Solid hulls of spaces of analytic functions and multipliers

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Definition

- If E and F are sequence spaces on $\mathbb{J}$ ($\mathbb{J} = \mathbb{Z}$ or $\mathbb{J} = \mathbb{Z}_+$), then $\lambda = \{\lambda_n\} \in \omega := \omega(\mathbb{J})$ is said to be a multiplier from E into F provided

$$\{\lambda_n x_n\} \in F \quad \text{for every} \quad \{x_n\} \in E.$$

  The set of all multipliers from E into F is denoted by $\mathcal{M}(E, F)$.

- $\lambda = \{\lambda_n\} \in \omega$ is said to be a multiplier from the space $X \subset H(\mathbb{D})$ (resp., $X \subset L_1(\mathbb{T})$, $\mathbb{T} := [0, 2\pi]$) into a sequence space F provided $\lambda \in \mathcal{M}(\widehat{X}, F)$, where

$$\widehat{X} := \Big\{\{\hat{f}(n)\};\ f = \sum_n \hat{f}(n)u_n \in X\Big\},$$

  and $u_n(z) := z^n$, $z \in \mathbb{D}$, $n \in \mathbb{Z}_+$ (resp., $u_n(t) := e^{int}$, $t \in \mathbb{T}$, $n \in \mathbb{Z}$).

- A subset X of $\omega(\mathbb{J})$ is said to be solid provided

$$\big(\{x_n\} \in X \text{ and } \ |y_n| \leqslant |x_n| \ \text{ for all } n \in \mathbb{J}\big) \implies \{y_n\} \in X.$$

- If $A \subset \omega$ is a sequence space, then there is a largest solid set, $s(A)$, contained within it, and a smallest solid set, $S(A)$, containing it. We also have,

$$s(A) = \mathcal{M}(\ell_\infty, A),$$

$$S(A) = \big\{x \in \omega;\ |x| \leqslant |a| \text{ for some } a \in A\big\}.$$

Theorem (J.M. Anderson and A.L. Shields, 1976)

Let $X \subset \omega$ be any solid linear space. Then for any linear sequence space $A \subset \omega$ we have

$$\mathcal{M}(A, X) = \mathcal{M}(S(A), X),$$

$$\mathcal{M}(X, A) = \mathcal{M}(X, s(A)).$$

THE BASIC PROBLEM

Given a sequence space $A \subset \omega$, find $s(A)$ and $S(A)$.

In what follows if $X \subset H(\mathbb{D})$ (resp., $X \subset L_1(\mathbb{T})$, $\mathbb{T} := [0, 2\pi]$), then

$$s(X) := s(\widehat{X}) \quad \text{and} \quad S(X) := S(\widehat{X}).$$

Examples

- For $L_p := L_p(\mathbb{T})$ with $p = 1$, we have

$$\ell_2 = s(L_1) \subset \widehat{L_1} \subset S(L_1) = c_0$$

(the last equality follows follows from the Riemann-Lebesgue lemma, and the fact: $\{\{\hat{f}(n)x_n\};\, f \in L_1,\, \{x_n\} \in c_0\} = c_0$ (E. Hewitt, 1964)).

- If $1 < p \leqslant 2$, then

$$\ell_2 \subset s(L_p) \subset \widehat{L_p} \subset S(L_p) \subset \ell(p', 2)$$

(by $L_2 \subset L_p \subset L_1$, and $\widehat{L_p} \subset \ell(p', 2)$, $1/p + 1/p' = 1$ (C.N. Kellogg, 1971)).

- If $2 \leqslant p < \infty$, then

$$\ell(p', 2) \subset s(L_p) \subset \widehat{L_p} \subset S(L_p) = \ell_2$$

Theorem (J.P. Kahane, Y. Katznelson and de Leeuw, 1977)

If $\{c_n\}_{n\in\mathbb{Z}} \in \ell_2$, *then there exists* $f \in C(\mathbb{T})$ *and a constant* K *independent of* $\{c_n\}$ *such that*

$$|c_n| \leqslant |\hat{f}(n)|, \quad n \in \mathbb{Z}$$

and

$$\|f\|_{C(\mathbb{T})} \leqslant K \Big(\sum_{n\in\mathbb{Z}} |c_n|^2\Big)^{1/2}.$$

- Denote by $H^+(\mathbb{D})$ the subspace of all $f \in H(\mathbb{D})$ such that the radial limit function

$$f_*(t) := \lim_{r \to 1^-} f_r(t)$$

exists for almost all $t \in \mathbb{T}$, where $f_r(t) := f(re^{it})$ for $0 \leqslant r < 1$ and $t \in \mathbb{T}$. Given a complex quasi-Banach lattice X on $\mathbb{T}$, we define spaces

$$HX := \left\{ f \in H(\mathbb{D});\ \sup_{0 \leqslant r < 1} \|f_r\|_X < \infty \right\},$$

$$HX^+ := \left\{ f \in H^+(\mathbb{D});\ f_* \in X \right\},$$

equipped with the quasi-norms

$$\|f\|_{HX} := \sup_{0 \leqslant r < 1} \|f_r\|_X,$$

$$\|f\|_{HX^+} := \|f_*\|_X.$$

- If $X := L_p$, $1 \leqslant p \leqslant \infty$, we obtain a classical Hardy space $H_p := HL_p$.

Theorem (MM & PM, 2010)

Let X be a quasi-Banach lattice on $\mathbb{T}$ such that $HX^+ \hookrightarrow N^+$. Then HX^+ is a quasi-Banach space.

- In the proof we use a deep theorem of Khintchine and Ostrovski. As usual, $N^+ \subset H(\mathbb{D})$ stands for the space equipped with the F-norm

$$\|f\|_{N^+} = \sup_{0<r<1} \int_0^{2\pi} \log\left(1 + |f(re^{it})|\right) dt.$$

Theorem (Khintchine and Ostrovski)

Let $\{f_n\}_{n=1}^{\infty}$ *be a sequence of functions analytic in the unit disc* $\mathbb{D}$ *satisfying the following conditions:*

(i) *There exists a constant* $C > 0$ *such that*

$$\sup_{0<r<1} \int_0^{2\pi} \log^+ |f_n(re^{it})|\, dt \leqslant C, \quad n \in \mathbb{N}.$$

(ii) *On some set* $E \subset \mathbb{T}$ *of positive measure the sequence* $\{(f_n)_*\}$ *of the radial limits converges in measure to a function* ϕ.

Then the sequence $\{f_n\}$ *converges uniformly on compact subsets of* $\mathbb{D}$ *to a function* f, *and the sequence* $\{(f_n)_*\}$ *converges in measure on* E *to* f_*.

Theorem (MM & PM, 2010)

Assume that X is a r.i. quasi-Banach lattice on $\mathbb{T}$. Suppose that at least one of the following statements holds:

(i) *X has the Fatou property.*

(ii) *X is a real interpolation space with respect to (L_p, L_q), $0 < p < q \leqslant \infty$ (i.e., $X = (L_p, L_q)_E$ for some E).*

Then HX and HX^+ are quasi-Banach spaces and $HX = HX^+$ with the equivalence of quasi-norms.

Theorem (Hausdorff-Young)

If $f = \sum_{n=0}^{\infty} \hat{f}(n) u_n \in H_p$, $1 \leqslant p \leqslant 2$, *then* $\{\hat{f}(n)\} \in \ell_{p'}$, *where* $\frac{1}{p} + \frac{1}{p'} = 1$ *and*

$$\|\{\hat{f}(n)\}\|_{\ell_{p'}} \leqslant \|f\|_{H_p}.$$

In terms of multipliers, $\{1, 1, ..., \} \in \mathcal{M}(H_p, \ell_{p'})$.

Theorem (G.H. Hardy and J.E. Littlewood, 1926)

If $f = \sum_{n=0}^{\infty} \hat{f}(n) u_n \in H_1$, *then*

$$\sum_{n=0}^{\infty} \frac{|\hat{f}(n)|}{n+1} \leqslant \pi \|f\|_{H_1}.$$

In terms of multipliers, $\{1/(n+1)\} \in \mathcal{M}(H_1, \ell_1)$.

Theorem (R.E.A.C. Paley, 1933)

If $\{n_k\}$ is a lacunary sequence, then there exists a positive constant C such that if $f = \sum_{n=0}^{\infty} \hat{f}(n) u_n \in H_1$ we have

$$\Big(\sum_k |\hat{f}(n_k)|^2\Big)^{1/2} \leqslant C\|f\|_{H_1}.$$

In terms of multipliers, $\{\lambda_n\} \in \mathcal{M}(H_1, \ell_2)$, where $\lambda_n = 1$ if $n = n_k$, and $\lambda_n = 0$ if $n \neq n_k$.

Theorem (C. Fefferman, unpublished)

$\{\lambda_n\}_{n\in\mathbb{Z}_+} \in \mathcal{M}(H_1, \ell_1)$ *if and only if*

$$\sup_{N\geqslant 1} \sum_{j=0}^{\infty} \Big(\sum_{k=jN}^{(j+1)N-1} |\lambda_k| \Big)^2 < \infty.$$

Theorem (S.V. Kisliakov, 1981)

$$S(H_\infty) = \ell_2.$$

Theorem (M. Jevtić and M. Pavlović, 2006)

If $0 < p < 1$, *then* $\{x_n\} \in S(H_p)$ *if and only if*

$$\sum_{n=0}^{\infty} 2^{-n(1-p)} \Big(\sup_{0\leqslant k\leqslant 2^n} |x_k| \Big)^p < \infty.$$

Definition

- Let $I_0 := \{0\}$ and for $n \in \mathbb{N}$ set $I_n := \{k \in \mathbb{N};\ 2^{n-1} \leqslant k < 2^n\}$. For a given quasi-normed sequence lattices E, F on $\mathbb{Z}_+$ the space

$$\ell(E, F) := \left\{x \in \omega;\ \left\{\left\|x\chi_{I_n}\right\|_E\right\}_n \in F\right\}$$

equipped with the quasi-norm

$$\left\|x\right\| := \left\|\left\{\left\|x\chi_{I_n}\right\|_E\right\}_n\right\|_F$$

is called the dyadically blocked mixed sequence space of E and F.

- If $E = \ell_p$, $F = \ell_q$, then we obtain the well known space $\ell(p, q)$ introduced by C.N. Kellogg (1971).
- $\ell(\infty, F) := \ell(\ell_\infty, F)$.

Definition

- The galb G_X of a quasi-Banach space X is the space of all sequences $\{\lambda_n\} \in \omega(\mathbb{Z}_+)$ such that the series $\sum_{n\in\mathbb{Z}_+} \lambda_n x_n$ converges provided $\{x_n\}$ is a bounded sequence in X. G_X is a quasi-Banach sequence space when equipped with the quasi-norm

$$\big\|\{\lambda_n\}\big\|_{G_X} := \sup\left\{\Big\|\sum_{k=0}^{\infty}\lambda_k x_k\Big\|_X;\ \|x_k\|_X \leqslant 1\right\}.$$

Remark:

- *If X is a quasi-Banach space, then by the Aoki-Rolewicz Theorem there exists $0 < p \leqslant 1$ such that X is p-normable, and so $\ell_p \subset G_X$. When X is strictly p-normable (i.e., p-normable but not q-normable with any $q > p$), then $G_X = \ell_p$ (N.J. Kalton, unpublished, 1981).*

Theorem (MM & PM, 2010)

Let X be a quasi-Banach lattice on $\mathbb{T}$ such that $L_p \cap L_q \hookrightarrow X$, where $0 < p, q < 1$. If HX^+ is a quasi-Banach space then we have

$$\ell(\infty, G_X(v)) \subset S(HX^+),$$

where $v := \{2^n \phi(2^{-n/p}, 2^{-n/q})\}_{n \in \mathbb{Z}_+}$ and ϕ is the characteristic function of X with respect to (L_p, L_q), i.e.,

$$\phi(s,t) := \sup\left\{\|f\|_X;\ \|f\|_{L_p} \leqslant s,\ \|f\|_{L_q} \leqslant t\right\}, \quad s, t > 0.$$

Definition

- Let (Ω, Σ, μ) be a measure space and $\{\Omega_n\}_{n\in\mathbb{Z}_+} \subset \Sigma$ be a measurable partition of Ω, i.e.,

$$\Omega = \bigcup_{n\in\mathbb{Z}_+} \Omega_n \ \text{ and } \ \Omega_i \cap \Omega_j = \emptyset \ \text{ for } \ i \neq j.$$

- For a given quasi-Banach lattice Y a discretization Y^d of Y consists of all sequences $\{\xi_n\}$ such that $\sum_{n\in\mathbb{Z}_+} \xi_n \chi_{\Omega_n} \in Y$. Note that Y^d is a quasi-Banach sequence lattice equipped with the quasi-norm

$$\big\|\{\xi_n\}\big\|_{Y^d} := \Big\|\sum_{n=0}^{\infty} \xi_n \chi_{\Omega_n}\Big\|_Y.$$

Theorem (MM & PM, 2010)

Let $0 < p, q < 1$ *and* E, Y *be relative interpolation spaces with respect to* (H_p, H_q) *and* $\big(L_p(\mathbb{I}), L_q(\mathbb{I})\big)$. *Then for all* $f \in E$ *the function*

$$r \mapsto (1-r)^{-1} M_1(r, f) \in Y, \quad 0 < r < 1$$

where

$$M_1(r, f) := \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})| \, dt, \quad 0 < r < 1.$$

Theorem (MM & PM, 2010)

Assume that E and Y are relative interpolation spaces with respect to (H_p, H_q) and $\big(L_p(\mathbb{I}), L_q(\mathbb{I})\big)$, $0 < p, q < 1$. Then the following inclusion holds:

$$S(E) \subset \ell\big(\infty, Y^d(2^n)\big),$$

where Y^d is generated by the partition $\big\{[r_n, r_{n+1})\big\}_{n\in\mathbb{Z}_+}$ of the interval $[0,1)$ given by $r_n = 1 - 2^{-n}$, for each $n \in \mathbb{Z}_+$.

Theorem (MM & PM, 2010)

Let $\mathcal{F}$ *be an interpolation functor and* $0 < p_0, p_1 < 1$*. Then we have*

$$S(\mathcal{F}(H_{p_0}, H_{p_1})) = \ell\big(\infty, \mathcal{F}(\ell_{p_0}(v_0), \ell_{p_1}(v_1))\big),$$

where $v_j := \left\{2^{n(1-1/p_j)}\right\}_{n\in\mathbb{Z}_+}$ *for* $j = 0, 1$*.*

Theorem (MM & PM, 2010)

Assume that an r.i. quasi-Banach space X on $\mathbb{T}$ is a real interpolation space with respect to (L_p, L_q), $0 < p, q < 1$. Let $v = \{2^n \rho_X(2^{-n})\}_{n \in \mathbb{Z}_+}$, where ρ_X is the fundamental function of X. Then the following inclusions hold for the solid hull of HX with $Y := X(\mathbb{I})$

$$\ell(\infty, G_X(v)) \subset S(HX) \subset \ell(\infty, Y^d(2^n)),$$

where Y^d is generated by the partition $\{[r_n, r_{n+1})\}_{n \in \mathbb{Z}_+}$ of the interval $[0, 1)$ given by $r_n = 1 - 2^{-n}$, for each $n \in \mathbb{Z}_+$.

Definition

- Let $0 < p < \infty$, $0 < q \leqslant \infty$, $\alpha \in \mathbb{R}$. The Lorentz-Zygmunt space $L_{p,q}(\log L)^\alpha$ on $\mathbb{T}$ consists of all $f \in L^0(\mathbb{T})$ such that

$$\|f\| := \left(\int_{\mathbb{T}} \left(t^{1/p}(1 + |\log t|)^{|\alpha|} f^*(t)\right)^q \frac{dt}{t}\right)^{1/q} < \infty.$$

- In the case when $X = L_{p,q}(\log L)^\alpha$, the Hardy space HX is called Hardy-Lorentz-Zygmund space and is denoted by $H_{p,q}(\log L)^\alpha$. If $\alpha = 0$, then we obtain the Hardy-Lorentz spaces $H_{p,q}$ studied by M. Lengfield (2008), M. Jevtić and M. Pavlović (2009).

Theorem (MM & PM, 2010)

Let $0 < p < 1$, $\alpha \in \mathbb{R}$ *and* $v = \left\{ n^{1-1/p} \log^{\alpha}(n+1) \right\}_{n \in \mathbb{Z}_+}$. *Then for every* $0 < q \leqslant \infty$ *we have*

$$S\big(H_{p,q}(\log L)^{\alpha}\big) = \ell(\infty, q)(v).$$

Theorem (MM & PM, 2010)

Let $0 < p < 1$ *and* $\alpha \in \mathbb{R}$. *Then for any solid space* X, *we have*

$$\mathcal{M}\big(H_{p,\infty}(\log L)^{\alpha}, X\big) = X\big(n^{1/p-1} \log^{-\alpha}(n+1)\big).$$

Remark:

In the case when $\alpha = 0$ *the result was proved in a different way by Jevtić and Pavlović by using nested embeddings for Hardy-Lorentz spaces proved by Lengfield).*

Let $\varphi\colon [0,\infty) \to [0,\infty)$ be an Orlicz function, that is, a non-decreasing, and left-continuous positive function with $\varphi(0) = 0$.

The Orlicz space

$$L_\varphi(\mu) := \big\{x \in L_0(\mu);\ \exists\, \lambda > 0, \quad \varphi(|x|/\lambda) \in L_1(\mu)\big\},$$

If there exists $C > 0$ such that $\varphi(t/C) \leqslant \varphi(t)/2$ for $t > 0$, then L_φ is a quasi-Banach lattice equipped with the quasi-norm

$$\|x\|_\varphi = \inf \Big\{\lambda > 0;\ \int_\Omega \varphi(|x|/\lambda)\, d\mu \leqslant 1\Big\}.$$

- Given an Orlicz function φ,

$$\overline{\varphi}(t) := \limsup_{u\to 0^+} \frac{\varphi(tu)}{\varphi(u)}, \quad t > 0.$$

The lower, respectively, the upper Matuszewska-Orlicz indices of φ are defined by

$$\alpha_\varphi^0 = \lim_{t\to 0+} \frac{\ln \overline{\varphi}(t)}{\ln t}, \quad \text{respectively,} \quad \beta_\varphi^0 = \lim_{t\to\infty} \frac{\ln \overline{\varphi}(t)}{\ln t}.$$

- For a given Orlicz space L_φ we define Hardy-Orlicz space H_φ by $H_\varphi := HL_\varphi$. Since L_φ has the Fatou property, $H_\varphi = HL_\varphi^+$ with the equivalence of norms. However, it is well known that if φ is continuous and such that for any function $f \in H(\mathbb{D})$, $\varphi(|f|)$ is subharmonic on $\mathbb{D}$ (which is equivalent to $t \mapsto \varphi(e^t)$ is a convex function), then the following formula holds with the equality of quasi-norms

$$HL_\varphi = HL_\varphi^+.$$

Theorem (MM & PM, 2010)

Let φ be an Orlicz function such that $0 < \alpha_\varphi^0 \leqslant \beta_\varphi^0 < 1$. Then for the solid hull of the Hardy-Orlicz space H_φ the following equality holds

$$S(H_\varphi) = \ell(\infty, \ell_\phi),$$

where $\phi = \{\varphi_n\}$ with $\varphi_n(t) := 2^{-n}\varphi\big(2^n t\big)$ for $t \geqslant 0$ and $n \in \mathbb{Z}_+$.

Interpolation functor

- If A_0 and A_1 are quasi-Banach spaces such that

$$A_j \hookrightarrow \mathcal{X} \quad (j = 0, 1),$$

then $\overline{A} := (A_0, A_1)$ is called a quasi-Banach couple.

For a given quasi-Banach couple (A_0, A_1) we define:

- intersection $A_0 \cap A_1$ equipped with the quasi-norm

$$\|a\|_{A_0 \cap A_1} = \max\left\{\|a\|_{A_0}, \|a\|_{A_1}\right\}$$

- interpolation sum $A_0 + A_1$ equipped with the quasi-norm

$$\|a\|_{A_0 + A_1} = \inf\left\{\|a_0\|_{A_0} + \|a_1\|_{A_1};\ a = a_0 + a_1\right\}$$

- If $\overline{A} = (A_0, A_1)$ and $\overline{B} = (B_0, B_1)$ are couples of quasi-Banach spaces and $T\colon A_0 + A_1 \to B_0 + B_1$ is a linear operator such that $T|_{A_j}\colon A_j \to B_j$ $(j = 0, 1)$, then we write $T\colon \overline{A} \to \overline{B}$ and

$$\|T\|_{\overline{A}\to\overline{B}} := \max_{j=0,1} \|T|_{A_j}\|_{A_j\to B_j}.$$

- A map $\mathcal{F}\colon \overline{\mathcal{B}} \to \mathcal{B}$ is said to be an interpolation functor if for any $\overline{A}, \overline{B} \in \overline{\mathcal{B}}$ we have
  - (i) $A_0 \cap A_1 \subset \mathcal{F}(\overline{A}) \subset A_0 + A_1$ for any $\overline{A} \in \overline{\mathcal{B}}$
  - (ii) $T\colon \mathcal{F}(\overline{A}) \to \mathcal{F}(\overline{B})$ for any $T\colon \overline{A} \to \overline{B}$.
- $\mathcal{F}$ is said to be an exact interpolation functor if we have

$$\|T\|_{\mathcal{F}(\overline{A})\to\mathcal{F}(\overline{B})} \leqslant \|T\|_{\overline{A}\to\overline{B}}$$

K-method of interpolation

- K-functional of Peetre

$$K(t, a; \overline{A}) := \inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1};\ a = a_0 + a_1\}, \quad t > 0.$$

- If E is a quasi-Banach sequence lattice on $\mathbb{Z}$ such that

$$\ell_\infty \cap \ell_\infty(2^{-n}) \subset E,$$

then the K-space $(\overline{A}_E, \|\cdot\|)$ is defined by

$$\overline{A}_E = \{a \in A_0 + A_1;\ \{K(2^n, a; \overline{A})\} \in E\},$$

$$\|a\| = \big\|\{K(2^n, a; \overline{A})\}\big\|_E.$$

- $\overline{A} \mapsto \overline{A}_E$ is an exact interpolation functor.