Continuous embeddings of Besov-Morrey function spaces

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2nd International Workshop on
Interpolation Theory, Function Spaces and Related Topics

Santiago de Compostela, Spain, July 18-22, 2011

joint work with Leszek Skrzypczak (Poznań)

1. Function spaces
   - 1.1 Morrey spaces
   - 1.2 (Weighted) Besov spaces
   - 1.3 Besov-Morrey spaces

2. Embeddings of spaces on $\mathbb{R}^n$
   - 2.1 Unweighted case
   - 2.2 Some weighted case

3. Outlook

Morrey spaces on $\mathbb{R}^n$

Definition

Definition 1

$0 < q \le p < \infty$, Morrey space $M_{p,q}(\mathbb{R}^n)$: $f \in L_q^{\mathrm{loc}}(\mathbb{R}^n)$ with

$$\|f|M_{p,q}(\mathbb{R}^n)\| = \sup_{x\in\mathbb{R}^n, R>0} R^{\frac{n}{p}-\frac{n}{q}} \left(\int_{B(x,R)} |f(y)|^q \,\mathrm{d}y\right)^{1/q} < \infty$$

Rem.

- Ch. Morrey (1938), J. Peetre (1969), different notation/concepts, local/global approach
- $M_{p,q}(\mathbb{R}^n) = \begin{cases} \{0\}, & p < q \\ L_p(\mathbb{R}^n), & p = q \end{cases} \quad \curvearrowright \quad q < p$ refined (local) integrability
- $p = \infty \dashrightarrow M_{p,q}(\mathbb{R}^n) = L_\infty(\mathbb{R}^n)$

Convention: all spaces on $\mathbb{R}^n$ in the sequel

Morrey spaces

Basic properties

- (quasi-) Banach spaces (Banach spaces for $q \geq 1$)
- $L_p = M_{p,p} \hookrightarrow M_{p,q_1} \hookrightarrow M_{p,q_2}$ if $0 < q_2 \leq q_1 \leq p$
- $L_{p,\infty} \hookrightarrow M_{p,q}$, $0 < q < p < \infty$

Rem. Stummel(-Kato) classes $S_{\alpha,q}$: $f \in L_q^{\mathrm{loc}}$ with

$$\lim_{\varepsilon \to 0} \eta_{\alpha,q} f(\varepsilon) = \lim_{\varepsilon \to 0} \left(\sup_{x \in \mathbb{R}^n} \int_{B(x,\varepsilon)} |f(y)|^q |x-y|^{\alpha-n} \, \mathrm{d}y \right)^{1/q} = 0$$

Ragusa/Zamboni (2001): $0 < q < p < \infty$, $n\frac{q}{p} < \alpha < n$

$$\curvearrowright \quad M_{p,q} \hookrightarrow S_{\alpha,q}, \quad \eta_{\alpha,q} f(\varepsilon) \leq C \varepsilon^{\frac{\alpha}{q} - \frac{n}{p}} \|f|M_{p,q}\|$$

(Weighted) Besov Spaces $B^s_{p,r}(\mathbb{R}^n, w)$, $w \in \mathcal{A}_\infty$

$$\|f|L_p(\mathbb{R}^n, w)\| = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)\,\mathrm{d}x\right)^{1/p}, \quad 0 < p < \infty$$

$0 < p < \infty,\ 0 < r \le \infty,\ s \in \mathbb{R},\ (\varphi_j)_j$ smooth dyadic partition of unity

$$\left\|f|B^s_{p,r}(\mathbb{R}^n, w)\right\| = \left\|\left(2^{js}\left\|\mathcal{F}^{-1}(\varphi_j \mathcal{F} f)|L_p(\mathbb{R}^n, w)\right\|\right)_j |\ell_r\right\|$$

Rem.

- $B^s_{p,r}$, $0 < p, r \le \infty$, $s > n\left(\frac{1}{p} - 1\right)_+$, $m \in \mathbb{N}$, $m > s$:

$$\left\|f|B^s_{p,r}\right\| \sim \|f|L_p\| + \left(\int_0^1 t^{-sr} \sup_{|h| \le t} \|\Delta_h^m f|L_p\|^r \frac{\mathrm{d}t}{t}\right)^{\frac{1}{r}}$$

  $B^s_{\infty,\infty} = \mathcal{C}^s$, $s > 0$ Hölder-Zygmund spaces

- $\mathcal{A}_\infty = \bigcup_{p>1} \mathcal{A}_p$, $\qquad w \in \mathcal{A}_p$, $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$:

$$\left(\frac{1}{|B|}\int_B w(x)\,\mathrm{d}x\right)^{1/p} \left(\frac{1}{|B|}\int_B w(x)^{-p'/p}\,\mathrm{d}x\right)^{1/p'} \le A$$

Weighted Spaces of type $B^s_{p,r}(\mathbb{R}^n, w)$, $w \in \mathcal{A}_\infty$

Sequence spaces

$m \in \mathbb{Z}^n$, $\nu \in \mathbb{Z}$: n-dimensional dyadic cube $Q_{\nu,m} = \prod_{i=1}^n \left[\frac{m_i}{2^\nu}, \frac{m_i+1}{2^\nu}\right)$

$\chi^{(p)}_{\nu,m} = 2^{\frac{\nu n}{p}} \chi_{Q_{\nu,m}}$, $\quad w(Q) = \int_Q w(y)\,\mathrm{d}y$

$0 < p < \infty$, $0 < r \le \infty$, $w \in \mathcal{A}_\infty$, $\sigma \in \mathbb{R}$, $\quad \lambda = \{\lambda_{j,m}\}_{j,m} \subset \mathbb{C}$

$\lambda = \{\lambda_m\}_{m\in\mathbb{Z}^n} \in \ell_p(w)$

$$\iff \|\lambda|\ell_p(w)\| = \Big\| \sum_{m\in\mathbb{Z}^n} \lambda_m \chi_{0,m} | L_p(w) \Big\| \sim \Big(\sum_{m\in\mathbb{Z}^n} |\lambda_m|^p \; w(Q_{0,m}) \Big)^{\frac{1}{p}} < \infty$$

$\lambda \in b^\sigma_{p,r}(w)$

$$\iff \|\lambda|b^\sigma_{p,r}(w)\| = \Big\| \Big\{ 2^{j(\sigma - \frac{n}{p})} \Big\| \sum_{m\in\mathbb{Z}^n} \lambda_{j,m}\, \chi^{(p)}_{j,m} | L_p(w) \Big\| \Big\}_{j\in\mathbb{N}_0} | \ell_r \Big\| < \infty$$

$$\sim \Big\| \Big\{ 2^{j\sigma} \Big(\sum_{m\in\mathbb{Z}^n} |\lambda_{j,m}|^p \; w(Q_{j,m}) \Big)^{1/p} \Big\}_{j\in\mathbb{N}_0} | \ell_r \Big\|$$

Weighted Besov Spaces $B^s_{p,r}(\mathbb{R}^n, w)$, $w \in \mathcal{A}_\infty$

Wavelet characterisation

Proposition 2 (H./Skrzypczak, 2008)

$0 < p < \infty$, $0 < r \leq \infty$, $w \in \mathcal{A}_\infty$, $s \in \mathbb{R}$

$$B^s_{p,r}(w) \quad \text{isomorphic to} \quad \ell_p(w) \oplus \bigoplus_{j=1}^{2^n-1} b^\sigma_{p,r}(w), \quad \sigma = s + \frac{n}{2}$$

Rem. Daubechies wavelets

Besov-Morrey spaces on $\mathbb{R}^n$

Definition & basic properties

Definition 3

$0 < q \leq p < \infty$, $0 < r \leq \infty$, $s \in \mathbb{R}$, $(\varphi_j)_j$ dyadic partition of unity

Besov-Morrey space $MB^{s,r}_{p,q}(\mathbb{R}^n)$: $f \in \mathcal{S}'(\mathbb{R}^n)$ with

$$\left\| f \,|MB^{s,r}_{p,q}\right\| = \left\| \left(2^{js} \left\|\mathcal{F}^{-1}(\varphi_j \mathcal{F} f)|M_{p,q}\right\|\right)_j |\ell_r\right\| < \infty$$

Rem. different approaches, notation, ..., e.g. Kozono/Yamazaki (1994), Mazzucato (2003), Tang/Xu (2005), Sawano/Tanaka (2007-), Yuan/Sickel/Yang (2010-), ...

- independence of $(\varphi_j)_j$, quasi-Banach spaces (Banach for $q, r \geq 1$)
- $MB^{s,r}_{p,p} = B^s_{p,r}$, $\mathcal{S} \hookrightarrow MB^{s,r}_{p,q} \hookrightarrow \mathcal{S}'$, first embedding dense if $r < \infty$
- $MB^{s+\varepsilon,r_1}_{p,q} \hookrightarrow MB^{s,r_2}_{p,q}$, $\varepsilon > 0$; $MB^{s,r_1}_{p,q} \hookrightarrow MB^{s,r_2}_{p,q}$ if $r_1 \leq r_2$

Besov-Morrey spaces on $\mathbb{R}^n$

Sequence spaces & Wavelet characterisation

$$0 < q \le p < \infty,\ 0 < r \le \infty,\ \sigma \in \mathbb{R}, \qquad \lambda = \{\lambda_{j,m}\}_{j,m} \subset \mathbb{C}$$

$\lambda \in Mb^{\sigma,r}_{p,q}$

$$\iff \|\lambda|Mb^{\sigma,r}_{p,q}\| = \left\| \left\{ 2^{j(\sigma-\frac{n}{p})} \left\| \sum_{m\in\mathbb{Z}^n} \lambda_{j,m}\, \chi^{(p)}_{j,m} | M_{p,q} \right\| \right\}_{j\in\mathbb{N}_0} |\ell_r \right\| < \infty$$

$$\sim \left\| \left\{ 2^{j(\sigma-\frac{n}{p})} \sup_{\substack{\nu \le j \\ k \in \mathbb{Z}^n}} 2^{n(j-\nu)(\frac{1}{p}-\frac{1}{q})} \Bigg(\sum_{\substack{m \in \mathbb{Z}^n : \\ Q_{j,m} \subset Q_{\nu,k}}} |\lambda_{j,m}|^q \Bigg)^{\frac{1}{q}} \right\}_j |\ell_r \right\|$$

Proposition 4 (Sawano, 2008)

$$MB^{s,r}_{p,q} \quad \text{isomorphic to} \quad \ell_p \oplus \bigoplus_{j=1}^{2^n-1} Mb^{\sigma,r}_{p,q}, \qquad \sigma = s + \frac{n}{2}$$

Embeddings of Besov-Morrey spaces: the unweighted case

The 'classical' Besov space situation

$s_i \in \mathbb{R}$, $0 < r_i \le \infty$, $0 < p_i \le \infty$, $i = 1, 2$, $\delta = s_1 - \frac{n}{p_1} - s_2 + \frac{n}{p_2}$

$$\mathrm{id}_{BB} : B^{s_1}_{p_1,r_1} \hookrightarrow B^{s_2}_{p_2,r_2}$$

if, and only if,

$$p_1 \le p_2$$

and

$$\left\{2^{-j\delta}\right\}_{j\in\mathbb{N}_0} \in \ell_{r^*}, \quad \frac{1}{r^*} = \left(\frac{1}{r_2} - \frac{1}{r_1}\right)_+$$

The embedding id_{BB} is never compact.

Embeddings of Besov-Morrey spaces: the unweighted case

Sequence space version

Proposition 5

$s_i \in \mathbb{R}$, $0 < q_i \le p_i < \infty$, $0 < r_i \le \infty$, $i = 1, 2$, $\frac{1}{r^*} = \left(\frac{1}{r_2} - \frac{1}{r_1}\right)_+$

$$\mathrm{id}_{mm} : Mb^{s_1,r_1}_{p_1,q_1} \hookrightarrow Mb^{s_2,r_2}_{p_2,q_2}$$

if, and only if,

$$p_1 \le p_2, \qquad \frac{q_2}{p_2} \le \frac{q_1}{p_1}$$

and

$$\left\{2^{-j\delta}\right\}_{j\in\mathbb{N}_0} \in \ell_{r^*}$$

The embedding id_{mm} is never compact.

Embeddings of Besov-Morrey spaces: the unweighted case

Function spaces

Theorem 6

$s_i \in \mathbb{R}$, $0 < q_i \le p_i < \infty$, $0 < r_i \le \infty$, $i = 1, 2$, $\frac{1}{r^*} = \left(\frac{1}{r_2} - \frac{1}{r_1}\right)_+$

$$\mathrm{id}_{MM} : MB^{s_1,r_1}_{p_1,q_1} \hookrightarrow MB^{s_2,r_2}_{p_2,q_2}$$

if, and only if,

$$p_1 \le p_2, \qquad \frac{q_2}{p_2} \le \frac{q_1}{p_1}$$

and

$$\left\{2^{-j\delta}\right\}_{j\in\mathbb{N}_0} \in \ell_{r^*}$$

The embedding id_{MM} is never compact.

Rem. partial result by Sawano/Sugano/Tanaka (2009)

Embeddings of Besov-Morrey spaces: the unweighted case

Some consequences

$s_i \in \mathbb{R}$, $0 < r_i \le \infty$, $0 < q_i \le p_i < \infty$, $i = 1, 2$

$$\mathcal{M}B^{s_1,r_1}_{p_1,q_1} \hookrightarrow \mathcal{M}B^{s_2,r_2}_{p_2,q_2} \iff p_1 \le p_2,\ \frac{q_2}{p_2} \le \frac{q_1}{p_1},\ \left\{2^{-j\delta}\right\}_j \in \ell_{r^*}$$

- $q_1 \ge q_2$: $\quad \mathcal{M}B^{s_1,r_1}_{p_1,q_1} \hookrightarrow \mathcal{M}B^{s_2,r_2}_{p_2,q_2} \iff B^{s_1}_{p_1,r_1} \hookrightarrow B^{s_2}_{p_2,r_2}$
- $p_1 = q_1$: $\quad B^{s_1}_{p_1,r_1} \hookrightarrow \mathcal{M}B^{s_2,r_2}_{p_2,q_2} \iff B^{s_1}_{p_1,r_1} \hookrightarrow B^{s_2}_{p_2,r_2}$
- $p_2 = q_2$: $\quad \mathcal{M}B^{s_1,r_1}_{p_1,q_1} \hookrightarrow B^{s_2}_{p_2,r_2} \iff \mathcal{M}B^{s_1,r_1}_{p_1,q_1} = B^{s_1}_{p_1,r_1} \hookrightarrow B^{s_2}_{p_2,r_2}$,
  in particular,
  $$\mathcal{M}B^{s_1,r_1}_{p_1,q_1} \hookrightarrow B^{s_2}_{p_2,r_2} \implies p_1 = q_1$$

Rem. $\quad s \in \mathbb{R}$, $0 < r \le \infty$, $0 < q < p < \infty$: $\mathcal{M}B^{s,r}_{p,q} \not\hookrightarrow \bigcup_{\sigma,v,u} B^{\sigma}_{v,u}$

Embeddings of Besov-Morrey spaces: some weighted case

An example (H./Skrzypczak, 2008)

$s_i \in \mathbb{R}$, $0 < p_i, r_i \leq \infty$, $i = 1, 2$, $\frac{1}{r^*} = \left(\frac{1}{r_2} - \frac{1}{r_1}\right)_+$, $\frac{1}{p^*} = \left(\frac{1}{p_2} - \frac{1}{p_1}\right)_+$

$w(x) = |x|^\alpha \in \mathcal{A}_\infty \iff \alpha > -n$

$$\mathrm{id}^\alpha_{BB} : B^{s_1}_{p_1,r_1}(|x|^\alpha) \hookrightarrow B^{s_2}_{p_2,r_2}$$

if, and only if,

$$\left\{2^{-j(\frac{\alpha}{p_1} - \frac{n}{p^*})}\right\}_{j\in\mathbb{N}_0} \in \ell_{p^*}$$

and

$$\left\{2^{-j(\delta - \frac{\alpha}{p_1})}\right\}_{j\in\mathbb{N}_0} \in \ell_{r^*}$$

The embedding id^α_{BB} is compact if, and only if, $\delta > \frac{\alpha}{p_1} > \frac{n}{p^*}$.

Embeddings of Besov-Morrey spaces: some weighted case

An example

Proposition 7

$s_i \in \mathbb{R}$, $0 < r_i \leq \infty$, $i = 1, 2$, $0 < p_1 < \infty$, $0 < q_2 < p_2 < \infty$, $\alpha > -n$

$$\mathrm{id}^{\alpha}_{BM} : B^{s_1}_{p_1, r_1}(|x|^{\alpha}) \hookrightarrow MB^{s_2, r_2}_{p_2, q_2}$$

if, and only if,

$$\frac{\alpha}{p_1} \geq \frac{n}{p^*}$$

and

$$\left\{ 2^{-j(\delta - \frac{\alpha}{p_1})} \right\}_{j \in \mathbb{N}_0} \in \ell_{r^*}$$

The embedding $\mathrm{id}^{\alpha}_{BM}$ is compact if, and only if, $\delta > \frac{\alpha}{p_1} > \frac{n}{p^*}$, i.e., $\mathrm{id}^{\alpha}_{BM}$ is compact if, and only if, $\mathrm{id}^{\alpha}_{BB}$ is compact.

Embeddings of Besov-Morrey spaces: some weighted case

The general criterion

Proposition 8

$0 < p_1 < \infty$, $0 < q_2 \le p_2 < \infty$, $s_i \in \mathbb{R}$, $0 < r_i \le \infty$, $w \in \mathcal{A}_\infty$

$$\mathrm{id}^w_{BM} : B^{s_1}_{p_1,r_1}(w) \hookrightarrow MB^{s_2,r_2}_{p_2,q_2}$$

if, and only if,

$$\left\{ 2^{-j\delta} \sup_{\substack{\nu \le j \\ k \in \mathbb{Z}^n}} 2^{n(\frac{1}{p_2}-\frac{1}{q_2})(j-\nu)} \left\| \left\{ \left(\frac{|Q_{j,m}|}{w(Q_{j,m})} \right)^{\frac{1}{p_1}} : Q_{j,m} \subset Q_{\nu,k} \right\}_m |\ell_{u^*} \right\| \right\}_j \in \ell_{r^*}$$

where $\frac{1}{u^*} = (\frac{1}{q_2} - \frac{1}{p_1})_+$

Rem. $p_2 = q_2$: H./Skrzypczak (2008), Kühn/Leopold/Sickel/Skrzypczak (2006), also criterion for compactness

3. Outlook

- continuity / compactness of

$$\mathrm{id}^{w}_{MM} : MB^{s_1,r_1}_{p_1,q_1}(w_1) \hookrightarrow MB^{s_2,r_2}_{p_2,q_2}(w_2)$$

  i.e., when $q_i < p_i$, $i = 1, 2$
- more general / other weight classes
- situation on bounded domains: continuity, compactness, entropy numbers, approximation numbers, applications ...
- limiting / sharp embeddings
- spaces of type $MF^{s,r}_{p,q}$, ...
- Thank you very much for your attention!