On uniform non-ℓ_1^n-ness for direct sums of Banach spaces

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The purpose of this talk

We shall present some recent results on uniform non ℓ_1^n-ness for ψ-direct sums of Banach spaces and especially for the ℓ_1- and ℓ_∞-sums.

Some applications will be mentioned concerning super-reflexivity and FPP.

1. Preliminary definitions and facts

X: a real Banach space with dim $X \geq 2$.

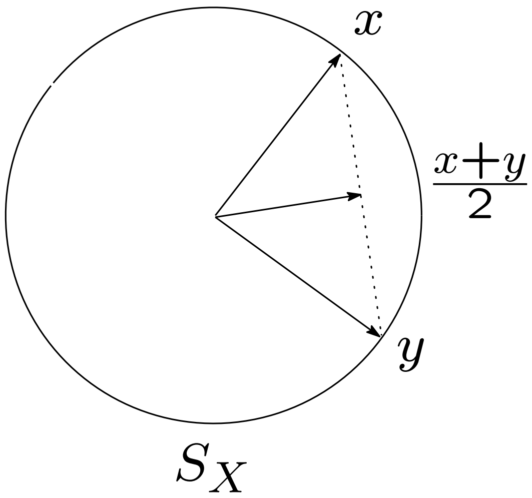
B_X: the closed unit ball of X

S_X: the unit sphere of X

Definition 1.1

X is called strictly convex (SC) provided

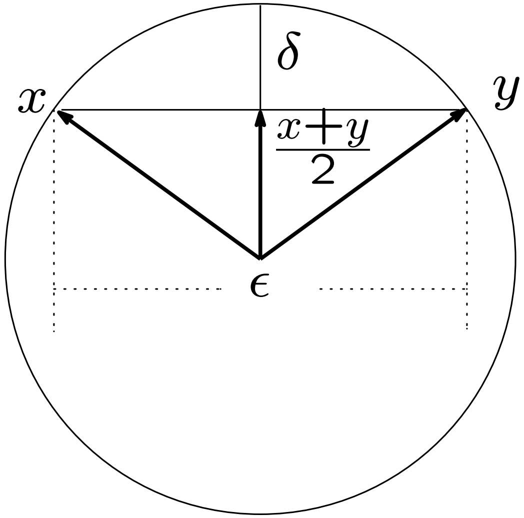
$$x,\ y \in S_X,\ x \neq y \implies \left\| \frac{x+y}{2} \right\| < 1$$



Definition 1.2

X is called uniformly convex (UC) provided for any ϵ $(0 < \epsilon < 2)$ there exists δ $(0 < \delta < 1)$ such that

$$x,\ y \in S_X, \|x-y\| \geq \epsilon \Rightarrow \left\|\frac{x+y}{2}\right\| \leq 1-\delta$$



Definition 1.3 (James, 1964)

X is called uniformly non-square (UNS)
provided there exists δ $(0 < \delta < 1)$ such that

$$x,\ y \in S_X,\ \left\|\frac{x-y}{2}\right\| \geq 1-\delta \Rightarrow \left\|\frac{x+y}{2}\right\| \leq 1-\delta$$

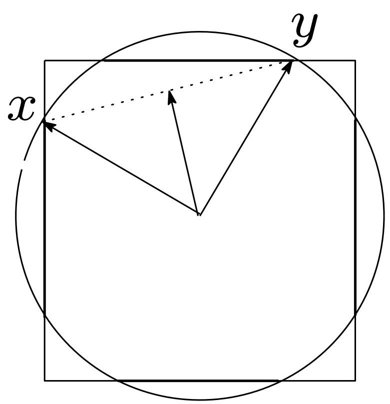
Or equivalently, provided $\exists\ \epsilon > 0$ and $\delta > 0$ such that

$$x, y \in S_X,\ \|x-y\| \geq \epsilon \quad \Rightarrow \quad \left\|\frac{x+y}{2}\right\| \leq 1-\delta$$

UNS vs UC

X: UC $\iff$ $\forall\,\varepsilon\ \exists\,\delta$ s.t. $\|x-y\| \geq \epsilon,\ x,y \in S_X \Rightarrow \left\|\dfrac{x+y}{2}\right\| \leq 1-\delta$

X: UNS $\iff$ $\exists\,\varepsilon,\ \delta$ s.t. $\|x-y\| \geq \epsilon,\ x,y \in S_X \Rightarrow \left\|\dfrac{x+y}{2}\right\| \leq 1-\delta$



In the above formulation of UNS we cannot let $\epsilon > 0$ tend to 0. That is, we have the same conclusion as UC for all $x, y \in S_X$ which are "apart from each other to some extent".
Thus UC implies UNS.

UNS vs SC

There is no implications between UNS and SC.

(We shall see this in the examples below.)

Example 1.1
(i) Let $1 < p < \infty$. Then L_p, ℓ_p are UC, and hence SC.
(ii) $L_1, L_\infty, \ell_1, \ell_\infty, c_0$ are not SC (not UC).

Example 1.2 Consider the following norms on $C[0, 1]$.

(i) $\|f\|_\infty = \max\{|f(t)| : 0 \leq t \leq 1\}$: not SC

(ii) $\|f\|_2 = \left\{\int_0^1 |f(t)|^2 \, dt\right\}^{1/2}$: SC (UC).

(iii) $\|f\|_0 := \|f\|_\infty + \|f\|_2$ is SC, but not UNS and hene not UC.

Example 1.3 Let $1 < \lambda < \sqrt{2}$.
Let $X_{2,\lambda} := (\ell_2, \|\cdot\|_\lambda)$, where

$$\|(\xi_n)\|_\lambda = \max\{\|(\xi_n)\|_2, \lambda\|(\xi_n)\|_\infty\}$$

Then $X_{2,\lambda}$ is UNS, but not SC.

Connection to approximation problems

Theorem A (Approximation to a finite dimensional subspace)

Let F be a finite dimensional subspace of X.
Then for any $x \in X$ there exists $y_0 \in F$ such that

$$\|x - y_0\| \leq \|x - y\| \quad \text{for all } y \in F.$$

(y_0 is called a best approximation of x in F.)

If X is strictly convex, we have a unique best approximation!

Theorem B (Approximation to a closed convex subset)

Let X be uniformly convex.

Let K be a nonempty closed convex subset of X.

Then for any $x \in X$ there exists a unique $y_0 \in K$ such that

$$\|x - y_0\| \leq \|x - y\| \quad \text{for all } y \in K$$

Connection to the fixed point property (FPP)

Theorem C (J.García-Falset et al., 2006)

If X is uniformly non-square, X has FPP for nonexpansive mappings.

Connection to reflexivity

定理 D (James, 1964)

Uniformly non-square spaces are reflexive.

Thus

$$X\text{: UC} \Longrightarrow X\text{: UNS} \Longrightarrow X\text{: reflexive}$$

Example 1.4 Consider the space

$$X_{2,\lambda} = (\ell_2, \|\cdot\|_\lambda), \quad 1 < \lambda < \sqrt{2},$$

where $\|(\xi_n)\|_\lambda = \max\{\|(\xi_n)\|_2, \lambda\|(\xi_n)\|_\infty\}$.

Then:

(i) If $1 < \lambda < \sqrt{2}$,

$X_{2,\lambda}$ is UNS, but not SC, whence not UC.

(ii) If $\lambda = \sqrt{2}$,

$X_{2,\sqrt{2}}$ is reflexive, but not UNS.

Connection to super-reflexivity

Theorem E

The following are equivalent.

(i) X is super-reflexive

(ii) X admits an equivalent UC norm (Enflo, 1972)

(iii) X admits an equivalent UNS norm (James, 1972)

2. Uniformly non-ℓ_1^n spaces

X: *UNS* $\iff$ $\exists \delta > 0$ s.t.

$$\|x\| = \|y\| = 1, \;\; \left\|\frac{x-y}{2}\right\| \geq 1 - \delta \implies \left\|\frac{x+y}{2}\right\| \leq 1 - \delta$$

$$\min\{\|x-y\|, \;\|x+y\| \leq 2(1-\delta)$$

X: *UNS* $\iff$ $\exists \delta > 0$ s.t. $\forall \; x, y \in S_X$

$$\min\{\|x+y\|, \;\|x-y\|\} \leq 2(1-\delta)$$

Definition 2.1

X is called *uniformly non-ℓ_1^n*

provided there exists ϵ $(0 < \epsilon < 1)$ such that

$\forall x_1, \cdots, x_n \in S_X$

$\exists \theta = (\theta_j)$ (an n-tuple of signs) for which

$$\left\| \sum_{j=1}^{n} \theta_j x_j \right\| \leq n(1-\epsilon).$$

If $n = 2$, *uniform non-ℓ_1^2-ness* coincides with *UNS-ness*.

If $n = 3$, *uniform non-ℓ_1^3* spaces are called *uniformly non-octahedral*.

If $n = 1$, the formal definition is possible, but no Banach space is *uniformly non-ℓ_1^1*.

Proposition A

X: uniformly non-ℓ_1^n $\Longrightarrow$ X: uniformly non-ℓ_1^{n+1}

Example 2.1

The space ℓ_1^n is uniformly non-ℓ_1^{n+1}, but not uniformly non-ℓ_1^n.

Why we discuss the uniform non ℓ_1^n-ness:

(i) X: UNS (uniformly non ℓ_1^2) $\Longrightarrow$ X: reflexive

X: uniformly non ℓ_1^3 $\Longrightarrow$ X: reflexive ?

This is not true ! (James [6])

(ii) X: UNS $\Longrightarrow$ X: FPP (2006)

X: uniformly non ℓ_1^3 $\Longrightarrow$ X: FPP ?

This is not known !

3. ψ-direct sums of Banach spaces

Absolute norms on $\mathbb{C}^2$

Definition 3.1 Let $\|\cdot\|$ be a norm on $\mathbb{C}^2$.

(i) $\|\cdot\|$: *absolute* $\Longleftrightarrow$ $\|(z,w)\| = \|(|z|,|w|)\| \quad (\forall z,\ w \in \mathbb{C})$

(ii) $\|\cdot\|$: *normalized* $\Longleftrightarrow$ $\|(1,0)\| = \|(0,1)\| = 1$

$N_a :=$ {all absolute normalized norms on $\mathbb{C}^2$}

Example 3.1 ℓ_p-norms ($1 \leq p \leq \infty$):

$$\|(z,w)\|_p = \begin{cases} (|z|^p + |w|^p)^{1/p} & \text{if } 1 \leq p < \infty, \\ \max(|z|, |w|) & \text{if } p = \infty. \end{cases}$$

Lemma A (Bonsall-Duncan, LN in 1973)

For all $\|\cdot\| \in N_a$

$$\|\cdot\|_\infty \leq \|\cdot\| \leq \|\cdot\|_1$$

Lemma B1 (B-D)

For any $\|\cdot\| \in N_a$ let

$$\psi(t) = \|(1-t,\, t)\| \quad (0 \le t \le 1) \tag{1}$$

Then $\psi(t)$ is convex (continuous) on $[0,1]$,

$$\begin{cases} \psi(0) = \psi(1) = 1, \\ \max\{1-t,\, t\} \le \psi(t) \le 1 \end{cases} \tag{2}$$

Ψ:={all convex functions on $[0,1]$ satisfying (2)}

To see the converse the following observation indicates how one should construct a norm in N_a from a given convex function $\psi \in \Psi$:

For a given $\|\cdot\| \in N_a$ let

$$\psi(t) = \|(1-t,\, t)\| \quad (0 \leq t \leq 1).$$

Then for all $(z, w) \neq (0, 0)$

$$\begin{aligned} \|(z,w)\| &= (|z|+|w|)\left\|\left(\frac{|z|}{|z|+|w|},\ \frac{|w|}{|z|+|w|}\right)\right\| \\ &= (|z|+|w|)\psi\left(\frac{|w|}{|z|+|w|}\right) \end{aligned}$$

Lemma B2 (B-D)

For any $\psi \in \Psi$ define

$$\|(z,w)\|_\psi = \begin{cases} (|z|+|w|)\psi\left(\dfrac{|w|}{|z|+|w|}\right) & \\ \qquad\qquad \text{if } (z,w) \neq (0,0), & \\ 0 \qquad\qquad \text{if } (z,w) = (0,0). & \end{cases} \tag{3}$$

Then $\|\cdot\|_\psi \in N_a$ and

$$\psi(t) = \|(1-t,\, t)\|_\psi \quad (0 \le t \le 1) \tag{1'}$$

By Lemmas B1 and B2

N_a and Ψ are in 1-1 correspondence with the equation (1'):

$$\|\cdot\| = \|\cdot\|_\psi \quad \longleftrightarrow \quad \psi$$

Example 3.2

The convex function corresponding to the ℓ_p-norm is given by

$$\psi_p(t) := \|(1-t,\ t)\|_p$$

$$= \begin{cases} \{(1-t)^p + t^p\}^{1/p} & \text{if } 1 \le p < \infty, \\ \max\{1-t,\ t\} & \text{if } p = \infty \end{cases}$$

Example 3.4 (Lorentz $\ell_{p,q}$ norm). Let $1 \le q \le p \le \infty$ and let

$$\|(z,\, w)\|_{p,q} := \begin{cases} \left\{z^{*q} + 2^{(q/p)-1} w^{*q}\right\}^{1/q} & \text{if } q < \infty, \\ \max\{z^*, 2^{1/p} w^*\} & \text{if } q = \infty, \end{cases}$$

where $\{z^*, w^*\}$ is the non-increasing rearrangement of $\{|z|, |w|\}$, that is, $z^* \ge w^*$.

Then $\|\cdot\|_{p,q} \in N_a$ and the corresponding function ($q < \infty$) is

$$\begin{aligned} \psi_{p,q}(t) &:= \|(1-t, t)\|_{p,q} \\ &= \begin{cases} \{(1-t)^q + 2^{q/p-1} t^q\}^{1/q} & \text{if } 0 \le t \le \frac{1}{2}, \\ \{t^q + 2^{q/p-1}(1-t)^q\}^{1/q} & \text{if } \frac{1}{2} \le t \le 1. \end{cases} \end{aligned}$$

If $p = q$, this norm coincides with the ℓ_p-norm.

ψ-Direct Sums of Banach Spaces

Definition 3.2 (Takahashi-Kato-Saito, JIA, 2002)

The ψ-direct sum $X \oplus_\psi Y$ of Banach spaces X and Y is the direct sum $X \oplus Y$ equipped with the norm:

$$\|(x, y)\|_\psi := \|(\|x\|, \|y\|)\|_\psi$$

Proposition 1.

X, Y: Banach spaces $\Rightarrow$ $X \oplus_\psi Y$ is a Banach space.

Example 3.5 (ℓ_p-sum).

The ℓ_p-sum $X \oplus_p Y := (X \oplus Y, \|\cdot\|_p)$ is the ψ_p-direct sum $X \oplus_{\psi_p} Y$.

Example 3.6 ($\ell_{p,q}$-sum).

Let $1 \leq q \leq p \leq \infty$.

The $\ell_{p,q}$-sum $X \oplus_{p,q} Y := (X \oplus Y,\ \| \cdot \|_{p,q})$ is the $\psi_{p,q}$-direct sum $X \oplus_{\psi_{p,q}} Y$

Theorem 3.1 (Takahashi-Kato-Saito 2002; Saito-Kato 2003)

(i) $X \oplus_\psi Y$: *SC*

$\Longleftrightarrow$ X, Y *are SC and* ψ *is strictly convex.*

(ii) $X \oplus_\psi Y$: *UC*

$\Longleftrightarrow$ X, Y *are UC and* ψ *is strictly convex.*

Theorem 3.2 (Kato-Saito-Tamura, MIA, 2004)

$X \oplus_\psi Y$: *UNS* $\iff$ X, Y are *UNS* and $\psi \neq \psi_1, \psi_\infty$.

Corollary 3.1

Let $1 \leq q \leq p \leq \infty$ *and neither* $p = q = 1$ *nor* $p = q = \infty$.

Then

(i) $X \oplus_{p,q} Y : SC \iff X, Y$: *SC*

(ii) $X \oplus_{p,q} Y : UC \iff X, Y$: *UC*

(iii) $X \oplus_{p,q} Y$: *UNS* $\iff X, Y$: *UNS*

The same is true for the ℓ_p*-sum* $X \oplus_p Y$, $1 < p < \infty$, *as the case* $p = q$.

Recall

Theorem 3.2

$X \oplus_\psi Y$: *UNS* $\iff$ X, Y: *UNS and* $\psi \neq \psi_1, \psi_\infty$.

This is extended to the uniform non-ℓ_1^n-ness.

Theorem 3.3 (Kato-Saito-Tamura, JNCA, 2010)

Assume that neither X nor Y is uniformly non-ℓ_1^{n-1}.

Then the following are equivalent.

(i) *$X \oplus_\psi Y$ is uniformly non-ℓ_1^n.*

(ii) *X and Y are uniformly non-ℓ_1^n and $\psi \neq \psi_1, \psi_\infty$.*

Remark 3.1

We cannot remove the condition,
neither X nor Y is uniformly non-ℓ_1^{n-1} for (i) $\Rightarrow$ (ii).

Theorem 3.2 implies that

$X \oplus_1 Y$ and $X \oplus_\infty Y$ cannot be UNS for all X and Y.

(Also recall that ℓ_1^2 and ℓ_∞^2 are not UNS.)

Theorem 3.3 indicates that

if either X or Y is uniformly non-ℓ_1^{n-1},

$X \oplus_1 Y$ and $X \oplus_\infty Y$ can be uniformly non-ℓ_1^n ($n \geq 3$).

Our next concern is these extreme cases !

4. ℓ_1- and ℓ_∞-sums

ℓ_1-sum

Theorem 4.1 (Kato-Tamura, Comment. Math. 2007; cf.JNCA)

The following are equivalent.

(i) $X \oplus_1 Y$ *is uniformly non-*ℓ_1^3*.*

(ii) X *and* Y *are uniformly non-square.*

Theorem 4.1 (ibidem)

The following are equivalent.

(i) $X \oplus_1 Y$ *is uniformly non-*ℓ_1^3.

(ii) X *and* Y *are uniformly non-square.*

Theorem 4.2 (ibidem)

The following are equivalent.

(i) $X \oplus_1 Y$ *is uniformly non-*ℓ_1^n.

(ii) *There exist* $n_1, n_2 \in \mathbb{N}$ *with* $n_1 + n_2 = n - 1$ *such that*

X *is uniformly non-*$\ell_1^{n_1+1}$ *and*

Y *is uniformly non-*$\ell_1^{n_2+1}$.

Corollary 4.1 (ibidem)

Let $X \oplus_1 Y$ *be uniformly non-*ℓ_1^n*.*

Then both of X *and* Y *are uniformly non-*ℓ_1^{n-1}*.*

ℓ_∞-sum

Recall the ℓ_1-sum case (Theorem 4.1):

$$X, Y\text{: UNS} \iff X \oplus_1 Y\text{: uniformly non-}\ell_1^3$$

Theorem 4.3 (ibidem.)

X, Y: *UNS* $\Longrightarrow$ $X \oplus_\infty Y$: *uniformly non-*ℓ_1^3

The converse is not true !

For three Banach spaces we obtain the next result,
which is interesting in contrast with the above ℓ_1-sum case.

Theorem 4,4 (Kato-Tamura, Comment. Math., 2009; cf.JNCA)

For three Banach spaces X, Y and Z

the following are equivalent.

(i) $(X \oplus Y \oplus Z)_\infty$ *is uniformly non-*ℓ_1^3*.*

(ii) X, Y *and* Z *are UNS.*

5. More satisfactory results on ℓ_1- and ℓ_∞-sums

Theorem 5.1 (Kato-Tamura, Comment Math., 2007)

The following are equivalent.

(i) $(X_1 \oplus \cdots \oplus X_m)_1$ *is uniformly non-*ℓ_1^n.

(ii) *There exist* m *positive integers* $n_1, \ldots, n_m$ *with* $n_1 + n_2 + \cdots + n_m = n - 1$ *such that*

$$X_i \text{ is uniformly non-}\ell_1^{n_i+1} \text{ for all } 1 \leq i \leq m.$$

Recall that

$(X_1 \oplus \cdots \oplus X_m)_1$ is not uniformly non-ℓ_1^m.

For the uniform non-ℓ_1^{m+1}-ness we have the following.

Theorem 5.2 (Kato-Tamura, CM, 2007)

The following are equivalent.

(i) $(X_1 \oplus \cdots \oplus X_m)_1$ *is uniformly non-*ℓ_1^{m+1}*.*

(ii) $X_1, \ldots, X_m$ *are uniformly non-square.*

This extends Theorem 4.1:

$$X \oplus_1 Y \text{ is uniformly non-}\ell_1^3 \iff X \text{ and } Y \text{ are UNS.}$$

Theorem 5.3 (Kato-Tamura, Comment. Math., 2009)

Let $n \geq 2$. The following are equivalent.

(i) $(X_1 \oplus \cdots \oplus X_{2^n-1})_\infty$ *is uniformly non-*ℓ_1^{n+1}.

(ii) $(X_1 \oplus \cdots \oplus X_m)_\infty$ *is uniformly non-*ℓ_1^{n+1} *for all* $m \leq 2^n - 1$.

(iii) $X_1, \ldots, X_{2^n-1}$ *are uniformly non-square.*

6. Some Appications

Super-reflexivity

Definition 6.1 (James, 1972)

X is called **super-reflexive**

$\overset{\textit{def}}{\Longleftrightarrow}$ $\Big[Y\text{: finitely representable in } X \Rightarrow Y\text{: reflexive} \Big]$

A Banach space Y is said to be **finitely representable** in X

$\overset{\textit{def}}{\Longleftrightarrow}$

$\forall\ \epsilon > 0$

$\forall\ F$: finite dimensional subspace of Y

$\exists\ E$: a finite dimensional subspace of X <u>with $\dim F = \dim E$</u>

such that

$d(F, E) := \inf\{\|T\|\|T^{-1}\| : T \text{ is an isomorphism of } F \text{ onto } E\} < 1+\epsilon.$

Uniformly non-square spaces are super-reflexive (James, 1972), while uniformly non-ℓ_1^3 spaces are not always reflexive (James, 1974).

Theorem 6.1 (K-Tam, CM, 2009)

Let X be uniformly non-ℓ_1^3.
If X is isometric to

(i) *an ℓ_1-sum of 2 Banach spaces, or*
(ii) *an ℓ_∞-sum of 3 Banach spaces,*

then X is super-reflexive.

Indeed, if $X \oplus_1 Y$ is uniformly non-ℓ_1^3, X and Y are UNS, whence super-reflexive. Therefore $X \oplus_1 Y$ is super-reflexive.

Fixed Point Property (FPP)

Definition 6.2

X has the FPP for nonexpansive mappings

$\overset{def}{\Longleftrightarrow}$

$\forall\ C$: nonempty closed bounded convex subset of X,
every nonexpansive mapping $T : C \to C$ has a fixed point.

(T is nonexpansive $\iff \|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$).

Theorem F (García-Falset et al., JFA, 2006)

X: UNS $\Rightarrow$ X: FPP

It's natural to ask
whether all uniformly non-octahedral spaces have FPP.
We have the following.

Theorem 6.2 (Kato-Tamura, CM, 2009)

Let X be uniformly non-octahedral (non-ℓ_1^3).

If X is isometric to an ℓ_∞-sum of 3 Banach spaces,

then X has FPP.

Indeed, García-Falset et al. (JFA, 2006) showed that

$$X : UNS \Longrightarrow R(1,X) < 2 \Longrightarrow X : FPP$$

If $(X_1 \oplus X_2 \oplus X_3)_\infty$ is uniformly non-ℓ_1^3, all X_1, X_2, X_3 are UNS by Theorem 4.4. Hence $R(1, X_i) < 2$. On the other hand, we can show

$$R(1, (X_1 \oplus X_2 \oplus X_3)_\infty) = \max_{1 \leq i \leq 3} R(1, X_i)$$

Therefore we have $R(1, (X_1 \oplus X_2 \oplus X_3)_\infty) < 2$, which implies that $(X_1 \oplus X_2 \oplus X_3)_\infty$ has *FPP*.

In the same way we have the following.

Theorem 6.3 (Kato-Tamura, CM, 2009)

Let X be uniformly non-ℓ_1^{n+1}.

If X is isometric to an ℓ_∞-sum of $2^n - 1$ Banach spaces,

then X has FPP.

The constant $R(1,X)$ by Domínguez Benavides (1994):

$$R(1,X) = \sup\left\{\liminf_{n\to\infty} \|x_n + x\|\right\},$$

where the supremum is taken over all $x \in B_X$ and all weakly null sequences $\{x_n\}$ in B_X such that $\lim_{n,m\to\infty; n\neq m} \|x_n - x_m\| \leq 1$.

Theorem 6.4 (Kato-Tamura, CM, 2007)

Let $X = (X_1 \oplus \cdots \oplus X_m)_1$ *be uniformly non-*ℓ_1^{m+1}*.*

Then $X_1, \cdots, X_m$ *have FPP.*

Indeed,
if $X = (X_1 \oplus \cdots \oplus X_m)_1$ is uniformly non-ℓ_1^{m+1},
all $X_1, \ldots, X_m$ are UNS by Theorem 5.2,
and hence have *FPP*.