MAXIMAL FUNCTION AND RELATED OPERATORS ON L^1

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Plan of the talk

1 Introduction

Notation:

L^p for the Lebesgue spaces,

$|E|$ the Lebesgue measure of a meas. set $E \subset \mathbb{R}^N$,

$B_R(x)$ an open ball centered at x with radius R.

Motivation:

- a question how strong are assumptions about the maximal function, varying from the finiteness of the maximal function at a single point to L_r integrability, $0 < r < 1$
- questions about the range of the maximal operator under minimal necessary hypothesis about the space on which it acts.

The presented material is mostly a joint work with A. Fiorenza (Univ. di Napoli).

For $f \in L^1_{\text{loc}}(\mathbb{R}^N), N \geq 1$; we consider the (*global, centered*) *maximal function*

$$Mf(x) = \sup_{R>0} \frac{1}{|B_R(x)|} \int_{B_R(x)} |f(y)|\, dy, \qquad x \in \mathbb{R}^N.$$

Let

$$\mathbb{D} = \{f \in L^1_{\text{loc}}(\mathbb{R}^N);\ Mf \not\equiv \infty\},$$

be the *domain of the maximal operator*.

$\mathbb{D}$ is a linear subspace of $L^1_{\text{loc}}(\mathbb{R}^N)$ (see Theorem 2.2 below).

Note that $L^1_{\text{loc}}(\mathbb{R}^N) \not\subset \mathbb{D}$: it suffices to consider, for instance, the function $x \mapsto \|x\|$ in $\mathbb{R}^N$.

Agreement: We shall work only with non-negative functions.

2 The domain

2.1 Theorem (Wiener 1939). *If* $f \in L^1(\mathbb{R}^N)$*, then* $Mf < \infty$ *a.e. in* $\mathbb{R}^N$.

Plainly $L^\infty(\mathbb{R}^N) \subset \mathbb{D}$. Hence $L^1(\mathbb{R}^N) + L^\infty(\mathbb{R}^N) \subset \mathbb{D}$. Therefore the Lebesgue, Orlicz, and Lorentz spaces are subsets of $\mathbb{D}$.

$\mathbb{D}$ is effectively larger than $L^1(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$, but it does not contain any space of the type

$$L^1_{\text{loc}}(\mathbb{R}^N) \cap (L^r(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)), \quad \text{where } 0 < r < 1.$$

2.2 Theorem. *Let* $f \in L^1_{\text{loc}}(\mathbb{R}^N)$*. Then* $f \in \mathbb{D}$ *is equivalent to any of the following conditions:*

(i) *there exists* $x_0 \in \mathbb{R}^N$ *such that* $Mf(x_0) < \infty$*;*

(ii) *there exists* $x_0 \in \mathbb{R}^N$ *such that*

$$\limsup_{R\to\infty} \frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} f(y)\, dy < \infty$$

(iii) *there exists* $K > 0$ *such that*

$$\limsup_{R\to\infty} \frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} f(y)\, dy = K < \infty$$

for every $x_0 \in \mathbb{R}^N$*;*

(iv) $Mf(x) < \infty$ *a.e. in* $\mathbb{R}^N$.

Two examples follow, showing that

$$L^1_{\text{loc}}(\mathbb{R}^N) \cap L^r(\mathbb{R}^N) \not\subset \mathbb{D} \not\subset L^1(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$$

if $0 < r < 1$. The functions f here live on sets of finite measure, therefore their level set $\{f(x) > \alpha\}$, $\alpha \geq 0$, have finite measure.

2.3 Example. Let $A_n = \{n - 1 < |x| < n\}$, $n \in \mathbb{N}$ and let F_n be any measurable subset of A_n such that $|F_n| = 2^{-n}$. Put $f = \sum_{n=1}^{\infty} a_n \chi_{F_n}$ where $a_n = 2^n$. Then $f \in L^1_{\text{loc}}(\mathbb{R}^N)$, $f \notin L^1(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$. At the same time $f \in \mathbb{D}$. Further, we have $f \in L^r(\mathbb{R}^N)$ for all $0 < r < 1$.

2.4 Example. Let us put $a_n = \left(2^{(r+1)/2r}\right)^n$ with some fixed $0 < r < 1$ in the previous example. Then $f \in L^1_{\text{loc}}(\mathbb{R}^N) \cap L^r(\mathbb{R}^N)$, $f \notin \mathbb{D}$. We observe that $f \notin L^1(\mathbb{R}^N)$.

2.5 Example. Functions in $\mathbb{D}$ can be very bad: for instance, the measure of every level set can be infinite, hence these functions cannot be rearranged. An example: Put

$$f(t) = \sum_{n=1}^{\infty} n\chi_{(n^3,n^3+1)}(t), \quad t \in \mathbb{R}^1.$$

Then $f \in L^1_{\text{loc}}(\mathbb{R}^1)$ and $f \in \mathbb{D}$, every level set of Mf is infinite, too. (Note in passing that $f \notin \text{BMO}(\mathbb{R}^1)$.

3 The range

Well known is:

- if $f \in \mathbb{D}$, then Mf is measurable lower semicontinuous function;
- $f \leq Mf$ and if $N > 2$ an equality can hold without f being a constant
- $M : L^1(\mathbb{R}^N) \to L^{1,\infty}(\mathbb{R}^N)$ is bounded;
- $M : L^p(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$, $1 < p \leq \infty$, is bounded.

An example showing that if $f \in L^1(\mathbb{R}^N)$, then generally we have not $Mf \in L^1_{\text{loc}}(\mathbb{R}^N)$:

3.1 Example. If $f \in \mathbb{D}$, then it may happen that $Mf \notin L^1_{\text{loc}}(\mathbb{R}^N)$. Put $f(x) = \chi_{(0,1/2)}(x)/(x \log^2 x)$, $x \in \mathbb{R}^1$ a.e.; then $f \in \mathbb{D}$, $Mf \notin L^1_{\text{loc}}(\mathbb{R}^N)$, $Mf \in L^1(\log L)^{-1}((0,1/2))$.

Example 2.5 shows that if $f \in \mathbb{D}$, then the measure of every level set of Mf can be infinite, considering a function f having the same property. Such a phenomenon may occur even if the measure of every level set of f is finite, namely, when $f \notin L^1(\mathbb{R}^N) + L^\infty(\mathbb{R}^N$, as in the Example 2.3) (the example with the concentric balls).

Two spaces near L_1

Let us recall the Kolmogorov inequality:

$$\|Mf\|^r_{L^r(A)} \leq \frac{c(N)|A|^{1-r}}{1-r}\|f\|^r_{L^1(\mathbb{R}^N)}, \tag{3.1}$$

true for every $f \in L^1(\mathbb{R}^N)$, $r \in]0,1[$, $A \subset \mathbb{R}^N$, $|A| < \infty$.

Hence $Mf \in L^r(A)$ with $0 < r < 1$, $f \in L^1(\mathbb{R}^N)$, $|A| < \infty$.

An extrapolation on the left hand side of (3.1) offers two reasonable candidates.

A characterization of logarithmic Lebesgue spaces, considered for $p \geq 1$ by Edmunds and Triebel, yields:

$$\int_0^{\varepsilon_0} \varepsilon^{\sigma-1} \|Mf\|_{L^{1-\varepsilon}(A)}\, d\varepsilon \leq c(N, |A|, \varepsilon_0, \sigma) \|f\|_{L^1(\mathbb{R}^N)}, \quad (3.2)$$

where $\varepsilon_0 \in (0,1)$ is arbitrary, $\sigma > 1$ is a parameter. The left hand side term of (3.2) is equivalent to the quasinorm in the (generalized) Orlicz space $L^1(\log L)^{-\sigma}(A)$. (Alternatively the abstract extrapolation Σ-method due to Milman can be employed.)

We have

3.2 Theorem. *If $f \in L^1(\mathbb{R}^N)$, then*

$$Mf \in \bigcap_{\sigma>1} L^1(\log L)^{-\sigma}(A) \ \text{ for every } A \subset \mathbb{R}^N, |A| < \infty.$$

3.3 Example. This is optimal in the scale of logarithmic Lebesgue spaces: Put

$$f(x) = \frac{1}{x|\log x| \log^2 |\log x|} \chi_{(0,a)}(x), \quad x \in \mathbb{R}^1,$$

where $a = \exp(-\exp(1))$. Then $f \in L^1(\mathbb{R}^N)$ and $Mf \notin L^1(\log L)^{-1}(]0, a[)$

The second approach based on (3.1): A bound for the quasi-norm of Mf in $L^{1)}(A)$, the grand L^1 space (Iwaniec and Sbordone, Greco):

$$\|Mf\|_{L^{1)}(A)} \leq c(N, |A|)\|f\|_{L^1(\mathbb{R}^N)},$$

where the quasinorm in $L^{1)}(A)$ is given by

$$\|g\|_{L^{1)}(A)} = \sup_{0<\varepsilon<1} \left(\varepsilon \frac{1}{|A|} \int_A |g(y)|^{1-\varepsilon}\, dy \right)^{1/(1-\varepsilon)} .$$

3.4 Proposition. *If $f \in L^1(\mathbb{R}^N)$, then it is $Mf \in L^{1)}(A)$ for every $A \subset \mathbb{R}^N$ of finite measure.*

The latter approach is better in terms of inclusions of functions spaces since

$$L^1(\log L)^{-1}(A) \subset L^{1)}(A) \subset \bigcap_{\sigma>1} L^1(\log L)^{-\sigma}(A)$$

for every A of finite measure (Capone, Fiorenza).

3.5 Theorem. *Assume that* $f \in L^1_{\text{loc}}(\mathbb{R}^N)$ *and let* $\varphi : [0,\infty[\to [0,\infty[$, *$\varphi$ strictly increasing,* $\varphi(\infty) = \infty$, $\lim_{t\to\infty} \varphi(t)/t^s = 0$ *for some* $0 < s < 1$. *Then the following statements are equivalent:*

(i) $f \in L^1(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$;

(ii) *there is* $\alpha > 0$ *such that* $f \in L^1(\{f > \alpha\})$;

(iii) *there is* $\alpha > 0$ *such that* $|\{Mf > \alpha\}| < \infty$;

(iv) *there is* $\alpha > 0$ *and* $0 < r < 1$ *such that* $Mf \in L^r(\{Mf > \alpha\})$;

(v) *there is* $\alpha > 0$ *such that* $Mf \in L^r(\{Mf > \alpha\})$ *for all* $0 < r < 1$;

(vi) *there is* $0 < r < 1$ *such that* $Mf \in L^r(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$;

(vii) $Mf \in \bigcap_{0<r<1} L^r(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$*;*

(viii) $\varphi(Mf) \in L^1(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$*;*

(ix) *there is* $\alpha > 0$ *such that* $|\{Mf > \alpha\}| < \infty$ *and* $Mf \in L^{1)}(\{Mf > \alpha\}) + L^\infty(\mathbb{R}^N)$*;*

(x) *there is* $\alpha > 0$ *such that* $|\{Mf > \alpha\}| < \infty$ *and*

$$Mf \in \bigcap_{\sigma>1} L^1(\log L)^{-\sigma}(\{Mf > \alpha\}) + L^\infty(\mathbb{R}^N).$$

3.6 Remark. The condition (ii) in Theorem 3.5 says that f is integrable over a special set of a finite measure. Examples 2.3 and 2.4 show that this cannot be replaced by integrability of f over any set of finite measure. Furthermore, the condition (iii) in Theorem 3.5 implies that all level sets of the maximal functions in the examples recalled are of infinite measure.

3.7 Corollary. *If $f \in L^{1-\varepsilon}(\mathbb{R}^N) \cap L^1_{\mathrm{loc}}(\mathbb{R}^N)$ for some $\varepsilon \in (0,1)$ and $Mf \in L^r(\{Mf > \alpha\})$ for some $0 < r < 1$, then in view of Theorem 3.5 we have $f \in L^1(\mathbb{R}^N)$.*

4 A survey of what can happen

For completeness let us recall the well-known variant of Stein's $L \log L$ theorem:

4.1 Theorem. *Let* $f \in L^1_{\text{loc}}(\mathbb{R}^N)$ *and* $\sigma \geq 1$*. Then the following statements are equivalent:*

(i) $f[\log(1+f)]^\sigma \in L^1(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$;

(ii) *there exists* $\alpha > 0$ *such that* $f[\log(1+f)]^\sigma \in L^1(\{f > \alpha\})$;

(iii) *there exists* $\alpha > 0$ *such that* $Mf \in L^1(\log L)^{\sigma-1}(\{f > \alpha\})$;

(iv) $Mf \in L^1(\log L)^{\sigma-1}(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$.

condition on f or Mf	**what can happen**
$f \in L^1_{\text{loc}}(\mathbb{R}^N)$	$f \notin \mathbb{D}$ (Ex. 2.4)
$\downarrow$	
$f \in \mathbb{D}$	$\lvert\{f > \beta\}\rvert = \infty, \beta \geq 0$ (Ex. 2.5)
$\downarrow$	
$\exists \alpha > 0 : \ \lvert\{f > \alpha\}\rvert < \infty$	$\lvert\{Mf > \beta\}\rvert = \infty, \beta \geq 0$ (Ex. 2.3)
$\downarrow$	
$\exists \alpha > 0 : \ \lvert\{Mf > \alpha\}\rvert < \infty$	$Mf \notin \frac{L^1}{\log L}(\{f > \beta\}), \beta \geq 0$ (Ex. 3.3)
$\downarrow$	
$\exists \alpha > 0 : \ Mf \in \frac{L^1}{\log L}(\{f > \alpha\})$	$Mf \notin L^1(\{f > \beta\}), \beta \geq 0$ (Ex. 3.1)
$\downarrow$	
$\exists \alpha > 0 : \ Mf \in L^1(\{f > \alpha\})$	

5 More general setup

(X, d, μ) a quasi-metric measure space with complete measure μ. By $\mathcal{M}$ we denote the set of all μ-measurable functions defined on X.

v, w are weight functions given on X i.e. measurable almost everywhere finite, locally integrable functions. For μ-measurable sets E we define the measures

$$v_\mu(E) = \int_E v(x)\, d\mu \text{ and } w_\mu(E) = \int_E w(x)\, d\mu.$$

Analogous claims can be established for a variety of operators, which satisfy two-weight inequality of the Kolmogorov type: For an arbitrary μ-measurable $E \subset X$ with finite measure and s, $0 < s < 1$, there holds

$$\int\limits_E |(Tf)(x)|^s v(x)\, d\mu \le c_2 \frac{(v_\mu E)^{1-s}}{1-s} \left(\int\limits_X \varphi(|f|)(x) w(x)\, d\mu \right)^s, \tag{5.1}$$

with a constant c_2 independent of f, E and s.

Modified maximal function

Let

$$\widetilde{M}f(x) = \sup_{r>0} \frac{1}{\mu B(x, N_0 r)} \int_{B(x,r)} |f(y)|\, d\mu$$

where $N_0 = a_1(1 + 2a_0)$ and the constants a_0 and a_1 are from the definition of a quasi-metric:

there exists a constant a_0 such that $d(x,y) \le a_0 d(y,x)$ for all x, y in X;

there exists a constant a_1 such that

$$d(x,y) \le a_1(d(x,z) + d(z,y))$$

for all $x, y, z \in X$.

Assume that there exists a constant c such that for all balls in X

$$\frac{1}{\mu B}\int_B v\,d\mu \le c\ \operatorname*{ess\,inf}_{x\in B} w(x). \tag{5.2}$$

Then the previous claims can be recovered.

Calderón-Zygmund singular integrals with non-doubling measures

The function $k : \mathbb{R}^n \times \mathbb{R}^n \backslash \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : x = y\} \to C$ is the *m-dimensional Calderón-Zygmund kernel* if there exist $c > 0$ and η, with $0 < \eta \leq 1$, such that

$$|k(x,y)| \leq \frac{c}{|x-y|^m}, \quad x,y \in \mathbb{R}^n,\ x \neq y,$$

and

$$|k(x,y) - k(x',y)| + |k(y,x) - k(y,x')| \leq \frac{c|x-x'|^\eta}{|x-y|^{m+\eta}}$$

if

$$|x-x'| \leq \frac{|x-y|}{2}.$$

Given a Borel measure μ on $\mathbb{R}^n$ and $f \in L^1_{\text{loc}}(\mathbb{R}^n, \mu)$, we define

$$T_\mu(x) := \int k(x,y)f(y)\,d\mu(y), \quad x \in \mathbb{R}^n \setminus \operatorname{supp}(f\,d\mu),$$

the *m-dimensional Calderón-Zygmund operator* (CZSIO) with the kernel k. Because of possible problems with convergence if $x \in \operatorname{supp}(f\,d\mu)$ we consider the ε-truncated operators T_ε, $\varepsilon > 0$:

$$T_{\mu,\varepsilon}f(x) := \int\limits_{|x-y|>\varepsilon} k(x,y)f(y)\,d\mu(y), \quad x \in \mathbb{R}^n,$$

and their uniform estimates with respect to $\varepsilon > 0$ in respective function spaces.

6 The local maximal function

The symbol Ω will now stand for an open bounded subset of $\mathbb{R}^N$, functions in Ω will be assumed to be measurable and non-negative.

The local maximal function of f is defined by

$$M_\Omega f(x) = \sup_{\substack{Q \ni x \\ Q \subset \Omega \\ Q \text{ cube}}} \frac{1}{|Q|} \int_Q f(y)\, dy, \qquad x \in \Omega,$$

where edges of cubes Q are parallel with coordinate axes.

Generally M_Ω preserves only some of the properties of M.

First consider $\Omega = Q_0$, cube in $\mathbb{R}^N$.

Put

$$\bar{f} = \begin{cases} f & \text{in } Q_0, \\ 0 & \text{in } \mathbb{R}^N \setminus Q_0, \end{cases}$$

then $M_{Q_0} f = (M_{\mathbb{R}^N} \bar{f})|_{Q_0}$.

Hence: $M_{Q_0} f < \infty$ a.e. in Q_0 iff $f \in L^1(Q_0)$. For $f \in L^1(Q_0)$ we have $M_{Q_0} f \in L^{1,\infty}(Q_0)$, $M_{Q_0} f \in L^{1)}(Q_0)$, $M_{Q_0} f \in \bigcap_{\sigma>1} L^1(\log L)^{-\sigma}(Q_0)$. In particular, $Mf \in \bigcap_{0<r<1} L^r(Q_0)$.

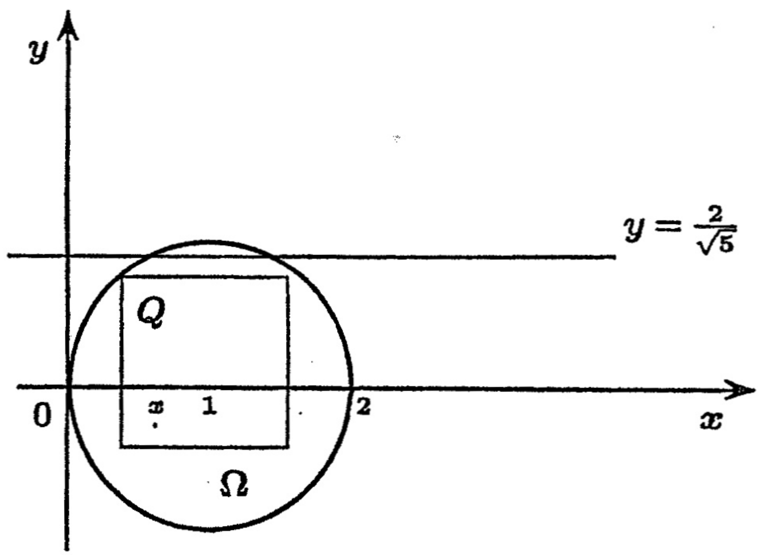
Further, $M_{Q_0} f$ need not be in $L_{1,\text{loc}}(Q_0)$ (one of previous examples).

The range of M_{Q_0} when f is “better” than $L^1(Q_0)$: $M_{Q_0}f \in L^1(Q_0)$ and f belongs to the Orlicz space $L_A(Q_0)$, where $\inf_{t>0} \frac{tA'(t)}{A(t)} > 1$, iff M_{Q_0} belongs to the same space. By the Stein's $L \log L$ theorem, $f \in L \log L(Q_0)$ iff $M_{Q_0}f \in L_1(Q_0)$.

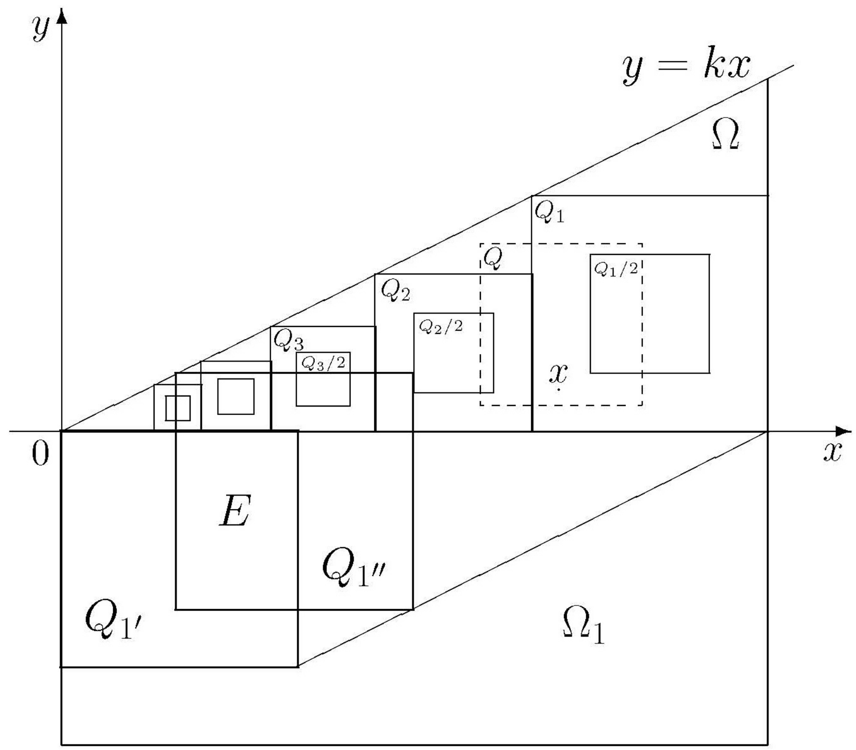
If Ω is a general domain, then M_Ω is different from $(M_{\mathbb{R}^N}\bar{f})\big|_\Omega$. In general $M_\Omega f \leq (M_{\mathbb{R}^N}\bar{f})\big|_\Omega$ and these functions need not be equivalent.

An example follows.

6.1 Example. Let $N = 2$, $\Omega = \{z = (x, y);\ |z - 1| < 1\}$, and let $f = 1$ in $\Omega \cap \{y > 2/\sqrt{5}\}$ and 0 otherwise in Ω. Then $(M_\Omega f)|_{\Omega\cap\{y\le 0\}} = 0$ while $(M_{\mathbb{R}^N}\bar{f})\big|_{\Omega\cap\{y\le 0\}} > 0$.



Hence: $\bar{f} \in \mathbb{D}$ is sufficient for $f \in \mathbb{D}_\Omega$. Nevertheless, next example shows that this assumption is too strong.



6.2 Example. Consider the situation illustrated by the upper part, that is, let $N = 2$ and consider a sequence of open cubes $Q_1, Q_2, \ldots$. Let Ω be the triangle domain whose boundary is contained in the positive axe x, the line $y = kx$, and the line containing the right vertical side of Q_1. Denote by $Q_1/2, Q_2/2, \ldots$, concentric cubes with sidelength equal to the half of the sides of $Q_1, Q_2, \ldots$. Let (a_i), be any sequence of positive real numbers such that

$$\sum_{i=1}^{\infty} a_i |Q_i| = \infty$$

and put

$$f = \sum_{i=1}^{\infty} a_i \chi_{Q_i/2}.$$

Then f is supported in a compact set and $f \notin L^1(\Omega)$. If we fix $x \in \Omega$, then every cube Q such that $Q \ni x$, $Q \subset \Omega$, intersect at most two of the cubes Q_i, thus M_Ω is finite a.e.

Hence: f need not be integrable over every compact subset of its support in order to have $M_\Omega f < \infty$ a.e. in Ω. Of course f must be integrable over cubes contained in Ω, this is, however, not sufficient for $M_\Omega f < \infty$ a.e. in Ω. Indeed, f can be integrable over cubes in Ω, and still M_Ω need not be a.e. finite—an example follows.

6.3 Example. Consider Ω_1 as Ω from Example 6.2 united with a rectangle pasted from below to Ω, with the left vertical side on the axe y and the upper horizontal side on the axe x. Put

$$f = \sum_{i=1}^{\infty} \frac{i}{|Q_i|} \chi_{Q_i/2}.$$

Let $Q_1^{'}$ be the translation of Q_1 having the left upper corner on the origin, and $Q_1^{''} \neq Q_1^{'}$ be any fixed translation of Q_1, contained in Ω_1, such that the left upper corner of $Q_1^{''}$ stays on the line $y = kx$ and such that the set $E = Q_1^{'} \cap Q_1^{''}$ has positive measure. Then for every Q_i, i sufficiently large, there exists a translation of Q_1 containing $E \cup Q_i$, therefore, if $x \in$

E, since $M_\Omega f(x) \geq \frac{1}{|Q_1|} \int\limits_{Q_i} f(y)\,dy = \frac{i}{4|Q_1|}$, we have $M_\Omega f = \infty$ in E. Hence $M_\Omega f$ is not finite a.e. while f is integrable over every cube contained in Ω.

The point is: a sequence of cubes with averages of f blowing up.

6.4 Theorem. *Let $f \in L^1(Q)$ for all $Q \subset \Omega$. Then the following statements are equivalent:*

(i) $M_\Omega f < \infty$ *a.e. in* Ω;

(ii) $$\sup_{\substack{|Q|>\varepsilon \\ Q \text{ cube} \\ Q \subset \Omega}} \frac{1}{|Q|} \int\limits_Q f(y)\,dy < \infty \qquad \text{for all } \varepsilon > 0.$$

6.5 Remark. If $\Omega = Q_0$, then (ii) is equivalent to $f \in L^1(Q_0)$.

The range of the local maximal function

If $f \in \mathbb{D}_\Omega$, then plainly M_Ω is lower semicontinuous and $f \leq M_\Omega f$ a.e. in Ω. On the other hand, in contrast to the behaviour of $M_{\mathbb{R}^N}$ and M_{Q_0}, it is not generally true that $M_\Omega f \in L^r(\Omega)$.

Consider Example 6.2 with $a_i = i \exp(1/|Q_i|^2)$. Then

$$\int_\Omega f^r \, dx = \sum_{i=1}^{\infty} \frac{|Q_i|}{4} i^r \exp\left(\frac{r}{|Q_i|^2}\right) = \infty, \qquad 0 < r < 1,$$

therefore $M_\Omega f \notin L^r(\Omega)$. But if Ω is a cube Q_0, then $M_{Q_0} f \in L^r(Q_0)$ for all $0 < r < 1$.

Also if Ω is a cube Q_0, then in contrast to the behaviour of $M_{\mathbb{R}^N}$ we have $\mathrm{BMO}(Q_0) \subset \mathbb{D}_{Q_0}$ (generally not true).

In spite of this we have

6.6 Theorem. *Let* $f \in L^1(\Omega)$. *Then* $M_\Omega f \in L^{1)}(\Omega)$, *therefore,* $M_\Omega f \in \bigcap\limits_{\sigma>1} L^1(\log L)^{-\sigma}(\Omega)$

6.7 Remark. If Ω is a cube Q_0 and $M_{Q_0} f \in L^1(\log L)^{-1}(Q_0)$, then the L^1 norm of f can be estimated as follows:

$$\int\limits_{Q_0} f(x)\,dx \le 2^{N+1} \int\limits_{Q_0} \frac{M_{Q_0} f(x)}{\log\left(e + \frac{M_{Q_0} f(x)}{|M_{Q_0} f(x)|_{Q_0}}\right)}\,dx,$$

where $|M_{Q_0} f(x)|_{Q_0} = \frac{1}{|Q_0|} \int\limits_{Q_0} M_{Q_0} f(x)\,dx$.

A historical comment

In 1910 F. Riesz proved the following characterization of L^p spaces on a cube: *Let* $1 < p < \infty$*. Then* f *belongs to* $L^p(Q)$ *iff there exists* $C > 0$ *such that for any decomposition* $\{Q_i\}$ *of* Q *into cubes* Q_i*, i.e.* $Q = \bigcup_i Q_i$ *one has*

$$\left(\sum_i \frac{1}{|Q_i|^{p-1}}\left(\int_{Q_i} |f(y)\,dy|\right)^p\right)^{1/p} \le C. \tag{6.1}$$

After easy manipulation with (6.1) (passing to dyadic cubes if necessary) we see that $M_Q f(x) \le \|f\|_p$ for $p \le 2$. Note that the left hand side of (6.1) is equivalent to $\|f\|_p$.

Hence for cubes and $p \le 2$ the maximal theorem was in fact known before WWI. Maybe it is discussed somewhere in the literature.

A loosely related problem: Plainly, one can replace decompositions $\{Q_i\}$ above (into cubes) by decompositions into measurable subsets of Q, say $\{A_i\}$. It is well known that $f \in L^{p,\infty}$ iff there exists C' such that

$$\sup_{A \subset Q} \frac{1}{|A|^{1-1/p}} \int_A |f(y)|\, dy \le C'. \tag{6.2}$$

Relation (6.2) can be clearly replaced by

$$\sup_{\{A_i\}} \sup_{A_i} \frac{1}{|A_i|^{1-1/p}} \int_{A_i} |f(y)|\, dy \le C', \tag{6.3}$$

that is, by sup of the expression in (6.2) taken over all decompositions $\{A_i\}$.

Real interpolation of L^p and $L^{p,\infty}$ leads to appropriate Lorentz space $L^{p,q}$.

The question is about an interpolation formula which might be perhaps derived from (6.1) *with* A_i *instead of* Q_i *and* (6.3)*. Would this be a formula for a quasinorm in* $L^{p,q}$ *without use of rearrangements?* Note that decompositions into cubes or have been treated in the literature (various clones of Morrey-Campanato spaces etc.)