2nd International Workshop on Interpolation Theory, Function Spaces and Related Topics, SANTIAGO DE COMPOSTELA, Spain 2011.

Estimates for covering numbers in Schauder's theorem about adjoints of compact operators .

Michael Cwikel

Technion – Israel Institute of Technology, Haifa

http://www.math.technion.ac.il/~mcwikel/ compact

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Joint work with Eliahu Levy

BUT I WILL ALSO REPORT HERE ON TWO OTHER RECENT RESEARCH PROJECTS

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Interpolation of cocompact imbeddings.

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2nd International Workshop on Interpolation Theory, Function Spaces and Related Topics, SANTIAGO DE COMPOSTELA, Spain 2011..

CALDERÓN COUPLES OF p-CONVEXIFIED BANACH LATTICES

Eliran Avni

Technion – Israel Institute of Technology, Haifa

arXiv:1107.3238

FIRST SOME DETAILS ABOUT ELIRAN'S WORK.

Let X_0, X_1 be Banach lattices of measurable functions on some measure space (Ω, Σ, μ).
They automatically form a Banach couple. Suppose they are also a Calderón couple.
Can we manufacture a new Calderón couple from (X_0, X_1)?
Recall, for every Banach lattice X and every $p \in [1, \infty)$, the ***p-convexification*** of X is the space $X^{(p)}$ of all measurable functions $f : \Omega \to \mathbb{R}$ such that $|f|^p \in X$. It is also a Banach lattice, normed by $\|f\|_{X^{(p)}} = \||f|^p\|_X^{1/p}$.
Example: Of course $\left(L^1\right)^{(p)} = L^p$.
Conjecture: If (X_0, X_1) is a Calderón couple of Banach lattices, then so is $\left(X_0^{(p)}, X_1^{(p)}\right)$.

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Theorem (Eliran Avni): If (X_0, X_1) is a positive Calderón couple of Banach lattices, then so is $\left(X_0^{(p)}, X_1^{(p)}\right)$.
(Preliminary version for sequence spaces.)
Definition: A couple of Banach lattices (X_0, X_1) is a positive Calderón couple if, for every non negative $f, g \in X_0 + X_1$ such that

$$K(t, g; X_0, X_1) \leq K(t, f; X_0, X_1) \text{ for all } t > 0\,,$$

there exists a bounded positive linear positive operator $T : (X_0, X_1) \to (X_0, X_1)$ such that $Tf = g$.

In view of Eliran's result, and the result about another new Calderón couple which Evgeniy presented in his very attractive talk yesterday, let me show you the building where we create all these Calderón couples.

Let me also recall our very recent conference in that building, celebrating Evgeniy's research and teaching over a 50 year period. I would like to think of this conference as also being, in some way, a kind of continuation of that celebration.

Center for Mathematical Sciences

First announcement

Functional Analysis

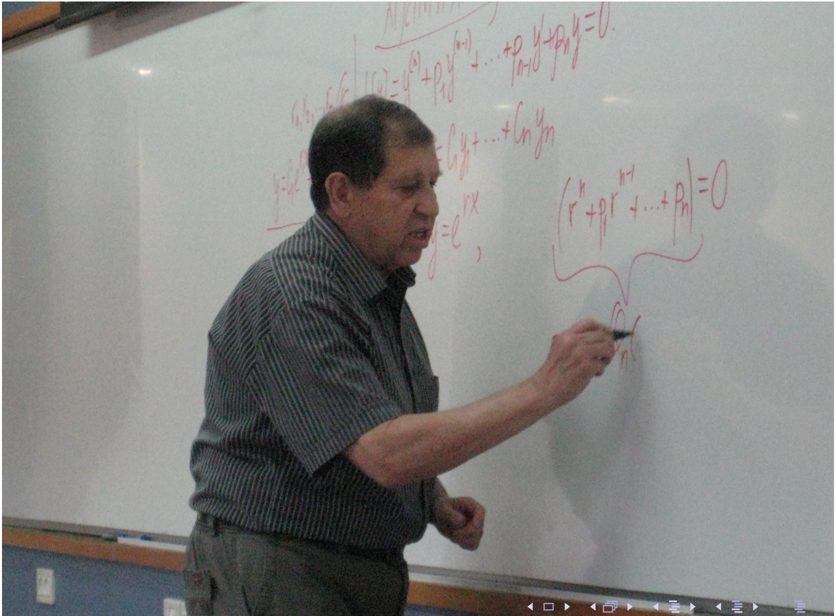
A conference in honour of Evgeniy Pustylnik

We will meet at the Technion to mark and celebrate 50 years
of Evgeniy's inspiring research and teaching.
The main events will be on Thursday May 19, 2011,
with some additional activities on May 18 and 20.

List of speakers:

Jonathan Arazy, University of Haifa
Yuri Brudnyi, Technion

You saw and heard Evgeniy in action yesterday. Here he is, similarly inspiring his students at the Technion just a few weeks ago.



And here he is at the very beginning of that career whose 50th anniversary was recently marked.

Remark: Yesterday when Evgeniy mentioned the useful role of ultrasymmetric spaces in various problems that he and other mathematicians are considering, he modestly omitted to mention that those spaces are his creation.

Now to my next topic:

2nd International Workshop on Interpolation Theory, Function Spaces and Related Topics, SANTIAGO DE COMPOSTELA, Spain 2011.

Estimates for covering numbers in Schauder's theorem about adjoints of compact operators .

Michael Cwikel and Eliahu Levy

Technion – Israel Institute of Technology, Haifa

arXiv:0810.4240v1

A rather general Arzelà-Ascoli-Schauder theorem.

The following theorem contains the classical theorems of Arzelà-Ascoli and of Schauder. It can be considered as a special case of considerably more abstract results presented in a paper by Robert G. Bartle;; and which have their roots in earlier work of R. S. Phillips, Šmulian and Kakutani.
A very nice and simple and different proof of it has been given by Eliahu Levy.

This theorem is the tool for "dualizing" in my proof that

$(*, *)$ ▶ (*Lattice Couple with Fatou property or order continuity*).

Theorem 1. Let A and B be two sets and let $h : A \times B \to \mathbb{C}$ be a function with the properties that

$$\sup_{a \in A} |h(a,b)| < \infty \text{ for each fixed } b \in B, \text{ and}$$

$$\sup_{b \in B} |h(a,b)| < \infty \text{ for each fixed } a \in A.$$

Define $d_A(a_1, a_2) := \sup_{b \in B} |h(a_1, b) - h(a_2, b)|$ for each pair of elements a_1 and a_2 in A.
Define $d_B(b_1, b_2) = \sup_{a \in A} |h(a, b_1) - h(a, b_2)|$ for each pair of elements b_1 and b_2 in B.
Then

$$(A, d_A) \text{ and } (B, d_B) \text{ are semimetric spaces}$$

(Well, that's obvious.)
and

(A, d_A) is totally bounded if and only if (B, d_B) is totally bounded.

By using the ideas of Eliahu Levy's proof we can get a quantitative version of the previous theorem.

For each $\epsilon > 0$ let $N_A(\epsilon)$ be the minimum number of d_A balls of radius ϵ required to cover A, and let $N_B(\epsilon)$ be the minimum number of d_B balls of radius ϵ required to cover B. We can find estimates connecting these two quantities.

If $N_A(\epsilon) < \infty$ then $N_B(\rho) < \infty$ for each $\rho > 2\epsilon$, and we can obtain an upper bound for $N_B(\rho)$ depending only on ρ and the quantity $\sup_{a\in A, b\in B} |h(a,b)| := C$.

In fact

$$N_B(2\epsilon + \delta) \leq \left(\frac{C}{\delta}\right)^{N_A(\epsilon)} .$$

If A is an absolutely convex subset of a linear space V over the reals and if the semimetric d_A is given by $d_A(a_1, a_2) = p(a_1 - a_2)$ for some seminorm p on V, then

$$N_B(2\epsilon + \delta) \leq \left(\frac{2\epsilon N_A(\epsilon)}{\delta}\right)^{N_A(\epsilon)} .$$

ESTIMATES FOR COVERING NUMBERS IN SCHAUDER'S THEOREM ABOUT ADJOINTS OF COMPACT OPERATORS

MICHAEL CWIKEL AND ELIAHU LEVY

Theorem 1. *Let A and B be two sets and let $h : A \times B \to \mathbb{C}$ be a function with the properties that*

$$(0.1) \qquad \sup_{a \in A} |h(a,b)| < \infty \text{ for each fixed } b \in B, \text{ and}$$

$$(0.2) \qquad \sup_{b \in B} |h(a,b)| < \infty \text{ for each fixed } a \in A.$$

Define $d_A(a_1, a_2) := \sup_{b \in B} |h(a_1, b) - h(a_2, b)|$ *for each pair of elements* a_1 *and* a_2 *in* A.

Define $d_B(b_1, b_2) = \sup_{a \in A} |h(a, b_1) - h(a, b_2)|$ *for each pair of elements* b_1 *and* b_2 *in* B.

Then (obviously!)

$$(0.3) \qquad (A, d_A) \text{ and } (B, d_B) \text{ are semimetric spaces}$$

and (not so obviously)

$$(0.4) \qquad (A, d_A) \text{ is totally bounded if and only if } (B, d_B) \text{ is totally bounded.}$$

Two slightly different covering numbers

Let (E, d) be a semimetric space.
For each $\epsilon > 0$, the ***intrinsic covering number*** $N_E(\epsilon)$ is the least positive integer n for which there exists a finite subset $F \subset E$ of cardinality n such that

$$\min_{y \in F} d(x, y) \leq \epsilon \text{ for each } x \in E \,, \tag{0.5}$$

i.e., $N_E(\epsilon)$ is the smallest n such that E is contained in some union of n closed balls of radius ϵ with centres in E
(Relevant for applications to (quantitative!) Schauder's theorem.)
For each subset G of E we define the *diameter* of G to (of course!) be the quantity

$$\operatorname{diam}(G) = \sup_{x,y \in G} d(x, y) \,.$$

For each each $\epsilon > 0$ we define the ***diameter covering number*** $N_E^{\Delta}(\epsilon)$ to be the smallest positive integer n for which there exist n subsets E_1, E_2, ..., E_n of E, each having diameter not exceeding 2ϵ and for which $E \subset \bigcup_{j=1}^{n} E_j$.

By the triangle inequality

$$N_E(2\epsilon) \le N_E^{\Delta}(\epsilon) \le N_E(\epsilon) \text{ for all } \epsilon > 0\,.$$

Both of these inequalities can be strict. We can even sometimes have $N_E(2\epsilon) = 1$ when $N_E^{\Delta}(\epsilon) = \infty$.

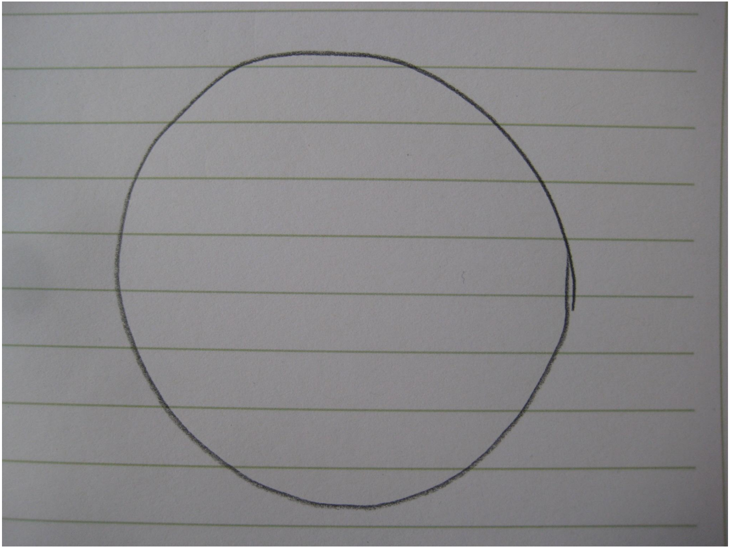
Various results here are obtained in terms of the covering numbers $N_{\mathcal{B}_{\mathbb{R}^n}}^{\Delta}(\epsilon)$ for $n = 1$ and $n = 2$, where $\mathcal{B}_{\mathbb{R}^n}$ is the closed euclidean unit ball of $\mathbb{R}^n$.

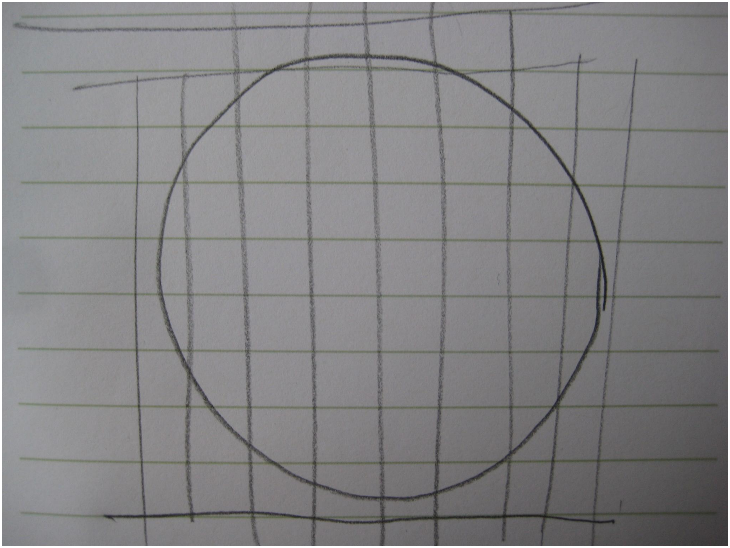
For $n = 1$, i.e., where $\mathcal{B}_{\mathbb{R}}$ is the closed interval $[-1, 1]$, it is a trivial to show that

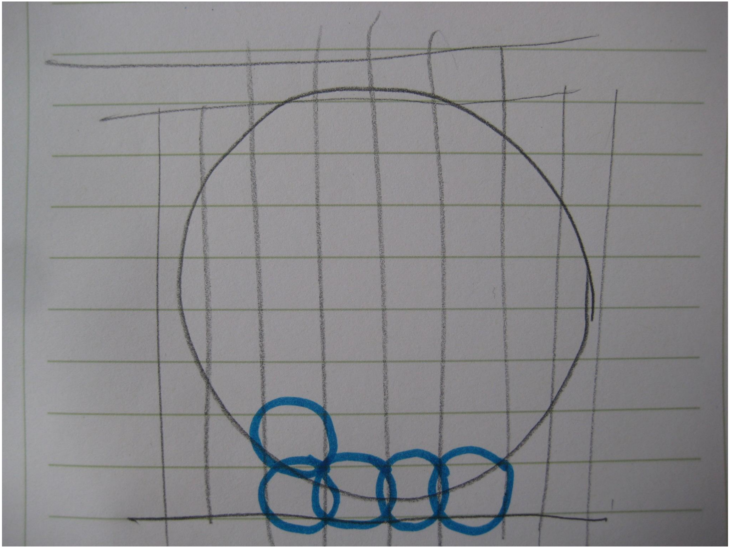
$$N_{[-1,1]}^{\Delta}(\epsilon) = N_{[-1,1]}(\epsilon) = \left\lceil \frac{1}{\epsilon} \right\rceil .$$

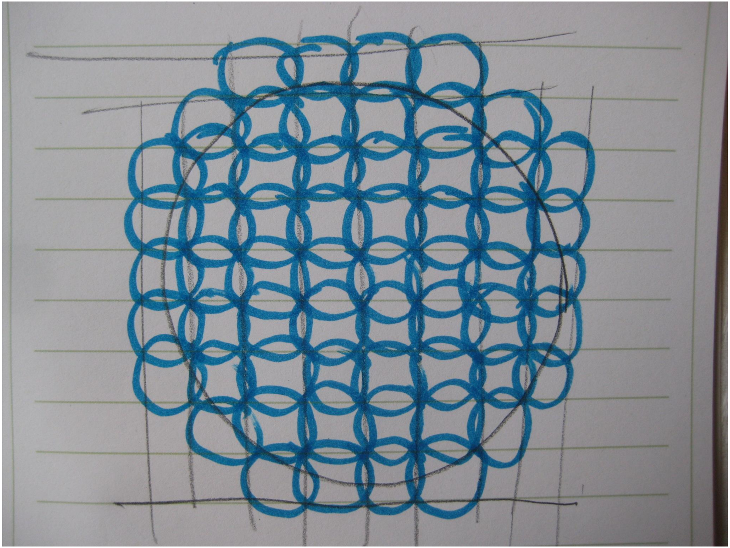
(Standard notation: For each $t \in \mathbb{R}$ we let $\lceil t \rceil$ denote the smallest integer which dominates t. This is the "*ceiling function*".)

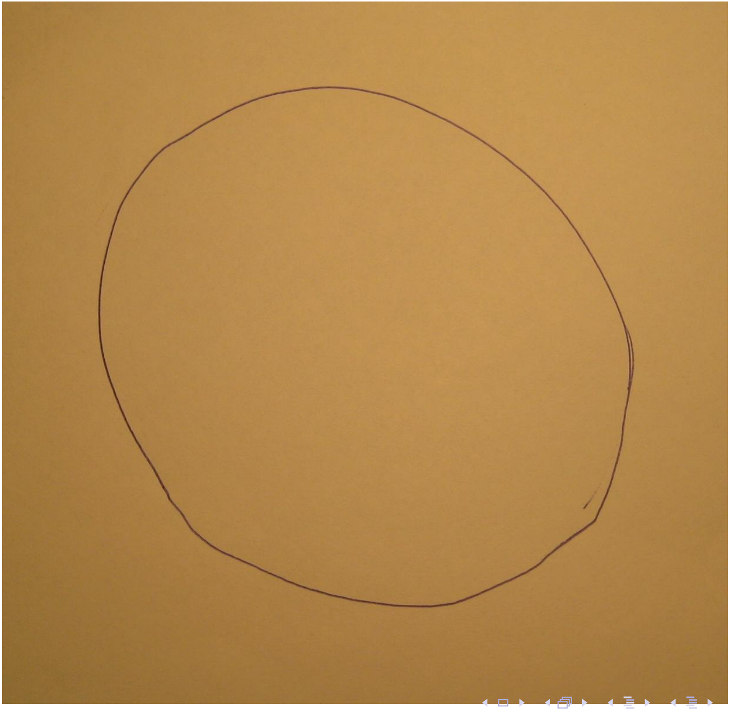
But, analogously, for $n = 2$, what is the minimal number of disks of radius ϵ needed to cover the unit disk??? "Hexagonal packing" seems? to be the best strategy for finding this???

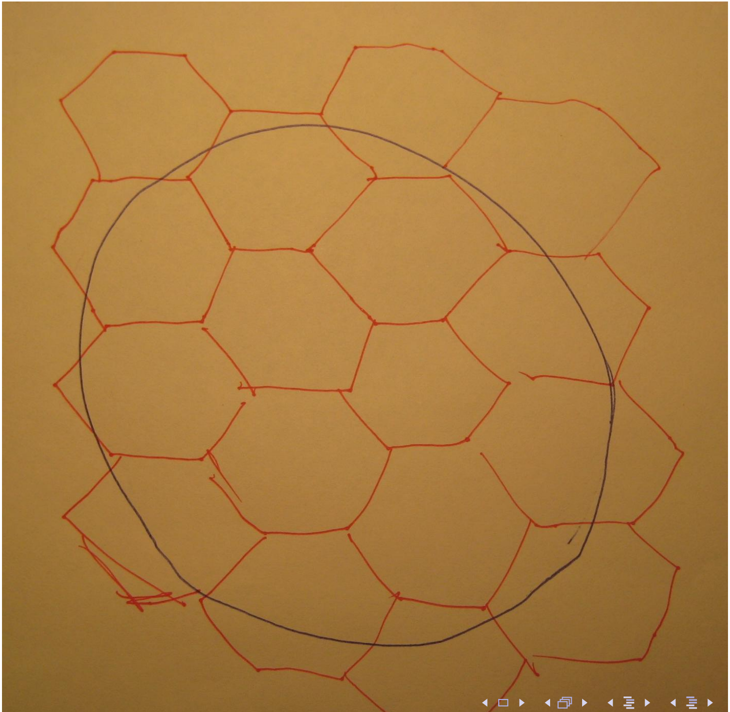


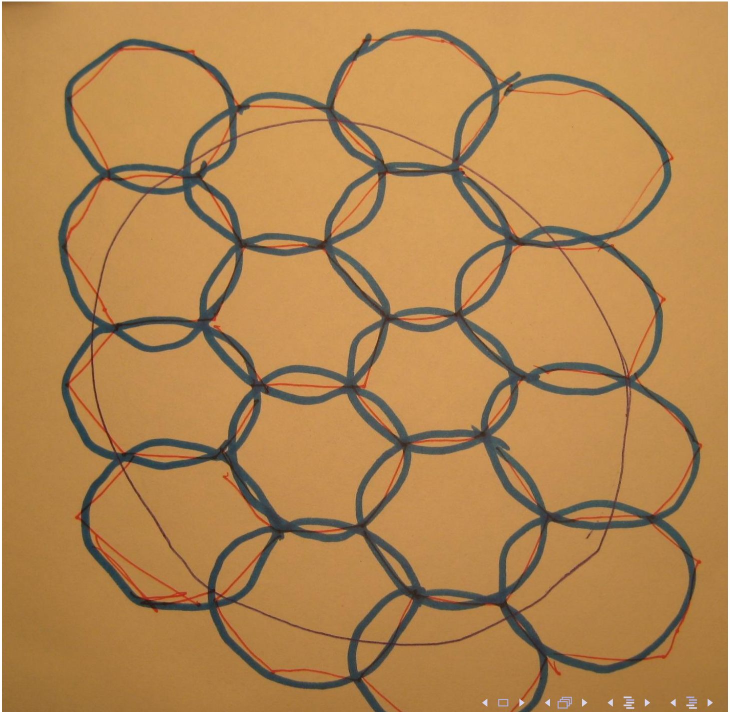












Theorem 2. *Let A and B be two sets and let $h : A \times B \to \mathbb{C}$ be a function with the properties stated in Theorem 1. Let d_A and d_B be the semimetrics defined on A and B respectively, as in Theorem 1.*
Suppose that the intrinsic covering number $N_A(\epsilon)$ is finite for some $\epsilon > 0$. Then
(i) The quantity $C := \sup_{a \in A, b \in B} |h(a,b)|$ is also finite.
(ii) The diameter covering number $N_B^{\Delta}(\rho)$ is finite for each $\rho > \epsilon$.
(EXAMPLES SHOW THAT THIS CANNOT BE WEAKENED TO $\rho \geq \epsilon$.)
(iii) Furthermore,

$$N_B^{\Delta}(\epsilon + \delta) \leq \left(\left\lceil \frac{\sqrt{2}C}{\delta} \right\rceil \right)^{2N_A(\epsilon)} \quad \textit{for each } \delta > 0\,, \tag{0.6}$$

and, if h is real valued, the following stronger estimate also holds.

$$N_B^{\Delta}(\epsilon + \delta) \leq \left(\left\lceil \frac{C}{\delta} \right\rceil \right)^{N_A(\epsilon)} \quad \textit{for each } \delta > 0\,. \tag{0.7}$$

(EXAMPLES SHOW THAT THIS ESTIMATE IS BEST POSSIBLE.)
(iv) By symmetry, the roles of A and B can be interchanged and so exactly analogous estimates hold for $N_A^{\Delta}(\epsilon + \delta)$ in terms of $N_B(\epsilon)$.

((Note that in this theorem we do not make any "compactness" or "total boundedness" assumptions about (A, d_A) or (B, d_B).))

Here is the obvious simplest way that we can apply Theorem 2.

A quantitative version of Schauder's theorem:

Let X and Y be Banach spaces and let $\mathcal{B}_X$ and $\mathcal{B}_{Y^*}$ be the closed unit balls of X and Y^* respectively.
Let $T : X \to Y$ be a bounded linear operator with adjoint $T^* : Y^* \to X^*$.
Take $A = \mathcal{B}_X$ and $B = \mathcal{B}_{Y^*}$ and choose $h(a, b) = b\,(Ta) = (T^*b)\,(a)$.
So $d_A(a_1, a_2) = \|Ta_1 - Ta_2\|_Y$ and $d_B(b_1, b_2) = \|T^*b_1 - T^*b_2\|_{X^*}$

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What are $N_A(\epsilon)$ and $N_B^{\Delta}(\epsilon)$ etc. in this context?

For each $\epsilon > 0$ let $N_T(\epsilon)$ denote the least number of closed balls in Y of radius ϵ with centres in $T\,(\mathcal{B}_X)$ which are required to cover the set $T\,(\mathcal{B}_X)$, and let $N_T^{\Delta}(\epsilon)$ denote the least number of subsets of Y each with Y-norm diameter not exceeding 2ϵ which are required to cover $T\,(\mathcal{B}_X)$.
Analogously, let $N_{T^*}(\epsilon)$ denote the least number of closed balls in X^* of radius ϵ with centres in $T^*\,(\mathcal{B}_{Y^*})$ which are required to cover the set $T^*\,(\mathcal{B}_{Y^*})$ and let $N_{T^*}^{\Delta}(\epsilon)$ denote the least number of subsets of X^* each with X^*-norm diameter not exceeding 2ϵ which are required to cover $T^*\,(\mathcal{B}_{Y^*})$.

Corollary 3. *Suppose that $N_T(\epsilon)$ is finite for some particular $\epsilon > 0$. Then $N^{\Delta}_{T^*}(\rho)$ is finite for all $\rho > \epsilon$ and the estimate*

$$N^{\Delta}_{T^*}(\epsilon+\delta) \leq \left(\left\lceil \frac{\sqrt{2}\,\|T\|_{X\to Y}}{\delta} \right\rceil \right)^{2N_T(\epsilon)} \tag{0.8}$$

holds for all $\delta > 0$.
If X and Y are real Banach spaces, then this estimate can be sharpened to

$$N^{\Delta}_{T^*}(\epsilon+\delta) \leq \left(\left\lceil \frac{\|T\|_{X\to Y}}{\delta} \right\rceil \right)^{N_T(\epsilon)} . \tag{0.9}$$

Furthermore, if $N_{T^*}(\epsilon)$ is finite for some $\epsilon > 0$, then $N^{\Delta}_{T}(\rho)$ is finite for all $\rho > \epsilon$ and the quantity $N^{\Delta}_{T}(\epsilon+\delta)$ can be estimated in terms of $N_{T^*}(\epsilon)$ via formulae exactly analogous to (0.8) and (0.9), where T and T^* are interchanged.

Apparently other results will give much better estimates than (0.8) and (0.9).
But here is a slightly more subtle variant of Corollary 3 for which, in some cases, our estimates are best possible.
With the perspective of Theorem 2 we can see that it may be just as appropriate and just as easy to work with the covering numbers of certain "significant" subsets of $T(\mathcal{B}_X)$ and of $T^*(\mathcal{B}_{Y^*})$, instead of working with the covering numbers of these sets themselves.
We will obtain new versions of the estimates (0.8) and (0.9) for $N_{T^*}^{\Delta}(\epsilon+\delta)$, which are stronger in the sense that the number $N_T(\epsilon)$ is replaced by a smaller, in some cases very much smaller number, which is the covering number of a suitable subset K of $T(\mathcal{B}_X)$.
Similarly the estimates for $N_T^{\Delta}(\epsilon+\delta)$, which were stated implicitly in Corollary 3, can be replaced by stronger results where $N_{T^*}(\epsilon)$ is replaced by the covering number of a suitable subset K^* of $T^*(\mathcal{B}_{Y^*})$.

Corollary 4. *Let X, Y, T, $N_T^{\Delta}(\epsilon)$, and $N_{T^*}^{\Delta}(\epsilon)$ all be as specified in the statement of Corollary 3.*
Let K be a "norming" subset of $T(\mathcal{B}_X)$, i.e., a subset with the property that

$$\sup\{|\langle u, y\rangle| : u \in K\} = \sup\{|\langle u, y\rangle| : u \in T(\mathcal{B}_X)\} \text{ for each } y \in Y^* . \tag{0.10}$$

Analogously, let K^* be a subset of $T^*(\mathcal{B}_{Y^*})$ with the property that

$$\sup\{|\langle x, v\rangle| : v \in K^*\} = \sup\{|\langle x, v\rangle| : v \in T^*(\mathcal{B}_{Y^*})\} \text{ for each } x \in X . \tag{0.11}$$

For each $\epsilon > 0$ let $N[K, \epsilon]$ be the least number of closed balls in Y of radius ϵ with centres in K which are required to cover the set K.
Analogously, let $N[K^*, \epsilon]$ denote the least number of closed balls in X^* of radius ϵ with centres in K^* which are required to cover the set K^*.

Suppose that $N[K,\epsilon]$ *is finite for some particular* $\epsilon > 0$.
Then $N^{\Delta}_{T^*}(\rho)$ *is finite for all* $\rho > \epsilon$ *and the estimate*

$$N^{\Delta}_{T^*}(\epsilon + \delta) \leq \left(\left\lceil \frac{\sqrt{2}\,\|T\|_{X \to Y}}{\delta} \right\rceil \right)^{2N[K,\epsilon]} . \tag{0.12}$$

holds for all $\delta > 0$.
If X *and* Y *are real Banach spaces then this estimate can be sharpened to*

$$N^{\Delta}_{T^*}(\epsilon + \delta) \leq \left(\left\lceil \frac{\|T\|_{X \to Y}}{\delta} \right\rceil \right)^{N[K,\epsilon]} . \tag{0.13}$$ *(BEST POSSIBLE!!)*

Analogously,
Suppose that $N[K^*, \epsilon]$ *is finite for some particular* $\epsilon > 0$.
Then $N_T^{\Delta}(\rho)$ *is finite for all* $\rho > \epsilon$ *and the estimate*

$$N_T^{\Delta}(\epsilon + \delta) \leq \left(\left\lceil \frac{\sqrt{2}\, \|T\|_{X \to Y}}{\delta} \right\rceil \right)^{2N[K^*, \epsilon]} \tag{0.14}$$

holds for all $\delta > 0$.
If X *and* Y *are real Banach spaces then this estimate can be sharpened to*

$$N_T^{\Delta}(\epsilon + \delta) \leq \left(\left\lceil \frac{\|T\|_{X \to Y}}{\delta} \right\rceil \right)^{N[K^*, \epsilon]} \tag{0.15}$$

..*(BEST POSSIBLE!!)*

Obviously Corollary 3 is nothing more than a special case of Corollary 4 since of course the sets $K = T(\mathcal{B}_X)$ and $K^* = T^*(\mathcal{B}_{Y^*})$ satisfy (0.10) and (0.11).
But it seems better and clearer to have begun this discussion by stating that special case separately.

Here is a natural example of a choice of K which satisfies (0.10) and for which $N[K, \epsilon]$ is very significantly smaller than $N_T(\epsilon)$.
Fix $n \in \mathbb{N}$. Let X and Y both be $\mathbb{R}^n$ equipped with the ℓ^1 norm, and let T be the identity operator on $\mathbb{R}^n$.
Let K be the subset of $\mathcal{B}_X$ which consists of the n points e_j for $j = 1, 2, ..., n$, where $e_1 = (1, 0, 0, ..., 0)$, $e_2 = (0, 1, 0, 0, ..., 0)$, ..., $e_n = (0, 0, ..., 0, 1)$.
Of course $N_T(\epsilon)$ is arbitrarily large for small values of ϵ. But $N[K, \epsilon] = n$ for all ϵ in the range $0 < \epsilon < 1$.
Of course in (0.10) we take $\langle \cdot, \cdot \rangle$ to be the usual inner product on $\mathbb{R}^n$, and so X^* and Y^* are both $\mathbb{R}^n$ equipped with the ℓ^∞ norm.
Clearly (0.10) holds here since, for each $y \in \mathbb{R}^n$, both sides of (0.10) equal $\|y\|_{\ell_n^\infty}$.
This example can be used to show that the estimate (0.13)

$$N^{\Delta}_{T^*}(\epsilon + \delta) \leq \left(\left\lceil \frac{\|T\|_{X \to Y}}{\delta} \right\rceil \right)^{N[K,\epsilon]}.$$

is best possible for certain values of the numbers ϵ and δ, and in fact for infinitely many such values, which can be taken arbitrarily small.

Now we get to the third and final topic of my talk.

2nd International Workshop on Interpolation Theory, Function Spaces and Related Topics, SANTIAGO DE COMPOSTELA, Spain 2011..

Interpolation of cocompact imbeddings.

Michael Cwikel and Kyril Tintarev

Before describing my results with Kyril it seems important to give you an overview of the general topic of co-compactness.

Here is a motivating example:

An important Sobolev embedding

Let $\mathcal{D}^{1,p}(\mathbb{R}^N)$ or $\dot{W}^{1,p}(\mathbb{R}^N)$ for $N > p$ denote the space which is the completion of C_0^∞ in the norm

$$\|u\| = \left(\int_{\mathbb{R}^N} |\nabla u|^p \mathrm{d}x \right)^{\frac{1}{p}}.$$

Consider this Sobolev embedding on $\mathbb{R}^N$:

$$(*) \qquad \mathcal{D}^{1,2}(\mathbb{R}^N) = \dot{W}^{1,2}\left(\mathbb{R}^N\right) \subset L^{2^\star}(\mathbb{R}^N)$$

where $2^\star = \frac{2N}{N-2}$ and $N > 2$.
(The square of the norm in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ is the quadratic form of the Laplacian, measures kinetic energy in quantum mechanics, or the energy of an electric field in electrodynamics, or thermodynamic energy etc.)
Is there an optimal function for this embedding?

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Is there an optimal function for this embedding?

Help!! The embedding (*) is not compact !!

But there are ways to overcome that.

The standard bubble (or instanton)

If there is a minimizer for this variational problem (on $\mathbb{R}^N$ with $N > 2$)

$$S_N = \inf_{\int |u|^{2^\star} dx = 1} \int_{\mathbb{R}^N} |\nabla u|^2 dx$$

then it will also be the optimal function that we are looking for.
The constant S_N is positive, since the space $\mathcal{D}^{1,2}(\mathbb{R}^N)$, is continuously imbedded into $L^{2^\star}(\mathbb{R}^N)$.
A solution of the Euler-Lagrange equation for the minimizer was found in 1931 by Bliss. It is $w(x) = \frac{C_N}{(1+|x|^2)^{\frac{N-2}{2}}}$ with a suitable normalization constant C_N (the "standard bubble"). But the question if the problem has a minimizer (w does not have to be one) remained open. The main technical difficulties for answering this are that,
(i) while the gradient norm is weakly lower semicontinuous, the functional $\int |u|^{2^\star} dx$ is not weakly continuous anywhere, and
(ii) the embedding $\mathcal{D}^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^\star}(\mathbb{R}^N)$ is not compact.
In 1976 G. Talenti proved existence of minimizers for this problem.

Existence of minimizers

Tools for a (streamlined) proof of Talenti's existence result:

1) A standard (Pólya-Szegő) rearrangement argument that reduces the problem to the radial subspace $\mathcal{D}^{1,2}_{\mathrm{rad}}(\mathbb{R}^N)$.

2) The Brezis-Lieb lemma: Whenever $u_k \to u$ weakly in L^p and also pointwise a.e., then

$$(BL) \qquad \int |u_k|^p = \int |u|^p + \int |u_k - u|^p + o(1)\,.$$

3) The following property: "CO-COMPACTNESS OF EMBEDDING":
Let $u_k \in \mathcal{D}^{1,2}_{\mathrm{rad}}(\mathbb{R}^N)$. If for every sequence $\{j_k\}_{k\in\mathbb{N}} \subset \mathbb{Z}$, the sequence $\left\{2^{\frac{N-2}{2}j_k} u_k(2^{j_k}x)\right\}_{k\in\mathbb{N}}$ weakly converges to zero, then $\{u_k\}_{k\in\mathbb{N}}$ converges to zero in $L^{2^\star}$ norm.
In other words: Weak convergence of u_k alone does not suffice for the $L^{2^\star}$-convergence. (The embedding $\mathcal{D}^{1,2}_{\mathrm{rad}}(\mathbb{R}^N) \hookrightarrow L^{2^\star}$ is not compact).

But weak convergence of the sequence under arbitrary rescalings, $u_k(x) \to 2^{\frac{N-2}{2}j_k} u_k(2^{j_k}x)$ does suffice.

The existence proof

Let $\{u_k\}_{k\in\mathbb{N}}$ be a minimizing sequence. Without loss of generality assume that it is weakly convergent to some $u \neq 0$.
Indeed, for any sequence $j_k \in \mathbb{Z}$, the sequence $\left\{2^{\frac{N-2}{2}j_k} u_k(2^{j_k}x)\right\}_{k\in\mathbb{N}}$ is also a minimizing sequence. However, if any such sequence converges weakly to zero, then $u_k \to 0$ in $L^{2^\star}$ and thus it is not a minimizing sequence.
Let $\int |u|^{2^\star} = t \in (0,1]$. Then, by the Brezis-Lieb lemma, $\int |u_k - u|^{2^\star} = 1 - t + o(1)$. At the same time, $S_N = \int |\nabla u_k|^2 + o(1) = \int |\nabla u|^2 + \int |\nabla(u - u_k)|^2 + o(1) \leq S_N t^{2/2^\star} + S_N(1-t)^{2/2^\star} + o(1)$.
This inequality holds only if $t = 1$, which implies that u is a minimizer. $\square$

Definition of co-compact embedding

What will it mean to say that the Banach space A is co-compactly embedded into the Banach space B?
First we need to choose a suitable group G of isometries of A. (In the previous example G was the a discrete subgroup of the group of normalized dilations.)
Then we have to define the notion of G-weak convergence in A.
DEFINITION: The sequence $\{u_n\}_{n\in\mathbb{N}}$ is G-weakly convergent to 0 if $g_n u_n$ converges weakly to 0 for EVERY choice of $g_n \in G$.
DEFINITION: Suppose $A \subset B$ continuously. Then this embedding is G-cocompact if every G-weakly convergent sequence in A is norm convergent in B.
EXAMPLE: If $G = \{\text{normalized dilations}\}$ then the embedding $\mathcal{D}^{1,2}(\mathbb{R}^N) \subset L^{2^\star}(\mathbb{R}^N)$ is G-cocompact.

Terry Tao has introduced the notion of "intermediate metric", closely related to G-weak convergence, and discusses this and "concentration compactness" (essentially co-compactness) in his blog.
http://terrytao.wordpress.com/2008/11/05/concentration-compactness-and-the-profile-decomposition/

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Concentration compactness and the profile decomposition

5 November, 2008 in expository, math.AP, math.GT | Tags: calculus of variations, concentration compactness, profile decomposition, randomness, structure | by Terence Tao

One of the most important topological concepts in analysis is that of *compactness* (as discussed for instance in my Companion article on this topic). There are various flavours of this concept, but let us focus on sequential compactness: a subset E of a topological space X is sequentially compact if every sequence in E has a convergent subsequence whose limit is also in E. This property allows one to do many things with the set E. For instance, it allows one to maximise a functional on E:

Proposition 1. (Existence of extremisers) Let E be a non-empty sequentially compact subset of a topological space X, and let

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One of the most important topological concepts in analysis is that of *compactness* (as discussed for instance in my Companion article on this topic). There are various flavours of this concept, but let us focus on sequential compactness: a subset E of a topological space X is sequentially compact if every sequence in E has a convergent subsequence whose limit is also in E. This property allows one to do many things with the set E. For instance, it allows one to maximise a functional on E:

Proposition 1. (Existence of extremisers) Let E be a non-empty sequentially compact subset of a topological space X, and let

Some more examples: Known cocompact embeddings

- Every compact embedding is G-cocompact for the group $G = \{\mathrm{Id}\}$ and therefore also for any larger group.
- The embedding $\ell^2(\mathbb{Z}) \hookrightarrow \ell^\infty(\mathbb{Z})$ is G-cocompact when G is the group of shifts by $\mathbb{Z}$.
- $C(\mathbb{R})$ is G-cocompactly embedded into itself when G is the group of shifts by $\mathbb{R}$. (Easy exercise.)
- Any Hilbert space H is G-cocompactly embedded into itself if G is the group of all unitary operators on H.

All of the above are easy exercises.

Now to hard analysis.....

Some more "serious" cocompact embeddings

- The subcritical Sobolev embeddings

$$W^{k,p}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N),\; N > kp,\; p < q < p^\star = \frac{pN}{N-kp},$$

  are cocompact with respect to shifts by $\mathbb{Z}^N$. (Essentially, this was proved already by Lieb in 1982 for $k = 1$). This property, (cf. Talenti's result above) is used in variational problems involving semilinear elliptic equations.
- The critical Sobolev imbedding $\mathcal{D}^{k,p}(\mathbb{R}^N) \hookrightarrow L^{p^\star}(\mathbb{R}^N)$ is G-cocompact where G is the product group of shifts by $\mathbb{R}^N$ and (discrete) normalised dilations. (Normalized dilations are the maps h_t defined for each fixed $t > 0$ by $h_t u(x) = t^{\frac{N-kp}{p}} u(tx)$.)
- Sobolev imbeddings involving the Laplace-Beltrami operator on a complete Riemannian manifold M are cocompact with respect to isometries of the manifold, if M is cocompact (in the "classical" sense of an infinite egg carton.)

A cocompactness result of Terence Tao

Terence Tao, A pseudoconformal compactification of the nonlinear Schrödinger equation and applications.
New York J. Math. 15 (2009) 265–282.
The Strichartz embedding (related to the time dependent Schrödinger equation for a free particle)

$$\|e^{it\Delta}u\|_{L^q(\mathbb{R}^{N+1})} \le C\|u\|_{L^2(\mathbb{R}^N)},\ q = \frac{2N+2}{N},$$

is G-cocompact where G is a product of operator groups of normalized dilations, space shifts, "time shifts" $(\hat{u}(\xi) \mapsto e^{i\tau|\xi|^2}\hat{u}(\xi),\ \tau \in \mathbb{R})$, and Fourier variable shifts".

The terminology cocompactness was not used in the original formulation of these results. They were obtained and studied within the framework of "concentration compactness", a collection of methods for dealing with problems arising in PDE where the relevant embedding is not compact.

Profile decomposition

In presence of a cocompact imbedding, the defect of compactness (difference between the sequence and its limit) admits a rather rigid structure, often called a profile decomposition.
In a very particular special case this was shown by Struwe in 1984. ("Global compactness").
Subsequently, descriptions of profile decompositions for bounded sequences in Sobolev spaces were given by P.-L. Lions, 1987 (subcritical case), and by Solimini, 1995 (critical case).
In general Hilbert space H a profile decomposition theorem was produced by Schindler and Tintarev, 2002.

One can think of the investigations of profile decompositions as an attempt to capture some of the features of the Banach-Alaoglu theorem in a more general setting.

Now I shall briefly describe some features of my paper with Kyril.

Interpolation of cocompact imbeddings.

Michael Cwikel and Kyril Tintarev

Our main theorem deals with persistence of cocompactness for interpolated spaces. It can be considered as a sort of counterpart to results about persistence of compactness for operators mapping between "real method" or "complex method" interpolation spaces, in particular obtained by Alberto Calderón and by Arne Persson, in which hypotheses having a partial analogy with the hypotheses of our main theorem are imposed.

Remark 0.1. Note however that the compactness results of Calderón and Persson were subsequently found to also hold without these kinds of hypotheses and/or under other alternative hypotheses. An analogous complete removal of additional conditions in the case of cocompactness would mean that persistence of cocompactness under interpolation holds for all choices of the group G, which remains an open question. A negative answer to it would not surprise us.

As examples of applications of our main theorem, we prove the cocompactness of classical Peetre imbeddings of inhomogeneous Sobolev spaces with fractional indices of smoothness into L^pspaces, relative to the group $G = \mathcal{D}_{\mathbb{R}^N}$ of shifts $u \mapsto u(\cdot - y)$. Analogous results for imbeddings of Besov spaces are also given.

1. Statements of the main results

In all that follows, whenever we deal with Banach spaces, whose elements are functions $u : \mathbb{R}^N \to \mathbb{C}$ and whose norms are translation invariant, we will always choose the group G in our defintions of G-weak convergence and G-cocompactness to be the set of lattice shifts. In other words, we take

$$G = \mathcal{D}_{\mathbb{Z}^N} := \{g_y\}_{y\in\mathbb{Z}^N} \text{ where } g_y u = u(\cdot - y)\,. \tag{1.1}$$

Whenever we deal here with a Banach couple (A_0, A_1) we will always associate a group G to that couple, and the elements g of G will always be assumed to be linear operators $g : A_0 + A_1 \to A_0 + A_1$, such that

$$g(A_j) \subset A_j \text{ and } g : A_j \to A_j \text{ is an isometry for } j = 0, 1\,. \tag{1.2}$$

Lemma. *Let (A_0, A_1) be a Banach couple and let G be a group of linear maps $g : A_0 + A_1 \to A_0 + A_1$ satisfying (1.2). Then each $g \in G$ is also an isometry on $A_0 + A_1$. Moreover, for every $p \in [1, \infty)$, $\theta \in (0, 1)$, the restriction of g to $(A_0, A_1)_{\theta,p}$, respectively $[A_0, A_1]_\theta$, is an isometry on $(A_0, A_1)_{\theta,p}$, respectively $[A_0, A_1]_\theta$.*

Proof. This follows immediately from the basic interpolation properties of the spaces $(A_0, A_1)_{\theta,p}$ and $[A_0, A_1]_\theta$ and $A_0 + A_1$ applied for the operators g and g^{-1}. $\square$

We now introduce a definition of an operator family whose properties (i) and (ii) below are reminiscent of various conditions imposed to obtain interpolation of compactness by Alberto Calderón and by Arne Persson. As we shall see below, the standard mollifiers in Sobolev spaces, equipped with lattice shifts, are an example of a family of operators M_t satisfying the definition.

Definition 1.1. Let (A_0, A_1) be a Banach couples with A_1 continuously imbedded in A_0 and let G be a group of linear operators $g : A_0 + A_1 \to A_0 + A_1$ which satisfies (1.2). Let A_1be continuously imbedded into some Banach space B_1. A family of bounded operators $\{M_t\}_{t\in(0,1)}$ from A_0 to A_1 is said to be a *family of G-covariant mollifiers* (relative to a space B_1) if it satisfies the following conditions:

(i) For $j = 0, 1$, the norm of M_t as a continuous map from A_j into itself is bounded independently of $t \in (0, 1)$, i.e., $\sup_{t\in(0,1)} \|M_t\|_{A_j \to A_j} < \infty$.

(ii) The function $\sigma(t) := \|I - M_t\|_{A_1 \to B_1}$ satisfies $\lim_{t\to 0} \sigma(t) = 0$.

(iii) For each $g \in G$, and $t \in (0, 1)$, there exists an element $h_{g,t} \in \mathcal{D}$ such that $gM_t = M_t h_{g,t}$.

Our main result is expressed in terms of general Banach couples.

Theorem 1.2. *Let* (A_0, A_1) *and* (B_0, B_1) *be Banach couples with* A_j *continuously imbedded in* B_j *for* $j = 0, 1$. *Suppose, further, that* A_1 *is continuously imbedded in* A_0. *Let* G *be a group of linear operators* $g : B_0 + B_1 \to B_0 + B_1$ *which satisfies (1.2) with respect to both of the couples* (A_0, A_1) *and* (B_0, B_1). *Assume that there exists a family of* G*-covariant mollifiers* $\{M_t : A_0 \to A_1\}_{t\in(0,1)}$. *(See Definition 1.1.) If, furthermore,* A_1 *is* G*-cocompactly imbedded into* B_1, *then, for every* $\theta \in (0, 1)$ *and* $q \in [1, \infty]$, *the space* $(A_0, A_1)_{\theta,q}$ *is* G*-cocompactly imbedded into* $(B_0, B_1)_{\theta,q}$ *and the space* $[A_0, A_1]_\theta$ *is* G*-cocompactly imbedded into* $[B_0, B_1]_\theta$.

We shall apply Theorem 1.2 to obtain cocompactness of interpolated imbeddings between certain function spaces. Our point of departure for doing this is the following cocompactness property of Sobolev imbeddings. It can be immediately shown to be an equivalent reformulation of Lemma 6 on p. 447 of Lieb's paper [19] and also of Lemma I.1 on p. 231 of P.-L. Lion's paper [21].

Theorem 1.3. *Suppose that $p \in (1, \infty)$. The Sobolev imbedding of $W^{1,p}(\mathbb{R}^N)$ into $L^q(\mathbb{R}^N)$, $p < q < p^*$, where $p^* = \frac{pN}{N-p}$ for $N > p$ and $p^* = \infty$ otherwise, is $\mathcal{D}_{\mathbb{Z}^N}$-cocompact.*

In the following elementary application of Theorem 1.2, we shall extend this property to the Sobolev imbedding of the spaces $W^{\alpha,p}(\mathbb{R}^N)$ for all $\alpha \in (0,\infty)$. We recall one of the equivalent definitions of the space $W^{\alpha,p}(\mathbb{R}^N)$, namely as the space of all functions $f : \mathbb{R}^N \to \mathbb{R}$ in $L^p(\mathbb{R}^N)$ whose Fourier transforms $\widehat{f}$ are such that $(1+|\xi|^2)^{\alpha/2}\widehat{f}(\xi)$ is also the Fourier transform of a function in $L^p(\mathbb{R}^N)$. This definition is valid for all real values of $\alpha > 0$, including non integer values.

We recall the Sobolev–Peetre imbedding theorem, which states that the continuous inclusion $W^{\alpha,p}(\mathbb{R}^N) \subset L^q(\mathbb{R}^N)$ holds whenever α is positive and $1 < p \le q \le p^*_\alpha$, where the critical exponent p^*_α is defined by

$$p^*_\alpha = \begin{cases} \frac{pN}{N-\alpha p} & , \quad N > \alpha p \\ \infty & , \quad N \le \alpha p \end{cases} . \tag{1.3}$$

When $\alpha = 1$ the prevalent notation is to write p^* instead of p^*_1 (as we did just above in Theorem 1.3).

Theorem 1.4. *Suppose that $\alpha \in (0,\infty)$ and $p \in (1,\infty)$. The Sobolev–Peetre imbedding of $W^{\alpha,p}(\mathbb{R}^N)$ into $L^q(\mathbb{R}^N)$ is $\mathcal{D}_{\mathbb{Z}^N}$-cocompact whenever $p < q < p^*_\alpha$. Moreover, the imbedding $W^{\alpha+\gamma,p}(\mathbb{R}^N) \subset W^{\gamma,q}(\mathbb{R}^N)$ is $\mathcal{D}_{\mathbb{Z}^N}$-cocompact for every $\gamma > 0$.*

We now state our third result, which is obtained by applying Theorem 1.2 to couples of Sobolev spaces, for which the real interpolation method yields Besov spaces. (Relevant definitions are recalled in Appendix A.) The continuity of the imbeddings considered in this theorem is due to Jawerth [17].

Theorem 1.5. *Suppose that $0 < \beta < \alpha < \infty$ and $1 < p_0 < p_1 < \infty$ and $q \in [1,\infty]$. If $\frac{N}{p_0} - \frac{N}{p_1} < \alpha - \beta$, then the continuous imbedding of $B^{\alpha,p_0,q}(\mathbb{R}^N)$ into $B^{\alpha,p_1,q}(\mathbb{R}^N)$ is $\mathcal{D}_{\mathbb{Z}^N}$-cocompact.*

Corollary 1.6. *Let α, β, p_0, p_1 and N be as in Theorem 1.5. Then the imbedding of $B^{\alpha,p_0,q_0}(\mathbb{R}^N)$ into $B^{\beta,p_1,q_1}(\mathbb{R}^N)$ is $\mathcal{D}_{\mathbb{Z}^N}$-cocompact whenever $1 \le q_0 \le q_1 \le \infty$.*

This corollary follows immediately from Proposition ??. We take $X_1 = B^{\alpha,p_0,q_0}$, $X_2 = B^{\beta,p_1,q_0}$ and $X_3 = B^{\beta,p_1,q_1}$. By Theorem 1.5, X_1 is $\mathcal{D}_{\mathbb{Z}^N}$-

This corollary follows immediately from Proposition ??. We take $X_1 = B^{\alpha,p_0,q_0}$, $X_2 = B^{\beta,p_1,q_0}$ and $X_3 = B^{\beta,p_1,q_1}$. By Theorem 1.5, X_1 is $\mathcal{D}_{\mathbb{Z}^N}$-cocompactly imbedded into X_2. The continuous imbedding $X_2 \subset X_3$ follows from (5.16) and (5.7).

Theorem 1.7. *Let $s > 0$, $1 < p < \infty$, $p < q_0 \le q < p_s^*$. Then the imbedding of $B^{s,p,q_0}(\mathbb{R}^N)$ into $L^q(\mathbb{R}^N)$ is $\mathcal{D}_{\mathbb{Z}^N}$-cocompact.*

2. The proof of Theorem 1.2

We consider the case of real interpolation. The proof for the complex case is completely analogous.

In view of the continuous imbedding $(A_0, A_1)_{\theta,q} \subset A_0 + A_1 = A_0$, it follows that, for each fixed t, the operator M_t is bounded from $(A_0, A_1)_{\theta,q}$ into A_1. Suppose that $u_k \overset{G}{\rightharpoonup} 0$ in $(A_0, A_1)_{\theta,q}$. Let $\{g_k\}_{k\in\mathbb{N}}$ be an arbitrary sequence in G. Then

$$g_k M_t u_k = M_t h_{g_k,t} u_k \tag{2.1}$$

by property (iii). Since $h_{g_k,t} u_k \rightharpoonup 0$ in $(A_0, A_1)_{\theta,q}$, we deduce that $M_t h_{g_k,t} u_k \rightharpoonup 0$ in A_1 for each fixed $t \in (0,1)$. The cocompactness of the imbedding $A_1 \subset B_1$ and (2.1) now imply that

$$\lim \|M_t u_k\|_{B_1} = 0 . \tag{2.2}$$

Muchas Gracias.

Thank you for your attention

תודה רבה.

א גרויסע דאנק.