# Remarks on entropy numbers

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- Let $\overline{X} = (X_0, X_1)$, $\overline{Y} = (Y_0, Y_1)$ be quasi-Banach couples
- $T : \overline{X} \to \overline{Y}$ means that

$$T : X_0 + X_1 \to Y_0 + Y_1 \text{ is linear}$$

and, for $i = 0, 1$,

$$\text{restriction of } T \text{ to } X_i \text{ is bounded from } X_i \text{ to } Y_i$$

- The $K-$functional: for all $t > 0$ and $x \in X_0 + X_1$,

$$K(t, x) = \inf \left\{ \|x_0\|_{X_0} + t \|x_1\|_{X_1} : x = x_0 + x_1; x_j \in X_j \right\}$$

- Real interpolation space $(X_0, X_1)_{\theta,q}$ $(\theta \in (0,1), q \in (0,\infty])$
- all $x \in X_0 + X_1$ with finite quasinorm

$$\|x\|_{\theta,q} := \begin{cases} \left( \int_0^\infty \left( t^{-\theta} K(t,x) \right)^q \frac{dt}{t} \right)^{1/q}, & 0 < q < \infty, \\ \sup_{t>0} \left\{ t^{-\theta} K(t,x) \right\}, & q = \infty. \end{cases}$$

- Compactness (Cwickel, Cobos-Kühn-Schonbek)
- $T : \overline{X} \to \overline{Y}$

$$T : X_0 \to Y_0 \text{ compact } \implies T : (X_0, X_1)_{\theta,q} \to (Y_0, Y_1)_{\theta,q} \text{ compact}$$

- Measure of non-compactness, entropy numbers
- Let $B(X, Y)$ be family of all bounded linear maps from $X$ to $Y$, $B(X) = B(X, X)$
- $B_X$ = closed unit ball in $X$

- Let $T \in B(X, Y)$, $n \in \mathbb{N}$
- $n^{th}$ (dyadic) entropy number of $T$

$$e_n(T) := \inf \left\{ \varepsilon > 0 : T(B_X) \text{ covered by } 2^{n-1} \; Y - \text{balls of radius } \varepsilon \right\}$$

- $\{e_n(T)\}$ is monotonic decreasing

$$\beta(T) := \lim_{n \to \infty} e_n(T)$$

is the measure of non-compactness of $T$; $\beta(T) = 0 \iff T$ compact

- $\beta$ behaves well under real interpolation

$$\beta\left(T_{\theta,q}\right) \leq C\beta\left(T_0\right)^{1-\theta}\beta\left(T_1\right)^{\theta}$$

Here

$$T_{\theta,q} = T : \left(X_0, X_1\right)_{\theta,q} \to \left(Y_0, Y_1\right)_{\theta,q}, \; T_i : X_i \to Y_i$$

- Cobos, Fernández-Martínez, Martínez (1999)
- Behaviour of entropy numbers under interpolation?
- Might conjecture

$$e_{m+n-1}\left(T_{\theta,q}\right) \leq C e_m\left(T_0\right)^{1-\theta} e_n\left(T_1\right)^{\theta}$$

- Supporting evidence
- True when one end-point space is fixed (Pietsch, Haroske-Triebel).
- Agrees with known results for concrete operators

- To explain background, some notation is needed
- $p \in (0, \infty]$, $I$ non-empty countable set, $f$ scalar-valued function on $I$

$$\|f\|_{p,I} = \left(\sum_{i \in I} |f(i)|^p\right)^{1/p} (p < \infty), \quad \|f\|_{\infty,I} = \sup_{i \in I} |f(i)|$$

$$l_p(I) = \left\{ f : \|f\|_{p,I} < \infty \right\}$$

- Often write $f_i$ instead of $f(i)$. When $I = \mathbb{N}$ or $\{1, 2, ..., n\}$ write $l_p$ or $l_p^n$.Points denoted by $\{f_i\}_{i \in \mathbb{N}}$, $f_i = f(i)$
- Lorentz spaces: $p \in (0, \infty)$, $u \in (0, \infty]$

$$\|x\|_{p,u} := \left(\sum_{i=1}^{\infty} \left(i^{1/p} x_i^*\right)^u i^{-1}\right)^{1/u} \ (u < \infty), \|x\|_{p,\infty} := \sup_{i \in \mathbb{N}} i^{i/p} x_i^*$$

- Here $\{x_i^*\}_{i \in \mathbb{N}}$ is non-increasing rearrangement of $\{|x_i|\}_{i \in \mathbb{N}}$

- Given $\theta \in (0,1)$, $u \in (0,\infty]$; $p_0, p_1 \in (1,\infty)$,

$$(l_{p_0}, l_{p_1})_{\theta,u} = l_{p,u},$$

where

$$1/p = (1-\theta)/p_0 + \theta/p_1.$$

- Lorentz spaces play a big part in our analysis

- Examples
- Let $p_0, p_1, q_0, q_1, u \in [1, \infty]$

$$\alpha > \max\left(1/q_0 - 1/p_0, 1/q_1 - 1/p_1, 0\right),$$

- $0 < \theta < 1$ and

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

  Consider the diagonal map $((a_k)) \longmapsto ((\lambda_k a_k))$, where $\lambda_k \sim k^{-\alpha}$
- Let $T_0 : l_{p_0} \to l_{q_0}$, $T_1 : l_{p_1} \to l_{q_1}$, $T : (l_{p_0}, l_{p_1})_{\theta,u} \to (l_{q_0}, l_{q_1})_{\theta,u}$ be its realisations
- Then $(l_{p_0}, l_{p_1})_{\theta,u} = l_{p,u}, (l_{q_0}, l_{q_1})_{\theta,u} = l_{q,u}$, and

$$e_k(T_i) \sim k^{1/q_i - 1/p_i - \alpha} \ (i = 0, 1), e_k(T) \lesssim k^{1/q - 1/p - \alpha},$$

  so that

$$e_{k+l-1}(T) \lesssim e_k^{1-\theta}(T_0)\, e_l^{\theta}(T_1).$$

- Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$ with smooth boundary
- Let $s_0, s_1, t_0, t_1 \in \mathbb{N}_0$, $1 < r, p < \infty; \theta \in (0,1), q \in (1, \infty)$ and suppose that $s_0 - t_0, s_1 - t_1 > n\left(1/p - 1/r\right)_+$.
- Let $id_i : W_p^{s_i}(\Omega) \to W_r^{t_i}(\Omega)$ $(i = 0, 1)$, $id : \left(W_p^{s_0}(\Omega), W_p^{s_1}(\Omega)\right)_{\theta,q} \to \left(W_r^{t_0}(\Omega), W_r^{t_1}(\Omega)\right)_{\theta,q}$ be the natural embeddings.
- Then

$$\left(W_p^{s_0}(\Omega), W_p^{s_1}(\Omega)\right)_{\theta,q} = B_{p,q}^{s}(\Omega), \left(W_r^{t_0}(\Omega), W_r^{t_1}(\Omega)\right)_{\theta,q} = B_{r,q}^{t}(\Omega$$

- where $s = (1-\theta)s_0 + \theta s_1$, $t = (1-\theta)t_0 + \theta t_1$.
- Also

$$e_k\left(id_i\right) \sim k^{-(s_i - t_i)/n} \; (i = 0, 1), e_k\left(id\right) \sim k^{-(s-t)/n}.$$

Hence

$$e_{k+l-1}(id) \lesssim e_k^{1-\theta}\left(id_0\right) e_l^{\theta}\left(id_1\right).$$

- Conjecture false (Netrusov-E)
- Basic idea: use sequence spaces, combinatorial arguments
- **Theorem** For any $\lambda \in (0,1)$ there exist Banach spaces $X_0, X_1, Y_0, Y_1$ and a linear map $T : X_0 + X_1 \to Y_0 + Y_1$ such that for all $u \in (0,\infty]$, $\theta \in (0,1)$ and $n \in \mathbb{N}$,

$$e_n\left(T : X_0 \to Y_0\right) \leq c_0 n^{-\lambda},\ e_n\left(T : X_1 \to Y_1\right) \leq c_1 n^{-\lambda}$$

and

$$e_n\left(T : (X_0, X_1)_{\theta,u} \to (Y_0, Y_1)_{\theta,u}\right) \geq cn^{-\lambda}(\log n)^{\lambda}.$$

- Indication of proof
- Let $E_i$ $(i \in \mathbb{N})$ be disjoint subsets of $\mathbb{N}$ with union $\mathbb{N}$, $\sharp E_i = 2^{2^i}$ for each $i$, arranged so that every element of $E_i$ is less than every element of $E_{i+1}$.
- Let $\lambda > 0$ and define a linear operator $T$ by

$$T\left((x_i)_{i=1}^{\infty}\right) = (y_j)_{j=1}^{\infty}, y_j = 2^{-i\lambda}x_j \ \ (j \in E_i, i \in \mathbb{N}).$$

- **Lemma** Let $p, q, u, v \in (0, \infty]$ be such that $\lambda := 1/p - 1/q > 0$ and set $\sigma = 1/u - 1/v$. View $T$ as a map from $l_{p,u}(\mathbb{N})$ to $l_{q,v}(\mathbb{N})$. Then

$$e_m(T) \geq cm^{-\lambda}(\log m)^{-\sigma+\lambda} \ (m \in \mathbb{N}).$$

- Sketch of proof
- By a change of variable the required estimate from below is equivalent to

$$e_{s2^s}(T) \geq c2^{-s\lambda}s^{-\sigma} \ (s \in \mathbb{N})$$

- Given any set $E$ and any $v \in \mathbb{N}$ with $v \leq \sharp E$, put

$$\mathcal{L}(E, v) = \{E_1 \subset E : \sharp E_1 = v\}$$

- Let $s \in \mathbb{N}$ be large; let $k_0$ = least natural number such that $s \leq 2^{k_0}$.
- Lemmas of combinatorial type give the following:

- There is a set $\mathcal{A} \subset A_{k_0+1} \times A_{k_0+2} \times ... \times A_s$, each $A_i \subset \mathcal{L}(E_i, 2^{s-1})$, such that
- (i) for any distinct $F_0, F_1 \in A_i$ $(i \in \{k_0 + 1, ..., s\})$,

$$\sharp\,(F_0 \cap F_1) \leq 2^{s-i-1};$$

- (ii) for any $f_0 = (f_{0,i})_{i=k_0+1}^{s}$, $f_1 = (f_{1,i})_{i=k_0+1}^{s} \in \mathcal{A}$, $f_0 \neq f_1$,

$$\sharp\,\{i \in \{k_0 + 1, ..., s\} : f_{0,i} = f_{1,i}\} \leq (s - k_0)/2;$$

- (iii) there is a positive absolute constant $c$ such that

$$\log \sharp A_i \geq c2^s \;\; (i \in \{k_0 + 1, ..., s\});$$

- (iv) there is a positive absolute constant $c_1$ such that

$$\log \sharp\mathcal{A} \geq c_1 2^s s.$$

- For any $f = (f_i)_{i=k_0+1}^{s} \in \mathcal{A}$, let

$$x_f := \left( \sum_{i=k_0+1}^{s} 2^{-(s-i)/p} \chi_{f_i} \right) s^{-1/u}.$$

It turns out that

$$\|x_f\|_{p,u} \leq c_2, \; f \in \mathcal{A},$$

and, for distinct $f_0, f_1 \in \mathcal{A}$,

$$\|Tx_{f_0} - Tx_{f_1}\|_{q,u} \geq c_3 2^{-s\lambda} s^{-\sigma}.$$

The result follows.

- **Completion of proof of theorem**
- Let $X_0 = l_{p_0}, X_1 = l_{p_1}, Y_0 = l_{q_0}, Y_1 = l_{q_1}$, where $p_0, p_1, q_0, q_1 \in (1, \infty)$ and

$$1/p_0 - 1/q_0 = 1/p_1 - 1/q_1 = \lambda > 0,$$

Use the result of Kühn: if $0 < p < q \leq \infty$, $\lambda := 1/p - 1/q > 0$ and $D$ is the diagonal map

$$D(x_i)_{i\in\mathbb{N}} = ((\log(i+1))^{-\lambda} x_i)_{i\in\mathbb{N}},$$

then $e_k(D) \asymp k^{-\lambda}$.

- After some manipulation, this result can be applied to give

$$e_n\left(T : l_{p_j} \to l_{q_j}\right) \leq c_j n^{-\lambda} \ (j = 0, 1)$$

- Note that

$$(l_{p_0}, l_{p_1})_{\theta,u} = l_{p,u} \text{ and } (l_{q_0}, l_{q_1})_{\theta,u} = l_{q,u},$$

where

$$1/p = (1-\theta)/p_0 + \theta/p_1, 1/q = (1-\theta)/q_0 + \theta/q_1.$$

By the last lemma,

$$e_n\left(T : (l_{p_0}, l_{p_1})_{\theta,u} \to (l_{q_0}, l_{q_1})_{\theta,u}\right) \geq cn^{-\lambda}(\log n)^{\lambda}.$$