Compactness of embeddings of Besov and Triebel-Lizorkin spaces defined on quasi-bounded domains

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Motivation

$$e_k\left(B^{s_1}_{p_1,q_1}(\Omega) \hookrightarrow B^{s_2}_{p_2,q_2}(\Omega)\right) \sim k^{-\frac{s_1-s_2}{n}}$$

if

$$\frac{s_1 - s_2}{n} > \left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+ .$$

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Function spaces and wavelet on domains Triebel 2008

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$$\gamma := \frac{s_1 - s_2}{b(\Omega)} + \frac{b(\Omega) - n}{b(\Omega)}\left(\frac{1}{p_1} - \frac{1}{p_2}\right) > \left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+ .$$

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Ω in $\mathbb{R}^n$ is called quasi-bounded if

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Corollary

Let Ω be an unbounded, uniformly E-porous domain in $\mathbb{R}^n$. If Ω is not quasi-bounded then an embedding

$$\bar{B}^{s_1}_{p_1,q_1}(\Omega) \hookrightarrow \bar{B}^{s_2}_{p_2,q_2}(\Omega)$$

is never compact.

Let Ω be an uniformly E-porous domain with $\Omega \neq \mathbb{R}^n$, $b(\Omega) < \infty$ and define $\widetilde{W}_p^k(\Omega) := \overline{C_0^\infty(\Omega)}^{W_p^k}$. Then we have

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And with one additional property concerning the box packing holds

$$e_k\Big(\widetilde{W}_{p_1}^{k_1}(\Omega) \hookrightarrow \widetilde{W}_{p_2}^{k_2}(\Omega)\Big) \sim k^{-\gamma} \quad .$$

Application

Let $\alpha > 0$ and

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$$\lambda_k(-\Delta) \sim \begin{cases} k^{\frac{2\alpha}{1+\alpha}} & \text{if} \quad 0 < \alpha < 1 \\ k \log k & \text{if} \quad \alpha = 1 \\ k^{\frac{2}{2}} & \text{if} \quad \alpha > 1 \, . \end{cases}$$

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A close set Γ is said to be uniformly porous if it is porous and $\Gamma = \operatorname{supp}\mu$ with a locally finite positive Radon measure μ on $\mathbb{R}^n$ and

$$\mu(B(\gamma, r)) \sim h(r)\,, \qquad \text{with} \qquad \gamma \in \Gamma, \quad 0 < r < 1\,,$$

where $h : [0, 1] \to \mathbb{R}$ is a continuous strictly increasing function with $h(0) = 0$ and $h(1) = 1$.

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$$\bar{B}^s_{p,q}(\Omega) = \begin{cases} \widetilde{B}^s_{p,q}(\Omega) & \text{if} \quad 0 < p \le \infty,\ 0 < q \le \infty,\ s > \sigma_p\,, \\ B^0_{p,q}(\Omega) & \text{if} \quad 1 < p < \infty,\ 0 < q \le \infty,\ s = 0\,, \\ B^s_{p,q}(\Omega) & \text{if} \quad 0 < p < \infty,\ 0 < q \le \infty,\ s < 0\,. \end{cases}$$

Theorem [Triebel 2008]

Let Ω be an uniformly E-porous domain in $\mathbb{R}^n$, $\Omega \neq \mathbb{R}^n$. Let $\left\{\Phi_r^j : j \in \mathbb{N}_0;\ r = 1, \ldots, N_j\right\}$ be a special u-wavelet system, constructed in [Tr08, Section 2.3 and 2.4] that is an orthonormal basis in $L_2(\Omega)$ and $u > \max(s, \sigma_p - s)$.

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$$f = \sum_{j=0}^{\infty} \sum_{r=1}^{N_j} \lambda_r^j 2^{-j\frac{n}{2}} \Phi_r^j, \qquad \lambda \in \ell_q\left(2^{j(s-\frac{n}{p})} \ell_p^{N_j}\right) .$$

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If $f \in \bar{B}^s_{p,q}(\Omega)$ then the representation is unique with $\lambda = \lambda(f)$

$$\lambda_r^j = \lambda_r^j(f) = 2^{j\frac{n}{2}} (f, \Phi_r^j),$$

where (,) is a dual pairing and

$$I :\ \bar{B}^s_{p,q}(\Omega) \ni f \mapsto \lambda(f) \in \ell_q\left(2^{j(s-\frac{n}{p})} \ell_p^{N_j}\right)$$

is an isomorphism.

Sequence spaces

Let $\beta = \{\beta_j\}_{j=0}^{\infty}$ be a sequence of positive numbers and let $\{N_j\}_j$ be a sequence with $N_j \in \mathbb{N} \cup \{\infty\}$.

$$\ell_q\Big(\beta_j \ell_p^{N_j}\Big) := \Bigg\{\lambda = \{\lambda_{j,r}\}_{j,r} : \quad \lambda_{j,r} \in \mathbb{C}\,,$$

$$\Big\| \lambda \Big| \ell_q\Big(\beta_j \ell_p^{N_j}\Big) \Big\| = \left(\sum_{j=0}^{\infty} \beta_j^q \left(\sum_{r=1}^{N_j} |\lambda_{j,r}|^p \right)^{q/p} \right)^{1/q} < \infty \Bigg\}$$

(usual modifications if $p = \infty$ and/or $q = \infty$).

If $N_j = \infty$ then we put $\ell_p^{N_j} = \ell_p$.

Box packing

Let $\Omega \subset \mathbb{R}^n$ be a nonempty open set $\Omega \neq \mathbb{R}^n$.

$$b_j(\Omega) = \sup \left\{ k : \bigcup_{\ell=1}^{k} Q_{j,m_\ell} \subset \Omega \,, \quad Q_{j,m_\ell} \right\}, \qquad j = 0, 1, \ldots$$

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- It holds $$2^n b_{j-2}(\Omega) \leq N_j \leq b_j(\Omega) \quad .$$

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If the measure $|\Omega|$ is finite then $b(\Omega) = n$.

If Ω is unbounded and not quasi-bounded then $b(\Omega) = \infty$.

But there exist also quasi-bounded domains such that $b(\Omega) = \infty$.

Example 1

Let $\alpha > 0$. We consider the open sets $\omega_\alpha \subset \mathbb{R}^2$ defined as follows

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Then

$$b_j(\omega_\alpha) \sim \begin{cases} 2^{j(\alpha^{-1}+1)} & \text{if} \quad 0 < \alpha < 1\,, \\ j2^{2j} & \text{if} \quad \alpha = 1\,, \\ 2^{2j} & \text{if} \quad \alpha > 1 \end{cases}$$

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and in consequence

$$b(\omega_\alpha) = \begin{cases} \alpha^{-1} + 1 & \text{if} \quad 0 < \alpha < 1\,, \\ 2 & \text{if} \quad \alpha \geq 1\,. \end{cases}$$

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$$b_j(\omega_\alpha) \sim \begin{cases} 2^{j(\alpha^{-1}+1)} & \text{if} \quad 0 < \alpha < 1\,, \\ j2^{2j} & \text{if} \quad \alpha = 1\,, \\ 2^{2j} & \text{if} \quad \alpha > 1 \end{cases}$$

and in consequence

$$b(\omega_\alpha) = \begin{cases} \alpha^{-1} + 1 & \text{if} \quad 0 < \alpha < 1\,, \\ 2 & \text{if} \quad \alpha \geq 1\,. \end{cases}$$

$\lim_{j\to\infty} b_j(\omega_\alpha) 2^{-jb(\omega_\alpha)}$ is a positive finite number if $\alpha \neq 1$.
If $\alpha = 1$ then the limit equals infinity.

Example 2

Let $\alpha > 0$. We consider the open set $\Omega_\alpha \subset \mathbb{R}^2$ defined as follows

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Again $\lim_{j\to\infty} b_j(\Omega_\alpha) 2^{-jb(\Omega_\alpha)}$ is a positive finite number if $\alpha \neq 1$, but if $\alpha = 1$ then the limit equals again infinity.

Remark

If there exist j_0 such that $N_{j_0} = \infty$ then the embedding $\ell_{q_1}\left(\beta_j\,\ell_{p_1}^{N_j}\right) \hookrightarrow \ell_{q_2}(\ell_{p_2}^{N_j})$ holds if and only if

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Corollary

Let Ω be an unbounded, uniformly E-porous domain in $\mathbb{R}^n$.
If Ω is not quasi-bounded then an embedding

$$\ell_{q_1}\left(2^{s_1-\frac{n}{p_1}-s_2+\frac{n}{p_2}}\, \ell_{p_1}^{N_j}\right) \hookrightarrow \ell_{q_2}(\ell_{p_2}^{N_j}) \quad \text{or}$$

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is never compact.

Theorem

Let Ω be an uniformly E-porous domain with $\Omega \neq \mathbb{R}^n$ and $b(\Omega) < \infty$. Let

$$\frac{s_1 - s_2}{b(\Omega)} + \frac{b(\Omega) - n}{b(\Omega)}\Big(\frac{1}{p_1} - \frac{1}{p_2}\Big) > \big(\frac{1}{p_1} - \frac{1}{p_2}\big)_+ .$$

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If

$$0 < \liminf_{j\to\infty} b_j(\Omega)2^{-jb(\Omega)} \le \limsup_{j\to\infty} b_j(\Omega)2^{-jb(\Omega)} < \infty$$

then

$$e_k\left(\bar{B}^{s_1}_{p_1,q_1}(\Omega) \hookrightarrow \bar{B}^{s_2}_{p_2,q_2}(\Omega)\right) \sim k^{-\gamma}$$

with

$$\gamma = \frac{s_1 - s_2}{b(\Omega)} + \frac{b(\Omega) - n}{b(\Omega)}\Big(\frac{1}{p_1} - \frac{1}{p_2}\Big).$$

The condition

$$0 < \liminf_{j\to\infty} b_j(\Omega)2^{-jb(\Omega)} \leq \limsup_{j\to\infty} b_j(\Omega)2^{-jb(\Omega)} < \infty$$

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$$2^n b_j(\Omega) \leq b_{j+1}(\Omega) \leq c_1 b_j(\Omega) \quad .$$

And $\quad s_1 - s_2 + \left(b(\Omega) - n\right)\left(\frac{1}{p_1} - \frac{1}{p_2}\right) > b(\Omega)\left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+ \quad$ by

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Then we obtain

$$e_{2b_k(\Omega)}\left(\bar{B}^{s_1}_{p_1,q_1}(\Omega) \hookrightarrow \bar{B}^{s_2}_{p_2,q_2}(\Omega)\right) \sim 2^{-k(s_1-s_2-n(\frac{1}{p_1}-\frac{1}{p_2}))}\, b_k(\Omega)^{-(\frac{1}{p_1}-\frac{1}{p_2})}$$

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or

$$e_k\left(\bar{B}^{s_1}_{p_1,q_1}(\Omega) \hookrightarrow \bar{B}^{s_2}_{p_2,q_2}(\Omega)\right) \sim \mathbb{B}^{-1}(2^{-k(s_1-s_2-n(\frac{1}{p_1}-\frac{1}{p_2}))})\, k^{-(\frac{1}{p_1}-\frac{1}{p_2})}.$$

Remark

Let $s_1, s_2 \in \mathbb{R}$, $0 < p_1, p_2, q_1, q_2 \leq \infty$ and we assume

$$\frac{s_1 - s_2}{n} > \left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+.$$

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For positive real γ, such that

$$\frac{s_1 - s_2}{n} \geq \gamma > \left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+ ,$$

there exists an uniformly E-porous quasi-bounded domain Ω in $\mathbb{R}^n$ such that

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Choose an example with $b(\Omega) = n \, \dfrac{\frac{s_1 - s_2}{n} - (\frac{1}{p_1} - \frac{1}{p_2})}{\gamma - (\frac{1}{p_1} - \frac{1}{p_2})} \ge n$.

Let Ω be a uniformly E-porous quasi-bounded domain with $|\Omega| = \infty$ and $b(\Omega) < \infty$ with one additional property, then

$$\widetilde{W}_{p_1}^{k_1}(\Omega) \hookrightarrow \widetilde{W}_{p_2}^{k_2}(\Omega)$$

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Applications to spectral theory on unbounded domains

Let Ω be a quasi-bounded uniformly E-porous domain
Then we have $\bar{B}^{2m}_{2,2}(\Omega) = \bar{F}^{2m}_{2,2}(\Omega) = \widetilde{W}^{2m}_2(\Omega)$, $m \in \mathbb{N}$.
Let

$$A(x, D) = \sum_{|\alpha| \leq 2m} a_\alpha(x)\partial^\alpha$$

be formally self-adjoint, uniformly strongly elliptic differential operator of order $2m$, with real valued coefficients $a_\alpha \in C^\infty(\Omega)$ that are uniformly bounded and uniformly continuous for $|\alpha| \leq 2m$.

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Moreover we assume that A is a positive self-adjoint in $L_2(\Omega)$.

Theorem

Let Ω be an uniformly E-porous domain in $\mathbb{R}^n$, $\Omega \neq \mathbb{R}^n$, such that $b(\Omega) < \infty$ and

$$0 < \liminf_{j \to \infty} b_j(\Omega) 2^{-jb(\Omega)} \leq \limsup_{j \to \infty} b_j(\Omega) 2^{-jb(\Omega)} < \infty.$$

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Let $\lambda_1, \lambda_2, \ldots$ be eigenvalues of A ordered by their magnitude and counted according to their multiplicities. Then

$$\lambda_k \sim k^{\frac{2m}{b(\Omega)}}.$$

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Let $\alpha > 0$ and

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Appendix

Wavelets on domains

Let $\psi_F \in C^u(\mathbb{R})$ and $\psi_M \in C^u(\mathbb{R})$, $u \in \mathbb{N}$ be real compactly supported Daubechies wavelets.
We assume also that ψ_M satisfy the moment conditions for all $v \in \mathbb{N}_0$, $v < u$.

$$\Psi^{j,L}_{G,m}(x) := 2^{(j+L)n/2} \prod_{a=1}^{n} \psi_{G_a}(2^{j+L}x_a - m_a),$$

$$G \in \{F, M\}^n, \qquad m = (m_1, \ldots, m_n) \in \mathbb{Z}^n$$

where $L \in \mathbb{N}_0$ is fixed such that

$$\operatorname{supp} \psi_F \subset (-\varepsilon, \varepsilon) \qquad \operatorname{supp} \psi_M \subset (-\varepsilon, \varepsilon)$$

for some sufficiently small $\varepsilon > 0$.
$\{F, M\}^{n*} = \{F, M\}^n \setminus \{\bar{F} = (F, \ldots, F)\}$.

$$\mathbb{Z}_\Omega := \{x_r^j \in \Omega : \ j \in \mathbb{N}_0; \ r = 1, \ldots, N_j\}, N_j \in \overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$$

such that

$$|x_r^j - x_{r'}^j| \geq c_1 2^{-j}, \quad r \neq r' \text{ and } \operatorname{dist}\left(\bigcup_{r=1}^{N_j} B\big(x_r^j, c_2 2^{-j}\big), \Gamma)\right) \geq c_3 2^{-j}.$$

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The system of functions

$$\{\Phi_r^j : \; j \in \mathbb{N}_0; \; r = 1, \dots, N_j\} \quad \text{with} \quad \operatorname{supp} \Phi_r^j \subset B(x_r^j, c_2 2^{-j})$$

is called u-wavelet system (with respect to Ω) if it consists of:

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$$\Phi_r^j = \Psi_{G,m}^{j,L} \qquad \operatorname{dist}(x_r^j, \Gamma) \geq c_4 2^{-j}, \quad G \in \{G, F\}^{n*}, \; m \in \mathbb{Z}^n$$

$$\mathbb{Z}_\Omega := \{x_r^j \in \Omega : \ j \in \mathbb{N}_0; \ r = 1, \ldots, N_j\}, N_j \in \overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$$

such that

$$|x_r^j - x_{r'}^j| \geq c_1 2^{-j}, \quad r \neq r' \text{ and } \operatorname{dist}\left(\bigcup_{r=1}^{N_j} B(x_r^j, c_2 2^{-j}), \Gamma)\right) \geq c_3 2^{-j}.$$

The system of functions

$$\left\{\Phi_r^j : \ j \in \mathbb{N}_0; \ r = 1, \ldots, N_j\right\} \quad \text{with} \quad \operatorname{supp} \Phi_r^j \subset B(x_r^j, c_2 2^{-j})$$

is called u-wavelet system (with respect to Ω) if it consists of:

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$$\Phi_r^j = \sum_{|m-m'| \leq K} d_{m,m'}^j \Psi_{\bar{F},m}^{j,L} \qquad \operatorname{dist}(x_r^j, \Gamma) < c_4 2^{-j} m = m(j, r) \in \mathbb{Z}^n,$$

$$\text{with} \quad \sum_{|m-m'| \leq K} d_{m,m'}^j \leq D, \ \operatorname{supp} \Psi_{\bar{F},m'}^{j,L} \subset B(x_r^j, c_2 2^{-j}).$$

Example 3

Let $\{r_j\}_j$ be an increasing sequence of positive numbers.
We assume that $r_0 > 1$ and put $R_j = R_{j-1} + 2^{j(n-1)} r_j$, $R_{-1} = 0$.
The sequence $\{R_j - R_{j-1}\}_j$ is also increasing.

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If $r_k = 2^{k^\alpha}$, $\alpha > 1$ then $b_j(\Omega)$ are finite but $b(\Omega) = \infty$.