Generalized trigonometric functions

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(Etwas allgemein machen heisst, es denken. G.W.F. Hegel)

Based on:

* David E. Edmunds, Petr Gurka, J.L., Properties of generalized Trigonometric functions, *Preprint*

* David E. Edmunds, J.L., Eigenvalues, Embeddings and Generalised Trigonometric Functions, *Lecture Notes in Mathematics 2016, Springer*

Let $1 < p, q < \infty$ and define a (differentiable) function $F_{p,q} : [0,1] \to \mathbf{R}$ by

$$F_{p,q}(x) = \int_0^x \frac{1}{\sqrt[p]{1-t^q}} dt, \qquad 0 \le x \le 1. \tag{1}$$

Since $F_{p,q}$ is strictly increasing it is a one-to-one function on $[0,1]$ with range $[0, \pi_{p,q}/2]$, where

$$\pi_{p,q} = 2\int_0^1 \frac{1}{\sqrt[p]{1-t^q}} dt, \qquad 0 \le x \le 1. \tag{2}$$

The inverse of $F_{p,q}$ on $[0, \pi_{p,q}/2]$ we denote by $\sin_{p,q}$ and extend as in the case of sin (p=q=2) to $[0, \pi_{p,q}]$ by defining

$$\sin_{p,q}(x) = \sin_{p,q}(\pi_{p,q} - x) \qquad \text{for } x \in [\pi_{p,q}/2, \pi_{p,q}];$$

further extension is achieved by oddness and $2\pi_{p,q}$-periodicity on the whole of $\mathbf{R}$. By this means we obtain a differentiable function on $\mathbf{R}$ which coincides with sin when $p = q = 2$.

Corresponding to this we define a function $\cos_{p,q}$ by the prescription

$$\cos_{p,q}(x) = \frac{d}{dx}\sin_{p,q}(x), \ \ x \in \mathbf{R}. \tag{3}$$

Clearly $\cos_{p,q}$ is even, $2\pi_{p,q}$-periodic and odd about $\pi_{p,q}$; and $\cos_{2,2} = \cos$. If $x \in [0, \pi_{p,q}/2]$, then from the definition it follows that

$$\cos_{p,q}(x) = (1 - (\sin_{p,q}(x))^q)^{1/p}. \tag{4}$$

Moreover, the asymmetry and periodicity show that

$$|\sin_{p,q}(x)|^q + |\cos_{p,q}(x)|^p = 1, \quad x \in \mathbf{R}. \tag{5}$$

We will use:

$\pi_p := \pi_{p,p}$, $\sin_p := \sin_{p,p}$ and $\cos_p := \cos_{p,p}$.

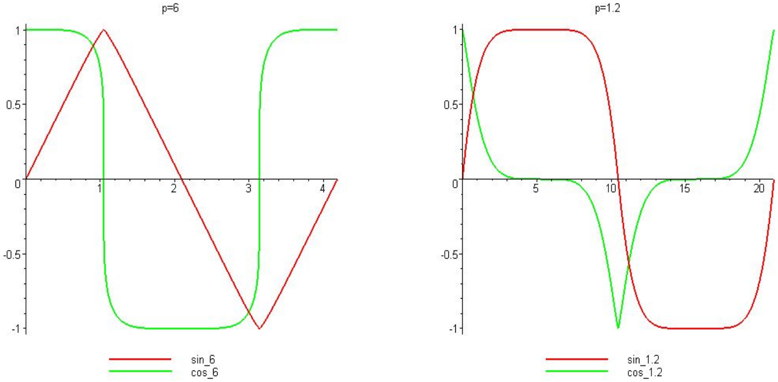


Figure: $\sin_{6,6}, \cos_{6,6}$ and $\sin_{1.2,1.2}, \cos_{1.2,1.2}$

$$\frac{\pi_p}{2} = p^{-1}\int_0^1 (1-s^p)^{-1/p} s^{1/p-1} ds = p^{-1}B(1-1/p, 1/p)$$

$$= p^{-1}\Gamma(1-1/p)\Gamma(1/p),$$

where B is the Beta function, Γ is the Gamma function and

$$\pi_p = \frac{2\pi}{p\sin(\pi/p)}. \tag{6}$$

Clearly $\pi_2 = \pi$ and, with $p' = p/(p-1)$,

$$p\pi_p = 2\Gamma(1/p')\Gamma(1/p) = p'\pi_{p'}. \tag{7}$$

Using (6) and (7) we see that π_p decreases as p increases, with

$$\lim_{p\to 1}\pi_p = \infty, \quad \lim_{p\to\infty}\pi_p = 2, \quad \lim_{p\to 1}(p-1)\pi_p = \lim_{p\to 1}\pi_{p'} = 2. \tag{8}$$

Properties of $\pi_{p,q}$

Lemma

Let $p, q \in (1,\infty)$. *Then*

$$p \mapsto \pi_{p,q} \quad \textit{is decreasing on } (1,\infty) \textit{ for any fixed } q \in (1,\infty),$$

$$q \mapsto \pi_{p,q} \quad \textit{is decreasing on } (1,\infty) \textit{ for any fixed } q \in (1,\infty),$$

$$q\pi_{p,q} = p'\pi_{q',p'}$$

$$\frac{\pi_{p,q}}{2} = \int_0^1 (1-t^q)^{-1/p} dt = \frac{p'}{q}\int_0^1 y^{-p'/p}(1-y^{p'})^{-1/q'} y^{p'/p} dy$$

$$= \frac{p'}{q}\int_0^1 (1-y^{p'})^{-1/q'} dy = \frac{p'}{q}\frac{\pi_{q',p'}}{2}.$$

Lemma

Let $p, q \in (1, \infty)$.

(i) *If* $p' \le q$ *then* $\pi_{p,q} \le \pi_{q'q}$.

(ii) *If* $p' > q$ *then* $\pi_{p,q} \le \frac{p'}{q}\pi_{p,p'}$.

(iii) $2 \le \pi_{p,p'} \le 4$.

$$\pi_{p,q} = \frac{2\,B(1/p',1/q)}{q} = \frac{2\,\Gamma(1/p')\,\Gamma(1/q)}{q\,\Gamma(1/p'+1/q)}.$$

On the interval $I = [0,1]$ let

$$Tf(x) := \int_0^x f(t)dt. \tag{9}$$

At first we consider T as a map from $L_2(0,1)$ into $L_2(0,1)$. It is obvious that T is compact and that there exists a function in $L_2(0,1)$ at which the norm of T is attained. In this case it is quite simple to show that $\|T|L_2(0,1) \to L_2(0,1)\| = 2/\pi$ and that the norm is attained when:

$$f(t) = \cos\left(\frac{\pi x}{2}\right)\frac{\pi}{2}$$

so that

$$Tf(t) = \sin\left(\frac{\pi x}{2}\right).$$

When $1 < p, q < \infty$ then again T is a compact map from $L_p(0,1)$ into $L_q(0,1)$ and there exists a function at which the norm is attained. In [Levin, Schmidt] it was proved that

$$\|T|L_p(0,1) \to L_q(0,1)\| =$$

$$= \frac{(p'+p)^{1-\frac{1}{p'}+\frac{1}{q}}(p')^{1/q}q^{1/p'}}{B(\frac{1}{p'},\frac{1}{q})}$$

and that the norm is attained when:

$$f(t) = \cos_{p,q}\left(\frac{\pi_{p,q}x}{2}\right)\frac{\pi_{p,q}}{2} \quad \text{and} \quad Tf(t) = \sin_{p,q}\left(\frac{\pi_{p,q}x}{2}\right).$$

Eigenfunctions for the p-Laplacian

We recall the definition of the p-Laplacian which is a natural extension of the Laplacian:

$$\Delta_p u = (|u'|^{p-2}u')'.$$

Evidently $\Delta_2 u = \Delta u$. Then the Dirichlet eigenvalue problem on $(0,1)$ is

$$\left.\begin{aligned} \Delta_p u + \lambda |u|^{p-2}u = 0 \quad \text{on } (0,1), \\ u(0) = 0, u(1) = 0. \end{aligned}\right\} \tag{10}$$

In [Elbert, Lindqvist, Drabek] it is shown that all eigenvalues of this problem are of the form

$$\lambda_n = (n\pi_p)^p \frac{p}{p'}$$

with corresponding eigenfunctions

$$u_n(t) = \sin_p(n\pi_p t).$$

Consider the pq-Laplacian and the corresponding eigenvalue problem

$$\left.\begin{array}{c} \Delta_p u + \lambda |u|^{q-2} u = 0 \quad \text{ on } (0,1), \\ u(0) = 0, u(1) = 0. \end{array}\right\} \tag{11}$$

Then all eigenvalues of this problem are of the form

$$\lambda_{n,\alpha} = (n\pi_{p,q})^q \frac{|\alpha|^{p-q} q}{p'}, \quad \alpha \in \mathbf{R} \setminus \{0\}$$

with corresponding eigenfunctions

$$u_{n\alpha}(t) = \frac{\alpha T}{n\pi_{p,q}} \sin_{p,q}(n\pi_{p,q} t).$$

The approximation theory

Let $1 < p < \infty$ and $-\infty < a < b < \infty$. Consider the Sobolev embedding on $I = [a, b]$,

$$E_0 : W_0^{1,p}(I) \to L^p(I) \tag{12}$$

where $W_0^{1,p}(I)$ is the Sobolev space of functions on the interval I with zero trace equipped with the following norm:

$$\|u|W_0^{1,p}(I)\| := \left(\int_a^b |u'(t)|^p dt \right)^{1/p}.$$

It is well known that (12) is a compact map and that more detailed information about its compactness plays an important role in different branches of mathematics. The properties of compact maps can be well described by using the Kolmogorov, Bernstein and Gel'fand n-widths together with the approximation numbers.

Definition

Let $T \in L(X, Y)$. Then **the n-th approximation and isomorphism** *of T are defined by*

$$\begin{aligned} a_n(T) &= \inf\{\|T - K\| : K \in L(X, Y), \mathrm{rank}(T) < n\}, \\ i_n(T) &= \sup\{\|A\|^{-1}\|B\|^{-1}\}, \end{aligned}$$

where the supremum is taken over all possible Banach spaces G and maps $A : Y \to G$ and $B : G \to X$ such that $I_G = ATB$ is an identity on G and $\dim(G) \geq n$.

Note that other important strict $s-$numbers as Gelfand, Bernstein, Kolmogorov and Mityagin numbers lies between the approximation and isomorphism numbers.

$$a_n(T) \geq d^n(T), b_n(T), d_n(T), m_n(T) \geq i_n(t)$$

Theorem

Let $\widetilde{s}_n$ *stands for any strict s-number (i.e.* $a_n, d_n, d^n, m_n, b_n, i_n$*). Let n be an integer, then*

$$\widetilde{s}_n(E_0) = \frac{|I|}{n\pi_p} \cdot \left(\frac{p'}{p}\right)^{1/p}$$

and

$$s_n(E_0) = \|(E_0 - R_n)g|L^p(I)\|, \quad \text{where } g(x) = \sin_p\left(\frac{x-a}{b-a}\pi_p n\right).$$

Here

$$R_n f = \sum_{i=1}^{n-1} P_i f$$

where

$$P_i f(x) = \chi_{I_i}(x) f\left(a + i\frac{|I|}{n}\right).$$

We can see from the previous theorem that the largest element in $BW_0^{1,p}(I) := \{f; \|f|W_0^{1,p}\| \le 1\}$ in the $L^p(I)$ norm is

$$f_1(x) := \frac{\sin_p\left(\frac{x-a}{b-a}\pi_p\right)}{\|\sin_p\left(\frac{x-a}{b-a}\pi_p\right)|W_0^{1,p}(I)\|}.$$

Let us approximate $BW_0^{1,p}(I)$ by a one-dimensional subspace in $L^p(I)$. The most distant element from the optimal one-dimensional approximation is

$$f_2(x) := \frac{\sin_p\left(\frac{x-a}{b-a}2\pi_p\cdot\right)}{\|\sin_p\left(\frac{x-a}{b-a}2\pi_p\right)|W_0^{1,p}(I)\|}.$$

We present below figures which show an image of $BW_0^{1,p}(I)$ restricted to a linear subspace $\mathrm{span}\{f_1, f_2, f_3\}$ in $L^p(I)$.
In the case $p = 2$ we obtain an ellipsoid (here the x, y, z axes correspond to f_1, f_2, f_3).

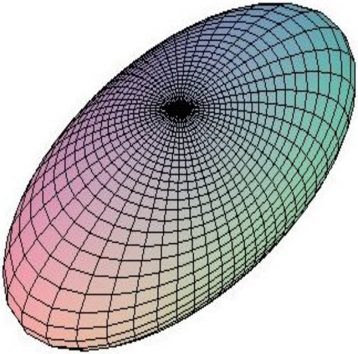


Figure 5: $p = 2$

When $p = 10$ and $p = 1.1$ we have the images below:

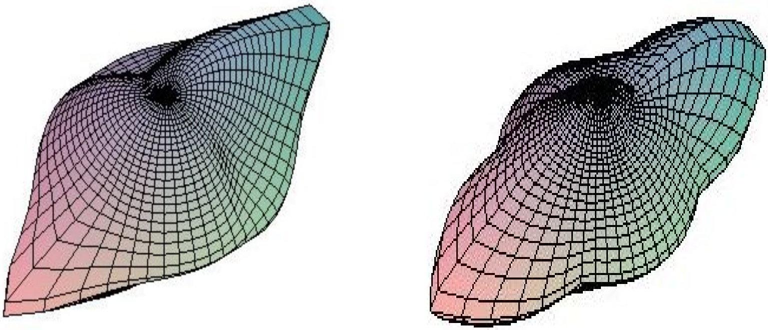


Figure 6: $p = 10$ $p = 1.1$

We can see that the main difference between Fig. 5 and Fig 6 is that the pictures in Fig. 6 are not convex. This suggests that possibly the functions f_1, f_2, f_3 are not orthogonal in the James sense.

Basis

Theorem

Let $p, q \in (1, \infty)$ *and let*

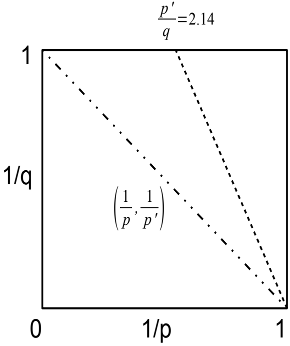
$$\frac{p'}{q} < \frac{4}{\pi^2 - 8} \approx 2.14. \tag{13}$$

Then the sequence $\left(\sin_{p,q}(n\,\pi_{p,q}\,t)\right)_{n\in\mathbf{N}}$ *is a a Schauder basis in* $L^r(0,1)$ *for any* $r \in (1, \infty)$.

The functions $f_{n,p}(x) := \sin_p(n\pi_p x)$ form a basis in $L_q(0,1)$ for every $q \in (1, \infty)$ if $p_0 < p < \infty$, where p_0 is defined by the equation

$$\pi_{p_0} = \frac{2\pi^2}{\pi^2 - 8}. \tag{14}$$

(i.e. $p_0 \approx 1.05$)


$\frac{p'}{q}=2.14$
1
1/q
$\left(\frac{1}{p},\frac{1}{p'}\right)$
0
1/p
1

Identities

Lemma

For all $x \in [0, \pi_{p,q}/2)$,

$$\frac{\mathrm{d}}{\mathrm{d}x} \cos_{p,q} x = -\frac{p}{q} (\cos_{p,q} x)^{2-p} (\sin_{p,q} x)^{q-1},$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \tan_{p,q} x = 1 + \frac{p\,(\sin_{p,q} x)^q}{q\,(\cos_{p,q} x)^p},$$

$$\frac{\mathrm{d}}{\mathrm{d}x} (\cos_{p,q} x)^{p-1} = -\frac{p(p-1)}{q} (\sin_{p,q} x)^{q-1},$$

$$\frac{\mathrm{d}}{\mathrm{d}x} (\sin_{p,q} x)^{p-1} = (p-1)\,(\sin_{p,q} x)^{p-1} \cos_{p,q} x.$$

Lemma

For all $y \in [0,1]$,

$$\cos_{p,q}^{-1} y = \sin_{p,q}^{-1}\left((1-y^p)^{1/q}\right),$$
$$\sin_{p,q}^{-1} y = \cos_{p,q}^{-1}\left((1-y^q)^{1/p}\right),$$
$$\frac{2}{\pi_{p,q}}\sin_{p,q}^{-1}\left(y^{1/q}\right) + \frac{2}{\pi_{q',p'}}\sin_{q',p'}^{-1}\left((1-y)^{1/p'}\right) = 1,$$
$$\left(\cos_{p,q}\left(\pi_{p,q}\, y/2\right)\right)^{p} = \left(\sin_{q',p'}\left(\pi_{q',p'}\,(1-y)/2\right)\right)^{p'}.$$

Lemma

Let $x \in [0, \pi_{4/3,4}/4)$. *Then*

$$\sin_{4/3,4}(2x) = \frac{2\sin_{4/3,4} x\,(\cos_{4/3,4} x)^{1/3}}{\left(1+4\,(\sin_{4/3,4} x)^4 (\cos_{4/3,4} x)^{4/3}\right)^{1/2}}. \qquad (15)$$