Optimal extensions for operators on Banach function spaces

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Problem

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- X Banach function space (B.f.s.)

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B.f.s.: Banach space $X \subset L^0(\Omega, \Sigma, \mu)$ such that

$g \in L^0$, $f \in X$ and $|g| \le |f| \quad \Rightarrow \quad g \in X$ and $\|g\|_X \le \|f\|_X$

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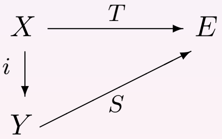

with S satisfying the same property $(*)$

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- E Banach space
- X Banach function space (B.f.s.)
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? Are there a B.f.s. Y and a linear operator S such that

$$\begin{array}{ccc} X & \xrightarrow{\ T\ } & E \\ {\scriptstyle i}\downarrow & \nearrow_{S} & \\ Y & & \end{array}$$

with S satisfying the same property $(*)$

If that is the case,

? Which is the largest of such B.f.s.' Y

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$$T\colon X(\mu) \to E \quad \text{linear} \quad \begin{cases} (\Omega, \Sigma) \text{ measurable space} \\ X(\mu) \text{ B.f.s. with a } \textbf{weak unit} \\ E \text{ Banach space} \end{cases}$$

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$$\nu_T\colon A \longrightarrow \nu_T(A) = T(\chi_A)$$

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- $\mathcal{R}_X = \{A \in \Sigma : \chi_A \in X(\mu)\} = \Sigma$ **iff** $L^\infty(\mu) \subset X(\mu)$
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$$0 \le f_n \uparrow f \text{ in the order of } X(\mu) \ \Rightarrow\ Tf_n \to Tf \text{ weakly in } E$$

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$(A_n) \subset \mathcal{R}_X$ disjoint sequence with $\cup A_n \in \mathcal{R}_X \;\Rightarrow\; \nu_T(\cup A_n) = \sum \nu_T(A_n)$

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A measurable function $f\colon \Omega \to \mathbb{R}$ is **integrable with respect to ν** if

(i) $\int |f| \, d|e^*\nu| < \infty$ for all $e^* \in E^*$

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$$\varphi = \textstyle\sum_{j=1}^n \alpha_j \chi_{A_j} \quad \Rightarrow \quad \int_A \varphi\, d\nu = \sum_{j=1}^n \alpha_j \nu(A_j \cap A)$$

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A set $A \in \mathcal{R}^{loc}$ is **ν-null** if $\nu(B) = 0$ for all $B \in \mathcal{R} \cap 2^A$

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$L^1(\nu) \subset L^1_w(\nu)$

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$L^1(\nu) = L^1_w(\nu)$ if $E \not\supset$ any copy of c_0

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- $L^1(\nu)$ and $L^1_w(\nu)$ are Banach spaces with norm

$$\|f\|_\nu = \sup_{e^* \in B_{E^*}} \int |f| \, d|e^*\nu|$$

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- $L^1(\nu)$ and $L^1_w(\nu)$ are B.f.s.' for a measure $\lambda\colon \mathcal{R}^{loc} \to [0, \infty]$,, $\lambda \approx \nu$

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- $L^1(\nu)$ is **order continuous**, i.e.

$$f_n, f \in L^1(\nu),, \ 0 \le f_n \uparrow f \ \nu\text{-a.e.} \ \Rightarrow \ f_n \to f \text{ in norm of } L^1(\nu)$$

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- The integration operator $I_\nu\colon L^1(\nu) \to E$, given by $I_\nu(f) = \int f\, d\nu$, is continuous with $\|I_\nu\| \le 1$.

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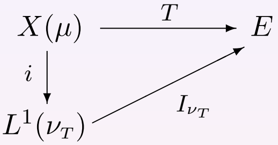
Theorem. If T is order-w continuous then



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$$\begin{array}{rcl} \nu_T : \mathcal{R}_X & \longrightarrow & E \\ A & \longrightarrow & \nu_T(A) = T(\chi_A) \end{array}$$

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$[f] \in L^0(\mu) \to [f] \in L^0(\nu_T)$ is well defined as $\mathcal{R}_X^{loc} = \Sigma$ and $\nu_T \ll \mu$

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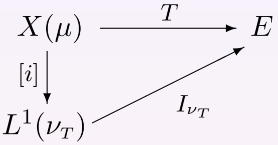

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$[f] \in L^0(\mu) \to [f] \in L^0(\nu_T)$ is well defined as $\mathcal{R}_X^{loc} = \Sigma$ and $\nu_T \ll \mu$

Extension for T

$T\colon X(\mu) \to E$ linear $\begin{cases} (\Omega, \Sigma) \text{ measurable space} \\ X(\mu) \text{ B.f.s. with a } \textbf{weak unit} \\ E \text{ Banach space} \end{cases}$

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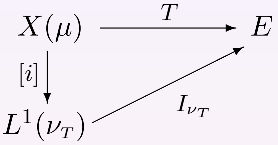

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$$X(\mu) \xrightarrow{\ T\ } E, \qquad X(\mu) \xrightarrow{\ [i]\ } L^1(\nu_T), \qquad L^1(\nu_T) \xrightarrow{\ I_{\nu_T}\ } E$$

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Remark.

- $T = I_{\nu_T} \circ [i]$ is continuous

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If X is order continuous

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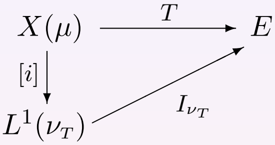

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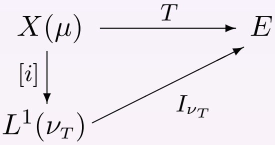

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$\Rightarrow$ $[i]\colon Z(\xi) \to L^1(\nu_T)$

and $S = I_{\nu_T}$ on $Z(\xi)$

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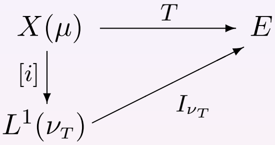

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For μ finite and $L^\infty(\mu) \subset X$:

G. P. Curbera & W. J. Ricker, **Optimal domains for kernel operators via interpolation**, Math. Nachr. **244** (2002), 47-63.

Example

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$T\colon L^p[0,\infty) \to L^p[0,\infty)$ the **Hardy operator** $(1 < p < \infty)$

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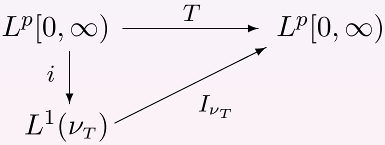

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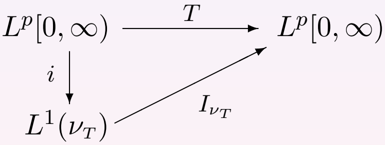

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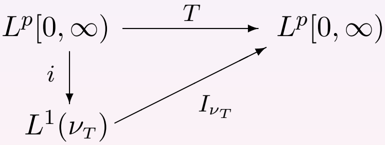

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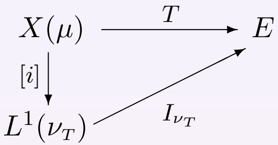
- $I_{\nu_T}(f)(x) = \frac{1}{x}\int_0^x f(y)\,dy$ for all $f \in L^1(\nu_T)$

Further extension for T

T order-w continuous



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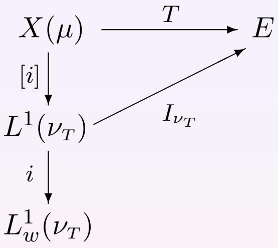

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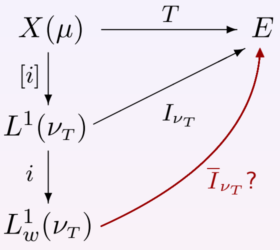

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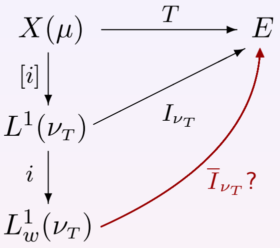

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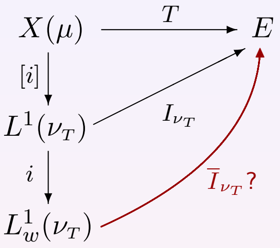

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$$E \not\supset \text{ any copy of } c_0 \;\Rightarrow\; L^1(\nu) = L^1_w(\nu)$$

Further extension for T

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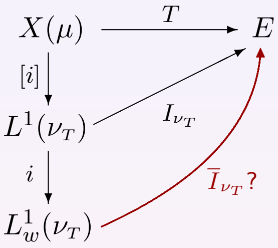

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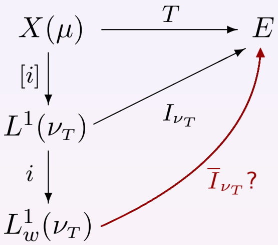

- The **Hardy operator** $T\colon L^p[0,\infty) \to L^p[0,\infty)$ has no further extension
- If $\overline{I}_{\nu_T}$ exists: $\quad \overline{I}_{\nu_T}$ order-w continuous $\quad \Leftrightarrow \quad L^1(\nu) = L^1_w(\nu)$

Extension of I_ν to $L^1_w(\nu)$

$\nu\colon \mathcal{R} \to E$ positive vector measure $\begin{cases} \mathcal{R} \text{ } \delta\text{-ring on } \Omega \\ E \text{ Banach lattice} \end{cases}$

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$$A \in \mathcal{R}^{loc},\ \ \|\nu\|(A) < \infty \qquad \Rightarrow \qquad A = (\cup_n A_n) \cup N \text{ with}$$

$$\|\nu\|(A) = \sup_{e^* \in B_{E^*}} |e^*\nu|(A) \qquad\qquad (A_n) \subset \mathcal{R} \text{ and } N \in \mathcal{R}^{loc}\ \nu\text{-null}$$

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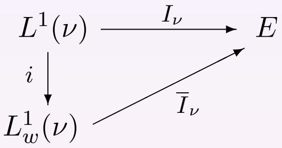

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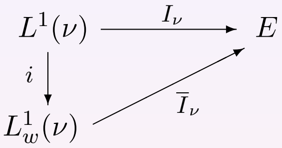

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ν locally σ-finite $\Leftrightarrow$ $L^1(\nu)$ is order dense in $L^1_w(\nu)$

J. M. Calabuig, O. D., M. A. Juan & E. A. Sánchez Pérez, **Banach lattice properties of L^1_w of a vector measure on a δ-ring**, preprint.

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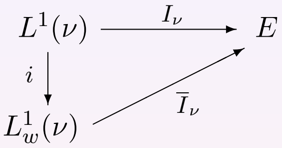

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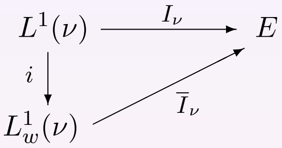

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$$\sup_n \|I_\nu(f_n)\|_E \le \sup_n \|f_n\|_\nu \le \|f\|_\nu < \infty,$$

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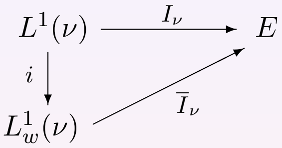

Proof. For each $0 \le f \in L^1_w(\nu)$ there exists $(f_n) \subset L^1(\nu)$,, $0 \le f_n \uparrow f$.

Then, $0 \le I_\nu(f_n) \uparrow$ and

$$\sup_n \|I_\nu(f_n)\|_E \le \sup_n \|f_n\|_\nu \le \|f\|_\nu < \infty,$$

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Extension of I_ν to $L^1_w(\nu)$

$\nu\colon \mathcal{R} \to E$ positive vector measure $\begin{cases} \mathcal{R} \text{ } \delta\text{-ring on } \Omega \\ E \text{ Banach lattice} \end{cases}$

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If $f \in L^1_w(\nu)$, then $\overline{I}_\nu(f) = \overline{I}_\nu(f^+) - \overline{I}_\nu(f^-)$ where $f = f^+ - f^-$.

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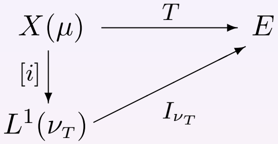
Remark. $\overline{I}_\nu$ is positive and order-order continuous, i.e.

$0 \le f_n \uparrow f$ in the order of $L^1_w(\nu)$ $\Rightarrow$ $\overline{I}_\nu(f_n) \uparrow \overline{I}_\nu(f)$ in the order of E

Further extension for T

Theorem. If $X(\mu)$ is an order continuous B.f.s. with a weak unit, E is a Banach lattice with the Fatou property and $T\colon X(\mu) \to E$ is a positive linear operator, then



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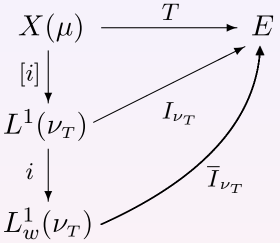

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$$\begin{array}{ccc} X(\mu) & \xrightarrow{\quad T \quad} & E \\ \big\downarrow [i] & \nearrow I_{\nu_T} & \uparrow \\ L^1(\nu_T) & & \\ \big\downarrow i & & \overline{I}_{\nu_T} \\ L^1_w(\nu_T) & \longrightarrow & \end{array}$$

The extension is **optimal** in the sense:

If $Z(\xi)$ is a B.f.s., $\xi \ll \mu$ and

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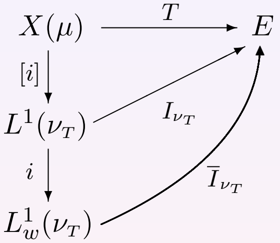

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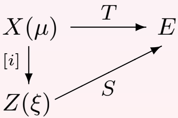

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$\Rightarrow$ $[i]\colon Z(\xi) \to L^1_w(\nu_T)$ and $S = \overline{I}_{\nu_T}$ on $Z(\xi)$

Example

$T\colon L^1[0,\infty) \to L^\infty[0,\infty)$ given by

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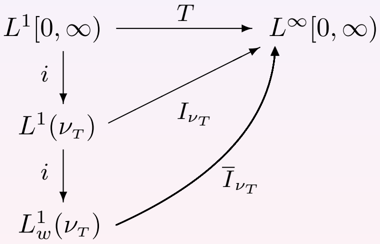

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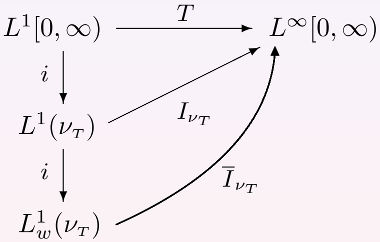

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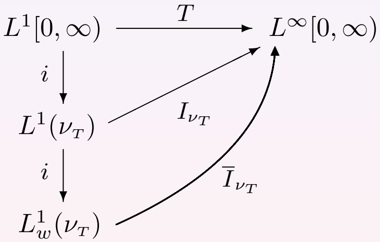

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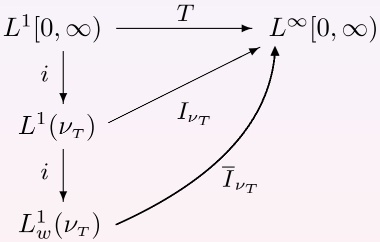

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- $\psi(x) = x+1, e^x, \ldots \ \Rightarrow\ L^1[0,\infty) \subsetneq L^1(\nu_T) \subsetneq L^1_w(\nu_T)$

Discrete case

$$T\colon X(\mu) \to E \quad \text{linear} \quad \begin{cases} (\Omega, \Sigma) \text{ measurable space} \\ X(\mu) \text{ B.f.s.} \\ E \text{ Banach space} \end{cases}$$

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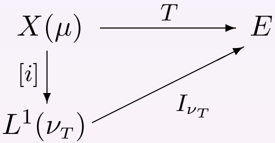

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$$\boxed{\mathcal{R}_X^{loc} = \mathcal{P}(\Omega)}$$

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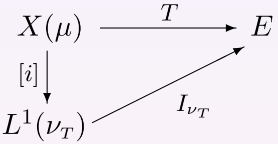

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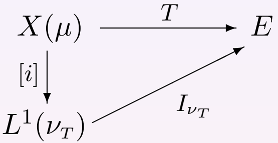

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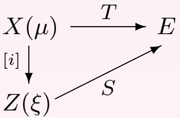

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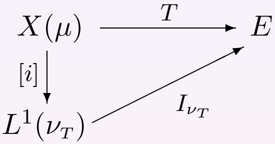

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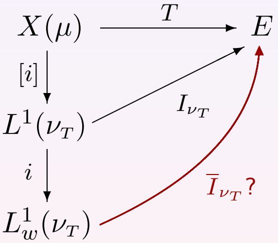

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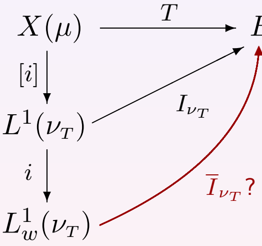

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$\Bigg\{$ Yes if E is a Banach lattice with the Fatou property, T is positive and ν_T is locally σ-finite

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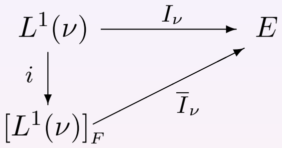

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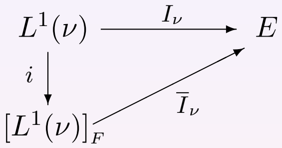

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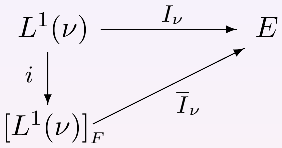

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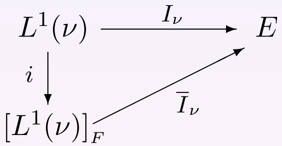
J. M. Calabuig, O. D., M. A. Juan & E. A. Sánchez Pérez, **Banach lattice properties of L^1_w of a vector measure on a δ-ring**, preprint.

Extension of I_ν

$\nu\colon \mathcal{R} \to E$ positive vector measure $\begin{cases} \mathcal{R} \ \delta\text{-ring on } \Omega \\ E \text{ Banach lattice} \end{cases}$

Proposition. If E has the Fatou property then



where $[L^1(\nu)]_F$ is the Fatou completion of $L^1(\nu)$.

Remark.

- $[L^1(\nu)]_F = L^1_w(\nu)$ if and only if ν is locally σ-finite.
- $[L^1(\nu)]_F = \{ f \in L^1_w(\nu) :\ \mathrm{Supp}(f) = (\cup A_n) \cup N,, \ A_n \in \mathcal{R},\ N\ \nu\text{-null} \}$.
- $L^1(\nu)$ is order dense in $[L^1(\nu)]_F$.

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$$\begin{array}{ccc} L^1(\nu) & \xrightarrow{\quad I_\nu \quad} & E \\ \downarrow{\scriptstyle i} & \nearrow{\scriptstyle \overline{I}_\nu} & \\ [L^1(\nu)]_F & & \end{array}$$

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- $L^1(\nu)$ is order dense in $[L^1(\nu)]_F$.
- $\overline{I}_\nu$ is positive and order-order continuous.

Discrete case

$$T\colon X(\mu) \to E \ \text{ linear } \begin{cases} (\Omega, \mathcal{P}(\Omega)) \\ X(\mu) \text{ B.f.s.,, } \chi_{\{\omega\}} \in X(\mu) \text{ and } T(\chi_{\{\omega\}}) \neq 0 \\ E \text{ Banach lattice} \end{cases}$$

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Theorem. $X(\mu)$ order continuous, E with the Fatou property, T positive. Then

$$\begin{array}{ccc} X(\mu) & \xrightarrow{\quad T \quad} & E \\ {\scriptstyle [i]}\big\downarrow & \nearrow_{I_{\nu_T}} & \\ L^1(\nu_T) & & \end{array}$$

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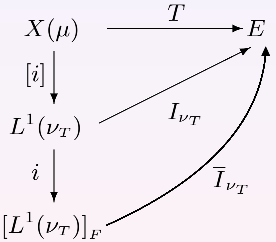

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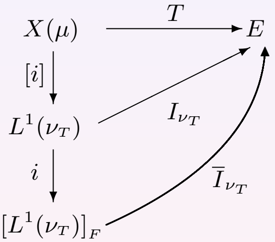
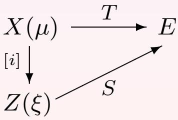


The extension is **optimal** in the sense:

$Z(\xi)$ B.f.s. with $Z(\xi)_a$ order dense, $\xi \ll \mu$ and



with S positive and order-order continuous

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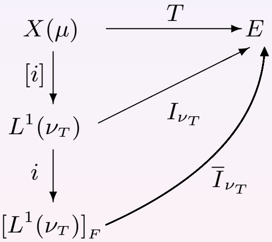

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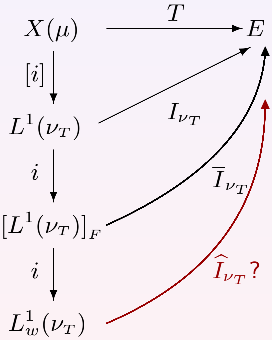

Extension of I_ν to $L^1_w(\nu)$

$\nu\colon \mathcal{R} \to E$ positive vector measure $\begin{cases} \mathcal{R} \ \delta\text{-ring on } \Omega \\ E \text{ Banach lattice} \end{cases}$

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Proposition. If E has the net-Fatou property, i.e.

$$\left.\begin{array}{c} (x_\tau) \subset E,\ \ 0 \le x_\tau \uparrow \\ \text{and } \sup_\tau \|x_\tau\|_E < \infty \end{array}\right\} \Rightarrow \begin{array}{c} \text{there exists } x = \sup_\tau x_\tau \text{ in } E \\ \text{and } \|x\|_E = \sup_\tau \|x_\tau\|_E \end{array}$$

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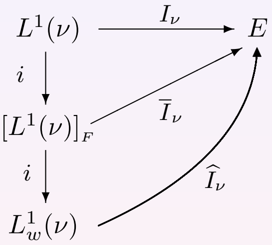

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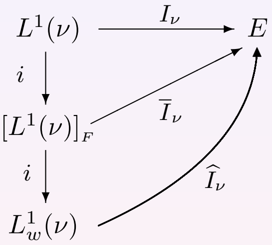

Proof. For each $0 \le f \in L^1_w(\nu)$ there exists $(f_\tau) \subset L^1(\nu)$,, $0 \le f_\tau \uparrow f$.

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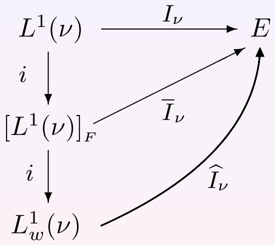

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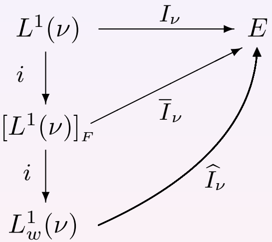

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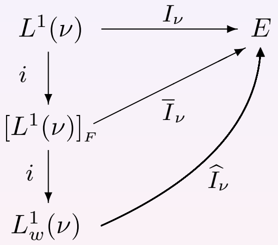

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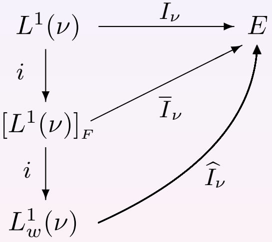

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Then, $0 \le I_\nu(f_\tau) \uparrow$ and $\sup_\tau \|I_\nu(f_\tau)\|_E \le \sup_\tau \|f_\tau\|_\nu \le \|f\|_\nu < \infty$

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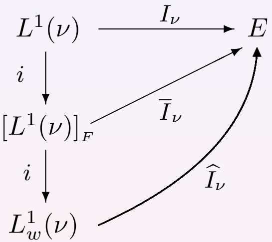

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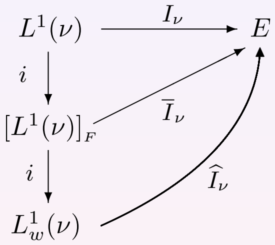

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If $f \in L^1_w(\nu)$, then $\widehat{I}_\nu(f) = \widehat{I}_\nu(f^+) - \widehat{I}_\nu(f^-)$ where $f = f^+ - f^-$.

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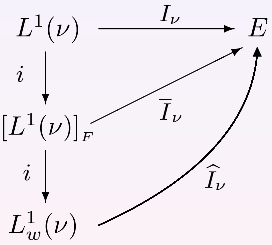

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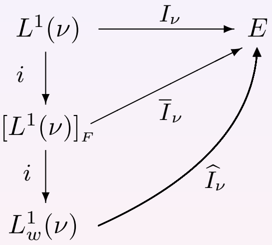

Remark. $\widehat{I}_\nu$ is positive

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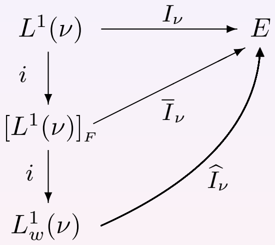


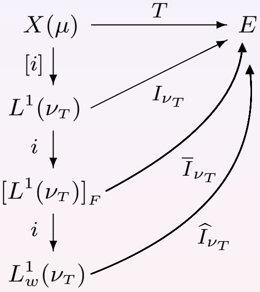
Remark. $\widehat{I}_\nu$ is positive and net-order-order continuous, i.e.

$$0 \le f_\tau \uparrow f \text{ in the order of } L^1_w(\nu) \ \Rightarrow\ \widehat{I}_\nu(f_\tau) \uparrow \widehat{I}_\nu(f) \text{ in the order of } E$$

Discrete case

$$T\colon X(\mu)\to E \quad \text{linear} \quad \begin{cases} (\Omega,\mathcal{P}(\Omega)) \\ X(\mu) \text{ B.f.s., } \chi_{\{\omega\}}\in X(\mu) \text{ and } T(\chi_{\{\omega\}})\neq 0 \\ E \text{ Banach lattice} \end{cases}$$

Theorem. $X(\mu)$ order continuous, E with the net-Fatou property, T positive,



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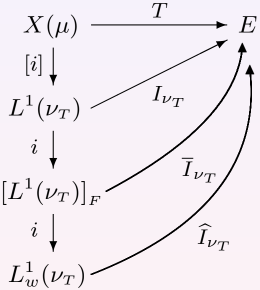

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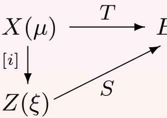

with S positive and net-order-order continuous

$\Rightarrow$ $[i]\colon Z(\xi) \to L^1_w(\nu_T)$ and $S = \widehat{I}_{\nu_T}$ on $Z(\xi)$

Example

I uncountable, $K\colon I \times I \to [0,\infty)$,, $K(\cdot,j) \neq 0 \;\; \forall\, j \in I$ and $\|K\|_\infty < \infty$.

$$\begin{aligned} T\colon \ell^1(I) &\longrightarrow \ell^\infty(I) \\ x = (x_j)_{j\in I} &\longrightarrow Tf(x) = \Big(\sum_{j\in I} x_j K(i,j)\Big)_{i\in I} \end{aligned}$$

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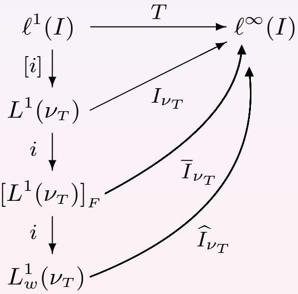

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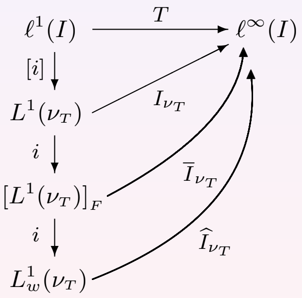

- $L^1_w(\nu_T) = \{ f\colon I \to \mathbb{R}„\; \sup_{i\in I} \sum_{j\in I} |f(j)| K(i,j) < \infty \}$

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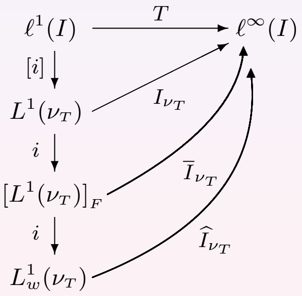

- $L^1_w(\nu_T)=\{f\colon I\to\mathbb{R}$„ $\sup_{i\in I}\sum_{j\in I}|f(j)|K(i,j)<\infty\}$
- $\widehat{I}_{\nu_T}(f)=\sum_{j\in I} f(j)K(\cdot,j)$ for all $f\in L^1_w(\nu_T)$

General case

$$T\colon X(\mu) \to E \quad \text{linear} \quad \begin{cases} (\Omega, \Sigma) \text{ measurable space} \\ X(\mu) \text{ B.f.s.} \\ E \text{ Banach space} \end{cases}$$

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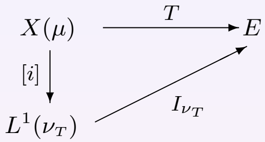

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with S order-w continuous such that

$$\sup_{B \in \mathcal{R}_X \cap \mathcal{P}(A)} |e^* S(\chi_B)| = 0 \ \Rightarrow \ e^* S(\chi_A) = 0$$

for every $e^* \in E^*$ and $A \in \Sigma$ with $\chi_A \in Z(\xi)$

$\Rightarrow$

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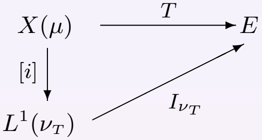

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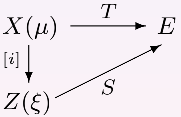

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for every $e^* \in E^*$ and $A \in \Sigma$ with $\chi_A \in Z(\xi)$

$\Rightarrow$ $[i]\colon Z(\xi) \to L^1(\nu_T)$ and $S = I_{\nu_T}$ on $Z(\xi)$

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