Hardy's Inequality and Curvature

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Hardy's inequality:

$$\int_\Omega |\nabla f(\mathbf{x})|^p d\mathbf{x} \geq c(n,p,\Omega) \int_\Omega \frac{|f(\mathbf{x})|^p}{\delta(\mathbf{x})^p} d\mathbf{x}, \quad f \in C_0^\infty(\Omega), \tag{1}$$

where Ω is a domain (an open connected set) in $\mathbb{R}^n$, $n \geq 2$, $1 < p < \infty$, and

$$\delta(\mathbf{x}) := \mathrm{dist}(\mathbf{x}, \mathbb{R}^\mathrm{n} \setminus \Omega) = \inf\{|\mathbf{x}-\mathbf{y}|, \ \mathbf{y} \in \mathbb{R}^\mathrm{n} \setminus \Omega\}.$$

- For a convex domain Ω in $\mathbb{R}^n$, $n \geq 2$, the optimal constant in (1) is

$$c(n,p,\Omega) = \left(\frac{p-1}{p}\right)^p. \tag{2}$$

- Brézis and Marcus (1997), Ω convex, $p = 2$,

$$\int_\Omega |\nabla f(\mathbf{x})|^2 d\mathbf{x} \geq \frac{1}{4} \int_\Omega \frac{|f(\mathbf{x})|^2}{\delta(\mathbf{x})^2} d\mathbf{x} + \lambda(\Omega) \int_\Omega |f(\mathbf{x})|^2 d\mathbf{x}, \tag{3}$$

  where

$$\lambda(\Omega) \geq \frac{1}{4\mathrm{diam}(\Omega)^2}.$$

Non-convex domains:

- a sharp constant in (1) is not known in general;
- (Ancona, 1986) for a planar simply connected domain:

$$c(2, 2, \Omega) \geq \frac{1}{16}; \tag{4}$$

- Laptev-Sobolev: "degrees of convexity";
- Annular regions: Avkhadiev-Laptev 2009.

Objective:

$$\int_{\Omega} |\nabla f(\mathbf{x})|^p d\mathbf{x} \geq c(n, p, \Omega) \int_{\Omega} \{1 + a(\delta, \partial\Omega)(\mathbf{x})\} \frac{|f(\mathbf{x})|^p}{\delta(\mathbf{x})^p} d\mathbf{x}, \qquad (5)$$

where $a(\delta, \partial\Omega)$ depends on δ and geometric properties of the boundary $\partial\Omega$ of Ω.

Skeleton $\mathcal{S}(\Omega)$ and ridge $\mathcal{R}(\Omega)$ of Ω

- $$\mathcal{S}(\Omega) := \{\mathbf{x} \in \Omega : \text{card } N(\mathbf{x}) > 1\}$$

  where $N(\mathbf{x}) := \{\mathbf{y} \in \partial\Omega : |\mathbf{y} - \mathbf{x}| = \delta(\mathbf{x})\}$, the set of *near points* of $\mathbf{x}$ on $\partial\Omega$.

- If $\mathbf{x} \in \Omega$ and $\mathbf{y} \in N(\mathbf{x})$ then $N(\mathbf{y} + t|\mathbf{x} - \mathbf{y}|) = \{y\}$ for all $t \in (0, \lambda)$, where
  $$\lambda := \sup\{t \in (0, \infty) : \mathbf{y} \in N(\mathbf{y} + t|\mathbf{x} - \mathbf{y}|)\}.$$

- $$\mathcal{R}(\Omega) := \{p(\mathbf{x}) : \mathbf{x} \in \Omega\}$$

  where $p(\mathbf{x}) := \mathbf{y} + \lambda|\mathbf{x} - \mathbf{y}|$ is called the *ridge point* of $\mathbf{x} \in \Omega$.

- (E/H, 1987): δ is differentiable at $\mathbf{x}$ if and only if $\mathbf{x} \notin \mathcal{S}(\Omega)$. In $\Omega \setminus \mathcal{S}(\Omega)$, $\nabla\delta$ is continuous and $\nabla\delta(\mathbf{x}) = (\mathbf{y} - \mathbf{x})/|\mathbf{y} - \mathbf{x}|$, where $N(\mathbf{x}) = \{\mathbf{y}\}$.
- $\mathcal{S}(\Omega)$ is of measure zero
- (D.H.Fremlin, 1997):

$$\mathcal{S}(\Omega) \subseteq \mathcal{R}(\Omega) \subseteq \overline{\mathcal{S}(\Omega)}$$

  We shall assume that $\mathcal{R}(\Omega)$ is closed relative to Ω and so $\mathcal{R}(\Omega) = \overline{\mathcal{S}(\Omega)}$

- if $\partial\Omega \in C^2$, then $\delta \in C^2$ on $\Omega \setminus \mathcal{R}(\Omega)$ ((Lewis/J.Li/Y.Li, 2011)

Lemma 1 (BEL 2011, Gilbarg/Trudinger, Appendix) Let Ω be a domain in $\mathbb{R}^n$, $n \geq 2$, with C^2 boundary, and set $\delta(\mathbf{x}) := \operatorname{dist}(\mathbf{x}, \partial\Omega)$. Then $\delta \in C^2(\Omega \setminus \mathcal{R}(\Omega))$ and

$$\Delta_{\mathbf{x}}\delta(\mathbf{x}) = \sum_{i=1}^{n-1} \left(\frac{\kappa_i}{1 + \delta\kappa_i} \right) \tag{6}$$

where the κ_i are the principal curvatures of $\partial\Omega$ at the near point $N(\mathbf{x})$ of $\mathbf{x}$. The equation (6) holds for all $\mathbf{x} \in \Omega \setminus \mathcal{R}(\Omega)$.

- The terms $\left[\kappa_i/(1+\delta\kappa_i)\right](\mathbf{y})$, $N(\mathbf{x}) = \{\mathbf{y}\}$, $i = 1, 2, \cdots, n-1$, in (6), are the principal curvatures of the level surface of δ through $\mathbf{x}$ at $\mathbf{x}$; $\frac{1}{n-1}\sum_{i=1}^{n-1}\left[\kappa_i/(1+\delta\kappa_i)\right](\mathbf{y})$ is the mean curvature of this level surface at $\mathbf{x}$.
- If Ω is convex, then $S(\bar{\Omega}^c) = R(\bar{\Omega}^c) = \emptyset$.
- If Ω is a convex domain with a C^2-boundary, then

$$\Delta\delta(\mathbf{x}) \geq 0 \text{ for all } \mathbf{x} \in \bar{\Omega}^{\mathrm{c}}, \delta(\mathbf{x}) = \operatorname{dist}(\mathbf{x}, \Omega), \tag{7}$$

$$\Delta\delta(\mathbf{x}) \leq 0 \text{ for all } \mathbf{x} \in \Omega \setminus \mathcal{R}(\Omega). \tag{8}$$

- If Ω is convex and $\partial\Omega \in C^2$, then for all i and with $N(\mathbf{x}) = \{\mathbf{y}\}$,

$$
\begin{aligned}
1 + \delta(\mathbf{x})\kappa_i(\mathbf{y}) &\geq 1, \text{for all } \mathbf{x} \in \overline{\Omega}^{\mathrm{c}},\ \delta(\mathbf{x}) = \operatorname{dist}(\mathbf{x}, \Omega) && (9)\\
1 + \delta(\mathbf{x})\kappa_i(\mathbf{y}) &> 0, \text{for all } \mathbf{x} \in \Omega \setminus \mathcal{R}(\Omega). && (10)
\end{aligned}
$$

- If Ω is convex and $\partial\Omega \in C^2$ then $\mathbf{x} \in \mathcal{R}(\Omega) \setminus \mathcal{S}(\Omega)$ if and only if for $N(\mathbf{x}) = \{\mathbf{y}\}$, and some i,

$$1 + \delta(\mathbf{x})\kappa_i(\mathbf{y}) = 0.$$

In this case $\mathcal{R}(\Omega)$ has Lebesgue measure zero.

Theorem 1 If $\partial\Omega \in C^2$, then for all $f \in \mathbb{C}_0^\infty(\Omega \setminus \mathcal{R}(\Omega))$

$$\int_\Omega |\nabla\delta \cdot \nabla f|^p d\mathbf{x} \geq \left(\frac{p-1}{p}\right)^p \int_\Omega \left\{1 - \frac{p\delta\tilde{\kappa}}{p-1}\right\} \frac{|f|^p}{\delta^p} d\mathbf{x} \tag{11}$$

where $\tilde{\kappa} := \sum_{i=1}^{n-1} \frac{\kappa_i}{1+\delta\kappa_i}$. If Ω is convex, then $\tilde{\kappa} \leq 0$. If $\mathcal{R}(\Omega)$ is "reasonably regular" then (11) holds for all $f \in C_0^\infty(\Omega)$.

Theorem 2 When Ω is a ball B_R, $\delta(\mathbf{x}) = R - |\mathbf{x}|$, $\kappa_i(\mathbf{x}) = -1/R, i = 1, 2, \cdots, n-1$, $\Delta\delta(\mathbf{x}) = -(n-1)/|\mathbf{x}|, \mathcal{R}(B_R) = \{0\}$ and for all $f \in C_0^\infty(\Omega)$,

$$\int_{B_R} |\nabla f(\mathbf{x})|^2 d\mathbf{x} \geq \frac{1}{4} \int_{B_R} \left\{\frac{(n-2)^2}{|\mathbf{x}|^2} + \frac{1}{\delta(\mathbf{x})^2} + \frac{2}{|\mathbf{x}|\delta(\mathbf{x})}\right\} |f(\mathbf{x})|^2 d\mathbf{x} \tag{12}$$

Theorem 3 Let $\Omega \subset \mathbb{R}^n, n \geq 2$, be convex with a C^2 boundary. Then for all $f \in C_0^\infty(\mathbb{R}^n \setminus \bar{\Omega})$,

$$\int_{\mathbb{R}^n\setminus\bar{\Omega}} |\nabla f(\mathbf{x})|^p d\mathbf{x} \geq \left(\frac{p-1}{p}\right)^p \int_{\mathbb{R}^n\setminus\bar{\Omega}} \left\{1 - \frac{p\tilde{\kappa}\delta}{p-1}\right\} \frac{|f(\mathbf{x})|^p}{\delta(\mathbf{x})^p} d\mathbf{x}, \tag{13}$$

where $\tilde{\kappa} = \sum_{i=1}^{n-1} \frac{\kappa_i}{1+\delta\kappa_i} \geq 0.$

Theorem 4 Let Ω be a convex domain in $\mathbb{R}^n$ with a C^2 boundary. Then for all $f \in \mathbb{C}_0^\infty(\mathbb{R}^n \setminus \bar{\Omega})$,

$$\int_{\mathbb{R}^n\setminus\bar{\Omega}} \delta^p |\nabla\delta \cdot \nabla f|^p d\mathbf{x} \geq \frac{1}{p^p} \int_{\mathbb{R}^n\setminus\bar{\Omega}} [1 + p\tilde{\kappa}\delta] |f|^p d\mathbf{x},$$

where $\tilde{\kappa} = \sum_{i=1}^{n-1} \frac{\kappa_i}{1+\delta\kappa_i} \geq 0.$

Non-convex domains: Torus

Example 1 Let $\Omega \subset \mathbb{R}^3$ be the interior of a ring torus with minor radius r and major radius $R > 2r$. Then $\Delta\delta < 0$ in $\Omega \setminus \mathcal{R}(\Omega)$ and

$$\begin{aligned} \int_\Omega |\nabla\delta \cdot \nabla f|^p d\mathbf{x} &\geq \left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|f|^p}{\delta^p} d\mathbf{x} \\ &+ \left(\frac{p-1}{p}\right)^{p-1} \int_\Omega \left(\frac{1}{(r-\delta)} - \frac{1}{\sqrt{x_1^2 + x_2^2}}\right) \frac{|f|^p}{\delta^{p-1}} d\mathbf{x} \end{aligned} \tag{14}$$

for all $f \in \mathbb{C}_0^\infty(\Omega)$, where $\mathbf{x} \in \Omega$ has co-ordinates (x_1, x_2, x_3), and the last integrand is positive.

Doubly connected domains

A domain $\Omega \subset \mathbb{R}^2 \equiv \mathbb{C}$ is *doubly connected* if its boundary is a disjoint union of 2 simple curves. If it has a smooth boundary then it can be mapped conformally onto an annulus $\Omega_{\rho,R} = B_R \setminus \overline{B_\rho} = \{z \in \mathbb{C} : \rho < |z| < R\}$, for some ρ, R.

Lemma 2 Let $\Omega_1 \subset \Omega_2 \subset \mathbb{C}$ and $B_\rho \subset B_R \subset \mathbb{C}$, $0 < \rho < R$, where B_r is the disc of radius r centered at the origin. Let

$$F : \Omega_2 \setminus \bar{\Omega}_1 \to B_R \setminus \overline{B_\rho}$$

be analytic and univalent. Then for $\mathbf{z} = x_1 + ix_2$, $\mathbf{x} = (x_1, x_2) \in \Omega_2 \setminus \bar{\Omega}_1$,

$$\mathfrak{F}(\mathbf{z}) := -\frac{|F'(\mathbf{z})|^2}{|F(\mathbf{z})|^2} + |F'(\mathbf{z})|^2 \left\{ \frac{1}{|F(\mathbf{z})| - \rho} + \frac{1}{R - |F(\mathbf{z})|} \right\}^2 \tag{15}$$

is invariant under scaling, rotation, and inversion. Hence, $\mathfrak{F}$ does not depend on the choice of the mapping F, but only on the geometry of $\Omega_2 \setminus \bar{\Omega}_1$.

Theorem 4 For $\Omega := \Omega_2 \setminus \bar{\Omega}_1 \subset \mathbb{R}^2$,

$$\int_\Omega |\nabla u(\mathbf{x})|^2 d\mathbf{x} \geq \frac{1}{4} \int_\Omega \mathfrak{F}(\mathbf{x}) |u(\mathbf{x})|^2 d\mathbf{x}.$$

Example 2 Let $\Phi(z) = (z-1)(z+1)$ and

$$\Omega = \{z : \rho^2 < |\Phi(z)| < R^2\}$$

for $0 < \rho < R$. The function $F(z) = \sqrt{\Phi(z)}$ is analytic and univalent in Ω and

$$F : \Omega \to \Omega_{\rho,R}.$$

A calculation gives

$$\begin{aligned} \mathfrak{F}(z) &= -\frac{|z|^2}{|z^2-1|^2} \\ &+ \frac{|z|^2}{|z^2-1|} \frac{(R-\rho)^2}{(\sqrt{|z|^2-1}-\rho)^2 (R-\sqrt{|z|^2-1})^2}. \end{aligned}$$