ON FRACTIONAL SOBOLEV INEQUALITIES, ISOPERIMETRY AND APPROXIMATION

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Basic definitions: Rearrangements

(Ω, d, μ) Metric space. μ Borel probability measure.
$u : \Omega \to \mathbb{R}$,

distribution function

$$\mu_u(t) = \mu \{x \in \Omega : |u(x)| > t\}, \ (t \geq 0).$$

decreasing rearrangement u_μ^* of u:

$$u_\mu^*(s) = \inf \{t : \mu_u(t) \leq s\}, \ (s \geq 0).$$

maximal function u_μ^{**} of u:

$$u_\mu^{**}(t) = \frac{1}{t} \int_0^t u_\mu^*(s) ds. \qquad (f+g)_\mu^{**}(t) \leq f_\mu^{**}(t) + g_\mu^{**}(t).$$

Modulus of the gradient:
$f \in Lip(\Omega)$

$$|\nabla f(x)| = \limsup_{d(x,y) \to 0} \frac{|f(x) - f(y)|}{d(x,y)},$$

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Symmetrization by truncation: Isoperimetry

$A \subset \Omega$, Borelian set

$$\mu^+(A) = \liminf_{h \to 0} \frac{\mu(A_h) - \mu(A)}{h},$$

$A_h = \{x \in \Omega : d(x, A) < h\}$.

The boundary measure is a natural generalization of the notion of surface area to the metric probability space setting.
An isoperimetric inequality measures the relation between $\mu^+(A)$ and $\mu(A)$ by means of the isoperimetric profile $I = I_{(\Omega,d,\mu)}$ defined as the pointwise maximal function $I_{(\Omega,d,\mu)} : [0,1] \to [0,\infty)$ such that

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Example: Isoperimetric Inequality on $\mathbb{R}^2$

Among all regions in the plane, enclosed by a piecewise C^1 boundary curve, with area A and perimeter L,

$$4\pi A \leq L^2.$$

If equality holds, then the region is a circle.

Symmetrization by truncation: Isoperimetry

$I_{(\Omega,d,\mu)}$ isoperimetric profile.

$J : [0,1] \to [0,\infty)$ continuous, concave function, symmetric about $1/2$ with $J(0) = 0$ st.

$$I_{(\Omega,d,\mu)}(t) \geq J(t), \ (t \in [0,1/2])$$

will be called an **isoperimetric estimator**

$\Omega \subset \mathbb{R}^n$ ("nice") $J(t) \simeq t^{(n-1)/n}$

$\mathbb{R}^n, d\gamma_n(x) = (2\pi)^{-n/2} e^{-|x|^2/2} dx, \ J(t) \simeq t \left(\log \frac{1}{t}\right)^{1/2}$

Condition 1. In what follows we shall assume (Ω, d, μ) has a nonzero isoperimetric estimator.

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Function spaces

$X = X(\Omega)$ Banach function space is a r.i. space if:

$$f \in X,\ g_\mu^* = f_\mu^* \Rightarrow g \in X \text{ and } \|g\|_X = \|f\|_X .$$

An r.i. space $X(\Omega)$ can be represented by a r.i. space on the interval $(0,1)$, with Lebesgue measure, $\bar{X} = \bar{X}(0,1)$, such that

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Function spaces

Examples:

- L^p-spaces

$$\|f\|_p = \left(\int_\Omega |f(x)|^p \, d\mu(x)\right)^{1/p} = \left(\int_0^\infty \mu_u(t) d(t^p)\right)^{1/p}$$

$$= \left(\int_0^1 f_\mu^*(t)^p dt\right)^{1/p}.$$

- Lorentz spaces $L^{p,q}$

$$\|f\|_{p,q} = \left(\int_0^1 \left(t^{1/p} f_\mu^*(t)\right)^q \frac{dt}{t}\right)^{1/q}. \qquad L^{p,1} \subset L^{p,p} = L^p \subset L^{p,\infty}.$$

- Others:

$$H_n(\Omega) = \left(\int_0^1 \left(\frac{f_\mu^*(t)}{\log(\frac{e}{t})}\right)^n \frac{dt}{t}\right)^{1/n}.$$

Orlicz spaces.

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Classically conditions on r.i. spaces are formulated in terms of the Hardy operators defined by

$$Pf(t) = \frac{1}{t}\int_0^t f(s)ds; \qquad Qf(t) = \frac{1}{t}\int_t^{\mu(\Omega)} f(s)\frac{ds}{s},$$

the boundedness of these operators on r.i. spaces can be simply described in terms of the so called Boyd indices defined by

$$\bar{\alpha}_X = \inf_{s>1} \frac{\ln h_X(s)}{\ln s} \quad \text{and} \quad \underline{\alpha}_X = \sup_{s<1} \frac{\ln h_X(s)}{\ln s},$$

where $h_X(s)$ denotes the norm of the dilation operator on $\bar{X}$ of the dilation operator E_s, $s > 0$, defined by

$$E_s f(t) = \begin{cases} f^*(\frac{t}{s}) & 0 < t < s, \\ 0 & s < t < 1 \end{cases}.$$

The operator E_s is bounded on $\bar{X}$ for every r.i. space $X(\Omega)$ and for every $s > 0$; moreover,

$$h_X(s) \leq \max(1, s). \tag{1}$$

For example, if $X = L^p$, then $\overline{\alpha}_{L^p} = \underline{\alpha}_{L^p} = \frac{1}{p}$.

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It is well known that if X is a r.i. space,

$$\begin{array}{l} P \text{ is bounded on } \bar{X} \ \Leftrightarrow \overline{\alpha}_X < 1, \\ Q \text{ is bounded on } \bar{X} \ \Leftrightarrow \underline{\alpha}_X > 0. \end{array} \tag{2}$$

Let X be a r.i. space,

$$\int_0^t f^*(s)ds \leq \int_0^t g^*(s)ds \to \|f\|_X \leq \|g\|_X$$

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Two Poincaré inequalities

$\Omega \subset \mathbb{R}^n$, "nice". $\left(\int_\Omega f = 0\right)$

Gagliardo-Nirenberg-Sobolev-Petre: $1 \leq p < n,\ q = \frac{pn}{n-p}$

$$\int_0^1 \left(t^{1/q} f^{**}(t)\right)^p \frac{dt}{t} \simeq \int_0^1 \left(t^{1/q} f^{*}(t)\right)^p \frac{dt}{t} \leq C \int_\Omega |\nabla f(x)|^p \, dx.$$

Gross' inequality: $\left(\int_{\mathbb{R}^n} f(x) d\gamma_n(x) = 0\right)$

$$\int_0^1 f^{**}_{\gamma_n}(t)^2 \log \frac{1}{t} dt \simeq \int_{\mathbb{R}^n} |f(x)|^2 \ln |f(x)| \, d\gamma_n(x) \leq \int_{\mathbb{R}^n} |\nabla f(x)|^2 \, d\gamma_n(x),$$

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$$\begin{aligned} f^{**}(t) &= \int_t^1 \left(-f^{**}\right)'(s)ds = \int_t^1 \left(f^{**}(s) - f^{*}(s)\right)\frac{ds}{s} + \|f\|_{L^1} \\ &= Q\left(f^{**}(\cdot) - f^{*}(\cdot)\right)(t) + \|f\|_{L^1} \end{aligned}$$

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Euclidean setting

$$\|f\|_{L^{p,q}} \preceq \left\| \overbrace{\left(\frac{f^{**}(t) - f^{*}(t)}{t} \right)}^{(-f^{**})'(t)} t^{1-1/n} \right\|_{L^{p,q}} \preceq \|\nabla f\|_{L^{p,q}}$$

Gaussian setting

$$\|f\|_{L^{2}} \preceq \left\| \overbrace{\left(\frac{f^{**}(t) - f^{*}(t)}{t} \right)}^{(-f^{**})'(t)} t\sqrt{\log \frac{1}{t}} \right\|_{L^{2}} \preceq \|\nabla f\|_{L^{2}}$$

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Question. Is there a relation (pointwise?) between

$(-f^{**})'(t)$, $J(t)$ and ∇f?

Symmetrization by truncation: The gradient

$I : [0,1] \to [0,\infty)$ isoperimetric estimator, there are equiv.

1. $$\forall A \subset \Omega,\ \text{Borel set},\ \mu^+(A) \geq I(\mu(A)). \text{ Isoperimetric}$$

2. $$\int_0^\infty I(\mu_f(s))ds \leq \int_\Omega |\nabla f(x)|\, d\mu(x).\ \text{ Ledoux}$$

3. $$(-f_\mu^*)'(s)I(s) \leq \frac{d}{ds}\int_{\{|f|>f_\mu^*(s)\}} |\nabla f(x)|\, d\mu(x).\ \text{ Maz'ya - Talenti}$$

4. $$\int_0^t ((-f_\mu^*)'(.)I(.))^*(s)ds \leq \int_0^t |\nabla f|_\mu^*(s)ds.\ \text{ Pólya-Szegö}$$

5. $$\left(-f_\mu^{**}\right)' I(t) = (f_\mu^{**}(t) - f_\mu^*(t))\frac{I(t)}{t} \leq |\nabla f|_\mu^{**}(t).\ \text{ Oscillation}$$

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$$f_{t_1}^{t_2}(x) = \begin{cases} t_2 - t_1 & \text{if } |f(x)| \geq t_2, \\ |f(x)| - t_1 & \text{if } t_1 < |f(x)| < t_2, \\ 0 & \text{if } |f(x)| \leq t_1. \end{cases}$$

$$\int_0^\infty I(\mu_{f_{t_1}^{t_2}}(s))ds \leq \int_\Omega \left|\nabla f_{t_1}^{t_2}(x)\right| d\mu.$$

$$\int_0^{t_2-t_1} I(\mu_{f_{t_1}^{t_2}}(s))ds \geq (t_2-t_1)\min\left(I(\mu\{|f| \geq t_2\}), I(\mu\{|f| > t_1\}\right).$$

For $s > 0$ and $h > 0$, pick $t_1 = f_\mu^*(s+h)$,$t_2 = f_\mu^*(s)$,

$$\begin{aligned}\left(f_\mu^*(s) - f_\mu^*(s+h)\right)\min(I(s+h), I(s)) &\leq \int_{\left\{f_\mu^*(s+h) < |f| \leq f_\mu^*(s)\right\}} |\nabla |f|(x)| \, d \\ &= \int_{\{|f| > f_\mu^*(t)\}} |\nabla |f|(x)| \, d\mu \\ &- \int_{\{|f| > f_\mu^*(s+h)\}} |\nabla |f|(x)| \, d\mu\end{aligned}$$

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thus

$$\frac{\left(f_\mu^*(s) - f_\mu^*(s+h)\right)}{h} \min(I(s+h), I(s))$$
$$\leq \frac{1}{h}\left(\int_{\{|f|>f_\mu^*(t)\}} |\nabla |f|(x)| \, d\mu - \int_{\{|f|>f_\mu^*(s+h)\}} |\nabla |f|(x)| \, d\mu\right)$$

But if f_μ^* absolutely continuous??

$$\int_{\{f_\mu^*(s+h)<|f|\leq f_\mu^*(s)\}} |\nabla |f|(x)| \, d\mu \overset{????}{=} \int_{\{f_\mu^*(s+h)<|f|<f_\mu^*(s)\}} |\nabla |f|(x)| \, d\mu$$

Then f_μ^* is absolutely continuous in $[a, b]$ $(0 < a < b < 1)$.

Condition 2. We assume that (Ω, μ) is such that for every $f \in Lip(\Omega)$, and every $c \in R$, we have that $|\nabla f(x)| = 0$, a.e. on the set $\{x : f(x) = c\}$.

Condition 1 and 2 are verified in all the classical cases: Euclidean, Gaussian, Riemannian manifolds with positive curvature as well as for doubling measures (homogeneous spaces).

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Condition 1 and 2 are verified in all the classical cases: Euclidean, Gaussian, Riemannian manifolds with positive curvature as well as for doubling measures (homogeneous spaces).

Integrability of solutions of elliptic equations

$$\begin{cases} -div(a(x,u,\nabla u)) = fw & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

where Ω is domain of $\mathbb{R}^n$ ($n \geq 2$), such that $\mu = w(x)dx$ is a probability measure on $\mathbb{R}^n$, or Ω has Lebesgue measure 1 if $w = 1$, and $a(x,\eta,\xi) : \ \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory function such that,

$$a(x,t,\xi).\xi \geq w(x)\left|\xi\right|^2, \quad \text{for a.e. } x \in \Omega \subset \mathbb{R}^n, \ \ \forall \eta \in \mathbb{R}, \ \ \forall \xi \in \mathbb{R}^n. \tag{4}$$

Example :$w = 1, a(x,t,\xi) = \xi$. Then (3) becomes

$$\begin{cases} \Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Theorem

Let $u \in W_0^1(w,\Omega)$ be a solution of (3). Let $\mu = w(x)dx$, and let $I = I_{(\mathbb{R}^n;\mu)}$ be the isoperimetric profile of $(\mathbb{R}^n;\mu)$. Then, the following inequalities hold

1.
$$\left(-u_\mu^*\right)'(t)I(t)^2 \leq \int_0^t f_\mu^*(s)ds, \ a.e. \tag{5}$$

2.
$$\int_t^{\mu(\Omega)} \left(|\nabla u|^2\right)_\mu^*(s)ds \leq \int_t^{\mu(\Omega)} \left(\left(-u_\mu^*\right)'(s)\int_0^s f_\mu^*(z)dz\right)ds. \tag{6}$$

$$R_I(h)(t) = \int_t^{\mu(\Omega)} \left(\frac{s}{I(s)}\right)^2 h(s)\frac{ds}{s}.$$

Let X, Y be two r.i. spaces on Ω such that,

$$\|R_I(h)\|_{\bar{Y}} \preceq \|h\|_{\bar{X}},$$

and, suppose that $\overline{\alpha}_X < 1$. Then, if u is a solution of (3) with datum $f \in X(\Omega)$, we have

$$\left\|u_\mu^*\right\|_{\bar{Y}} \preceq \left\|f_\mu^*\right\|_{\bar{X}}.$$

and

$$\left\|u_\mu^*\right\|_{\bar{Y}} \preceq \left\|\left(\frac{I(t)}{t}\right)^2 \left(u_\mu^{**}(t) - u_\mu^*(t)\right)\right\|_{\bar{X}} + \left\|u_\mu^*\right\|_{L^1} \preceq \left\|f_\mu^*\right\|_{\bar{X}}.$$

Moreover, if the operator $\tilde{R}_I(h)(t) = \left(\frac{I(s)}{s}\right)^2 \int_t^{\mu(\Omega)} \left(\frac{s}{I(s)}\right)^2 h(s)\frac{ds}{s}$ is bounded on $\bar{X}$, then

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The Euclidian case ($\Omega \subset \mathbb{R}^n$,$|\Omega| = 1$.)

$$\begin{cases} -div(a(x,u,\nabla u)) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{7}$$

with ellipticity condition,

$$a(x,t,\xi).\xi \succeq |\xi|^2, \quad \text{for a.e.} \quad x \in \Delta, \ \ \forall \eta \in \mathbb{R}, \ \ \forall \xi \in \mathbb{R}^n.$$

Let $X(\Omega)$ be an r.i. space such that $\bar{\alpha}_{\bar{X}} < 1$. Let u be a solution.

1. If $\underline{\alpha}_{\bar{X}} > 2/n$,
$$\left\| s^{-\frac{2}{n}} u^*(s) \right\|_{\bar{X}} \preceq \|f\|_{\bar{X}}.$$
2. If $\underline{\alpha}_{\bar{X}} \leq 2/n$,
$$\left\| s^{-\frac{2}{n}} (u^{**}(s) - u^*(s)) \right\|_{\bar{X}} + \|u\|_{L^1} \preceq \|f\|_{\bar{X}}.$$
3. If $\underline{\alpha}_{\bar{X}} > \frac{1}{2} + \frac{1}{n}$,
$$\left\| s^{-\frac{1}{n}} |\nabla u|^* (s) \right\|_{\bar{X}} \preceq \|f\|_X.$$

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Between exponential and Gaussian measure

Elliptic problems associated with Gaussian measures. Let $\alpha \geq 0$, $p \in [1,2]$ and $\gamma = \exp(2\alpha/(2-p))$, and let

$$\mu_{p,\alpha}(x) = Z_{p,\alpha}^{-1} \exp\left(-|x|^p (\log(\gamma + |x|)^\alpha\right) dx = \varphi_{\alpha,p}(x) dx, \qquad x \in \mathbb{R},$$

and

$$\varphi_{\alpha,p}^n(x) = \varphi_{\alpha,p}(x_1)\cdots\varphi_{\alpha,p}(x_n), \text{ and } \mu = \mu_{p,\alpha}^{\otimes n}.$$

Consider

$$\begin{cases} -div(a(x,u,\nabla u)) = f\varphi_{\alpha,p}^n & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{8}$$

with the ellipticity condition,

$$a(x,t,\xi).\xi \succeq \varphi_{\alpha,p}^n(x)\,|\xi|^2, \quad \text{for a.e.} \quad x \in \Omega, \ \forall \eta \in \mathbb{R}, \ \forall \xi \in \mathbb{R}^n,$$

where $\Omega \subset \mathbb{R}^n$ is an open set such that $\mu(\Omega) < 1$.

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where $\Omega \subset \mathbb{R}^n$ is an open set such that $\mu(\Omega) < 1$.

$$I_{\mu_{p,\alpha}^{\otimes n}}(s) \simeq s \left(\log \frac{1}{s} \right)^{1-\frac{1}{p}} \left(\log \log \left(e + \frac{1}{s} \right) \right)^{\frac{\alpha}{p}}, \quad 0 < s < \mu(\Omega).$$

Let u be a solution of (8) with datum $f \in X(\Delta)$. Assume that $\overline{\alpha}_{\bar{X}} < 1$. Then,

1. If $0 < \underline{\alpha}_{\bar{X}}$,

$$\left\| \left(\log \frac{1}{s}\right)^{2\left(1-\frac{1}{p}\right)} \left(\log\log\left(e+\frac{1}{s}\right)\right)^{2\frac{\alpha}{p}} u_{\mu}^{*}(s) \right\|_{\bar{X}} \preceq \|f\|_{X}.$$

2. If $0 = \underline{\alpha}_{\bar{X}}$,

$$\left\| \left(\log \frac{1}{s}\right)^{2\left(1-\frac{1}{p}\right)} \left(\log\log\left(e+\frac{1}{s}\right)\right)^{2\frac{\alpha}{p}} \left(u_{\mu}^{**}(s) - u_{\mu}^{*}(s)\right) \right\|_{\bar{X}} + \|u\|_{L^1} \preceq$$

3. If $\underline{\alpha}_{\bar{X}} > 1/2$,

$$\left\| \left(\log \frac{1}{s}\right)^{\left(1-\frac{1}{p}\right)} \left(\log\log\left(e+\frac{1}{s}\right)\right)^{\frac{\alpha}{p}} |\nabla u|_{\mu}^{*}(s) \right\|_{\bar{X}} \preceq \|f\|_{X}.$$

Weak assumptions

Condition 1: The isoperimetric profile $I_{(\Omega,d,\mu)}$ is a concave continuous function, increasing on $(0,1/2)$, symmetric about the point $1/2$ such that, moreover, vanishes at zero.
Condition 2: For every $f \in Lip(\Omega)$, and every $c \in R$, we have that $|\nabla f(x)| = 0$, $\mu-$a.e. on the set $\{x : f(x) = c\}$.
Condition 1': The isoperimetric profile $I_{(\Omega,d,\mu)}$ is a positive continuous function that vanishes at zero.

$I = I_{(\Omega,d,\mu)}$ isoperimetric.

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Theorem

Let (Ω, d, μ) *be a metric probability space that satisfies Conditions 1' and 2, and let* $1 \leq q < \infty$. *Then for* $f \in Lip(\Omega)$, *and for all* $t \in (0,1)$, *we have*

1. $$\int_0^t \left((-f_\mu^*)'(\cdot) w(\cdot) \right)^* (s) ds \leq \int_0^t |\nabla f|_\mu^* (s) ds.$$

2. $$(f_\mu^{**}(t) - f_\mu^*(t)) w(t) \leq \frac{1}{t} \int_0^t |\nabla f|_\mu^* (s) ds$$

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Theorem

Let (Ω, d, μ) be a metric probability space satisfying Condition 1'. Then for $f \in Lip(\Omega)$, we have

$$(f_\mu^{**}(t) - f_\mu^{*}(t))w(t) \leq \frac{1}{t}\int_0^t |\nabla f|_\mu^{*}(s)ds, \text{ for } t \in (0,1).$$

From here:

$$\left\| \left(f_\mu^{**}(t) - f_\mu^{*}(t) \right) w(t) \right\|_{\bar{X}} \leq \left\| |\nabla f|_\mu^{**} \right\|_{\bar{X}}$$

But this does not apply if $\bar{\alpha}_X = 1$.
What can be said for $\bar{\alpha}_X = 1$.

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What can be said for $\bar{\alpha}_X = 1$.

For simplicity let us assume that I is concave: Define

$$Q_I f(t) = \int_t^1 f(s) \frac{ds}{I(s)},$$

$f \in \bar{X}$, with supp$f \subset (0, 1/2)$, From the concavity of I, it follows that $s \preceq I(s)$, $s \in (0, 1/2)$, thus

$$Q_I f(t) = \int_t^{1/2} f(s) \frac{ds}{I(s)} \preceq Qf(t) = \int_t^{1/2} f(s) \frac{ds}{s}$$

therefore Q_I is bounded on X for any r.i space X such that $\underline{\alpha}_X > 0$.
Then, for all $g \in Lip(\Omega)$,

$$\left\| g - \int_\Omega g d\mu \right\|_X \preceq \|\nabla g\|_X .$$

Let $g \in Lip(\Omega)$. Write

$$g_\mu^*(t) = \int_t^{1/2} \left(-g_\mu^*\right)'(s)ds + g_\mu^*(1/2), \;\; t \in (0, 1/2].$$

$$\begin{aligned}
\|g\|_X = \left\|g_\mu^*\right\|_X &\preceq \left\|g_\mu^* \chi_{[0,1/2]}\right\|_X \\
&\preceq \left\| \int_t^{1/2} \left(-g_\mu^*\right)'(s)ds \right\|_X + g_\mu^*(1/2)\,\|1\|_{\bar{Y}} \\
&\leq \left\| \int_t^{1/2} \left(-g_\mu^*\right)'(s) I(s) \frac{ds}{I(s)} \right\|_X + 2\,\|1\|_{\bar{Y}}\,\|g\|_{L_1} \\
&\preceq \left\| \left(-g_\mu^*\right)'(s) I(s) \right\|_X + \|g\|_{L_1} \\
&\preceq \|\nabla g\|_X \; + \|g\|_{L_1}.
\end{aligned}$$

Lemma

Given $h \in Lip(\Omega)$ and bounded, there is a sequence $(h_n)_n$ of bounded lip. functions such that:

1. *For every $c \in R$, we have that $|\nabla h_n(x)| = 0$, μ−a.e. on the set $\{x : h_n(x) = c\}$.*
2. $$|\nabla h_n(x)| \leq (1 + \frac{1}{n}) \, |\nabla h(x)| \, .$$
3. $$h_n \underset{n \to 0}{\to} h \text{ in } L^1.$$
4. $$\int_0^t \left(((-h_n)^*)' (\cdot) I(\cdot) \right)^* (s) ds \leq \int_0^t |\nabla h_n|^* (s) ds.$$

Is it possible to obtain an inequality for all functions?

Euclidian case

$$f^{**}(t) - f^{*}(t) \leq c_n \frac{\omega_{L^1}(t^{1/n}, f)}{t}$$

where is X is a r.i. space

$$\omega_X(t, g) = \sup_{|h| \leq t} \|g(. + h) - g(.)\|_X .$$

Since

$$I(t) = t^{1-1/n}$$

this suggests

$$f^{**}(t) - f^{*}(t) \leq c_n \frac{\omega_{L^1}\left(\frac{t}{I(t)}, f\right)}{t}.$$

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$$\omega_X(t, g) = \sup_{|h| \leq t} \|g(. + h) - g(.)\|_X .$$

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Is it possible to obtain an inequality for all functions?

Euclidian case

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Metric spaces: $\omega_X(t, g)$???

In the euclidian case:

$$\omega_{L^1}(t,f) \simeq \inf\{\|f_0\|_1 + t\,\|\nabla f_1\| : f = f_0 + f_1\} := K(t, f; L^1, \mathring{W}^1_{L^1})$$

For each $f \in L_1$

$$f^{**}(t) - f^*(t) \leq 2\frac{K\left(\frac{t}{I(t)}, f; L^1, \overset{\circ}{W}{}^1_{L^1}\right)}{t},\ 0 < t < 1.$$

which implies (up to constant) isoperimetry.

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Mastylo (2010): There exists a universal constant $c > 0$, such that for every r.i. space $X(\Omega)$ with $\bar{\alpha}_X < 1$ and for all $f \in X + \mathring{W}^1_X$, we have

$$f_\mu^{**}(t) - f_\mu^{*}(t) \leq c \frac{K(\frac{t}{I(t)}, f)}{\phi_X(t)}, 0 < t < 1/4. \tag{9}$$

Question: Does (9) hold for all values of t, and without restrictions on the rearrangement invariant spaces X.

In that case, we are thus able to apply our result to sets of any measure, $0 < t < 1$, and, by means of considering $X = L^1$, we are able to show that the validity of (9) for all r.i. spaces is indeed equivalent to the isoperimetric inequality!

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Theorem

Let X be a r.i space on Ω. Then for each $f \in X$

$$f^{**}(t) - f^{*}(t) \leq 2\frac{K\left(\frac{t}{I(t)}, f; X, \overset{\circ}{W}{}^{1}_{X}\right)}{\phi_X(t)}, \ 0 < t < 1.$$

Theorem

The following are equivalent

i) Isoperimetric inequality:

$$I(\mu(A)) \preceq \mu^{+}(A), \text{for all Borel sets } A \text{ with } 0 < \mu(A) < 1.$$

ii) For each $f \in L_1$

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Theorem

Mastylo (2010) Let X be a r.i space, with $0 < \underline{\alpha}_X \leq \bar{\alpha}_X < 1$*. Let* $f \in X$*, then*

$$\left\| \left(f^*(s) - f^*(t) \right) \chi_{(0,t)}(s) \right\|_{\bar{X}} \leq cK\left(\frac{t}{I(t)}, f; X, \overset{\circ}{W}{}_X^1 \right), \; 0 < t < 1.$$

If $X = L^p$ $(1 < p < \infty)$, $\Omega = R^n$

$$\left(\int_0^t \left(f^*(s) - f^*(t) \right)^p ds \right)^{1/p} \leq cK\left(\frac{t}{I(t)}, f; L^p, \overset{\circ}{W}{}_{L^p}^1 \right) \preceq \omega_{L^p}\left(t^{1/n}, f \right)$$

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Theorem

Let X be a r.i space, with $0 < \underline{\alpha}_X$. Let $f \in X$, then the following statements are equivalent

1.
$$f_\mu^{**}(t) - f_\mu^*(t) \leq c\frac{K\left(\frac{t}{I(t)}, f; X, \overset{\circ}{W}{}_Y^1\right)}{\phi_X(t)},\ 0 < t < 1.$$

2.
$$\left\| \left(f_\mu^*(s) - f_\mu^*(t)\right)\chi_{(0,t)}(s) \right\|_{\bar{X}} \leq cK\left(\frac{t}{I(t)}, f; X, \overset{\circ}{W}{}_X^1\right),\ 0 < t < 1.$$

where $c = \|Q\|_{X \to X}$.

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Let $0 < t < 1$ fixed. Assume frist that f is bounded, let $h \in Lip(\Omega)$ such that $h \leq |f|$. Let $g \in \bar{X}'$ with $\|g\|_{\bar{X}'} = 1$. Notice $\bar{X}'$ is a r.i. space on $([0,1], m)$ (here m denotes the Lenesgue measure, we shall denote in what follows by g^* the rearrangment of g with respect to the m). Consider the decomposition

$$|f| = (|f| - h) + h.$$

Then

$$\begin{aligned} I &= \int_0^1 \left(f_\mu^*(s) - f_\mu^*(t)\right) \chi_{(0,t)}(s) g^*(s) ds \\ &\leq \||f| - h\|_X + \left\|\left(h_\mu^*(s) - h_\mu^*(t)\right) \chi_{(0,t)}\right\|_{\bar{X}} \end{aligned} \tag{11}$$

Let $(h_n)_n$ be the sequence to h, then

$$\begin{aligned}(h_n)^*_\mu(s)-(h_n)^*_\mu(t) &= \int_s^t \left(-(h_n)^*_\mu\right)'(z)dz \\ &= \int_s^t \left(-(h_n)^*_\mu\right)'(z)I(s)\frac{dz}{I(z)} \\ &\le \frac{t}{I(t)}\int_s^t \left(-(h_n)^*_\mu\right)'(z)I(s)\frac{dz}{z} \\ &\le \frac{t}{I(t)}\int_s^1 \left(-(h_n)^*_\mu\right)'(z)I(s)\frac{dz}{z}\end{aligned}$$

Since $0<\underline{\alpha}_X$

$$\begin{aligned}\left\|\left((h_n)^*_\mu(s)-(h_n)^*_\mu(t)\right)\chi_{(0,t)}(s)\right\|_{\bar{X}} &\le c\frac{t}{I(t)}\left\|\left(-(h_n)^*_\mu\right)'(z)I(s)\right\|_{\bar{X}} \\ &\le c\frac{t}{I(t)}\|\nabla h_n\|_X \\ &\le c\frac{t}{I(t)}(1+\frac{1}{n})\|\nabla h\|_{\bar{X}}\end{aligned}$$

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It has not been out intention to provide a comprehensive bibliography. Indeed, the topics discussed in this talk have been intensively studied for a long time, with a variety of different approaches. An extensive bibliography has been collected in the paper *Pointwise Symmetrization Inequalities for Sobolev functions and applications*.