{\Lambda(X)}\|a\|_2$$

Symmetric Weighted Khintchine inequality

- Regarding the Rademacher multiplicator space of X:
  - In general $\Lambda(X)$ is not symmetric and $\Lambda(X) \subseteq X$.
  - Example: $L^p(\log L)^{1/2} \subseteq \Lambda(L^p)$.
  - In particular: $\Lambda(L^p) \supseteq L^q([0,1])$, for all $q > p$.
- Extensions of the result are available, but with technical conditions on the structure of the support of the weight.

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