2nd International Workshop on Interpolation Theory, and Related Topics
Santiago de Compostela, Spain. July 18 to 22, 2011.

Complex interpolation of vector measures. A generalization of the Stein-Weiss formula.

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Partially supported by MTM2009-14483-C02 and *La Junta de Andalucía*.

Classical Results: Interpolation spaces of integrable function spaces with respect to a scalar positive measure.

- Complex interpolation (M. Riesz, 1926, G.O. Thorin, 1938, Calderón, 60', Lions, 60')

$$[L^{p_0}(\mu), L^{p_1}(\mu)]_{[\theta]} = [L^{p_0}(\mu), L^{p_1}(\mu)]^{[\theta]} = L^{p(\theta)}(\mu).$$

- Real interpolation (J. Marcinkiewicz, 1939, Lions, Peetre, 60')

$$(L^{p_0}(\mu), L^{p_1}(\mu))_{\theta,q} = L^{p(\theta),q}(\mu).$$

For $1 \leq p_0 \neq p_1 \leq \infty$ and $0 < \theta < 1$,

$$\frac{1}{p(\theta)} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

Classical Results: Interpolation spaces with change of measure.

E.M. Stein and G. Weiss, 1958

Generalizations of the theorems of M. Riesz and J. Marcinkiewicz to operators defined on L^p spaces with change of measure

- Complex interpolation
  $[L^{p_0}(f_0\mu), L^{p_1}(f_1\mu)]_{[\theta]} = L^{p(\theta)}(f_0^{1-\alpha} f_1^{\alpha}\mu), \ \alpha = \dfrac{\theta p(\theta)}{p_1}.$
- Real interpolation (the diagonal case, Stein-Weiss, Lions, Peetre)

$$(L^{p_0}(f_0\mu), L^{p_1}(f_1\mu))_{\theta,p(\theta)} = L^{p(\theta)}(f_0^{1-\alpha} f_1^{\alpha}\mu), \ \alpha = \frac{\theta p(\theta)}{p_1}$$

- Real interpolation (the off-diagonal case, Lizorkin, 1976 and Freitag, 1978, Gilbert, Peetre,...)

Classical Results: Interpolation spaces with change of measure.

Extensions

- (I. Asekritova, N. Kruglyak, L. Nikolova, St. Math. 2005) The Lizorkin-Freitag formula for several weighted L^p spaces.
- (M. Cwikel, Proc. Amer. Math. Soc. 1974) The Lions-Peetre formula

$$(L^{p_0}(A_0), L^{p_1}(A_1))_{\theta,q} = L^{p(\theta)}\left((A_0, A_1)_{\theta,q}\right), \; q = p(\theta)$$

  has no natural extension for $q \neq p(\theta)$.
- (Ferreyra, Proc. Amer. Math. Soc. 1997) The Stein-Weiss theorem cannot be extended to Lorentz spaces $L^{p,r}$ with change of measure.

What's about spaces of integrable functions with respect to a vector measure?

For a Banach space X, a measurable space (Ω, Σ) (where Σ is a $\sigma-$algebra of subsets of Ω) and a vector measure (a countably additive set function)

$$m : \Sigma \longrightarrow X$$

let us consider, for $1 \leq p < \infty$, the spaces

- $L^p_w(m)$ of scalar measurable functions f, on (Ω, Σ), such that $|f|^p$ is weakly integrable with respect to m.
- $L^p(m)$ of scalar measurable functions f, on (Ω, Σ), such that $|f|^p$ is integrable with respect to m.

Interpolation of $L^p(m)$ and $L^p_w(m)$ spaces

- Complex interpolation (A. Fernández, F. Naranjo, F. Mayoral, E.A. Sánchez-Pérez, Collect. Math., 2010)

$$[L^{p_0}(m), L^{p_1}(m)]_{[\theta]} = [L^{p_0}_w(m), L^{p_1}_w(m)]_{[\theta]} = L^{p(\theta)}(m)$$

$$[L^{p_0}(m), L^{p_1}(m)]^{[\theta]} = [L^{p_0}_w(m), L^{p_1}_w(m)]^{[\theta]} = L^{p(\theta)}_w(m)$$

- Real interpolation (A. Fernández, F. Naranjo and F. Mayoral, J. Math. Anal. Appl., 2011)

$$(L^{p_0}(m), L^{p_1}(m))_{\theta,q} = (L^{p_0}_w(m), L^{p_1}_w(m))_{\theta,q} = L^{p(\theta),q}(\|m\|).$$

- Complex interpolation with change of measure (R. del Campo, A. Fernández, F. Naranjo, F. Mayoral, E.A. Sánchez-Pérez, Acta Math. Sinica, 2011)

$$[L^{p_0}(m_0), L^{p_1}(m_1)]_{[\theta]} = ??$$

Outline

1. Spaces $L^p(m)$ and $L^p_w(m)$.
2. The interpolated vector measure.
3. Complex interpolation of $L^p(m)-$spaces.

1. Spaces $L^p(m)$ and $L^p_w(m)$.
   - Vector measures.
   - The function spaces.
   - Applications.

2. The interpolated vector measure.

3. Complex interpolation of $L^p(m)-$spaces.

Vector Measures

- (Ω, Σ) measurable space (Σ is a $\sigma-$algebra over a set Ω).
- $m : \Sigma \longrightarrow X$ (countably additive) vector measure in a Banach space X with dual X'.
- The semivariation of m. For $A \in \Sigma$,

$$\|m\|(A) := \sup\left\{\left|\langle m, x'\rangle\right|(A) : x' \in X', \|x'\| \leq 1\right\}$$

- $|\langle m, x'\rangle|$ is the variation measure of the scalar measure $\langle m, x'\rangle$ defined by $\langle m, x'\rangle(A) := \langle m(A), x'\rangle$,

$$\Sigma \xrightarrow{m} X \xrightarrow{x'} \mathbb{R}.$$

The space $L^p_w(m)$

• $L^p_w(m)$ is the space of all scalar measurable functions f on Ω such that $|f|^p$ is a *weakly integrable function* with respect to m. That is, $|f|^p$ is integrable with respect to each $|\langle m, x'\rangle|, x' \in X'$.

- It is a Banach lattice with the natural order $(a.e.)$ and the norm

$$\|f\|_p := \sup\left\{ \int_\Omega |f|^p d\left|\langle m, x'\rangle\right| : \|x'\| \leq 1 \right\}, \qquad f \in L^p_w(m).$$

- $L^p_w(m)$ has the Fatou property.

The space $L^p(m)$

- $L^p(m)$ is the space of all scalar measurable functions f on Ω such that $|f|^p$ is an *integrable function* with respect to m. That is, $f \in L^p_w(m)$ and for each $A \in \Sigma$ there exists $\int_A |f|^p \, dm \in X$ such that

$$\left\langle \int_A |f|^p \, dm, x' \right\rangle = \int_A |f|^p \, d\langle m, x' \rangle, \ \forall x' \in X'.$$

- $L^p(m)$ is an order-continuous closed ideal in $L^p_w(m)$.
- $L^p(m)$ is the closure, in $L^p_w(m)$, of the simple functions.
- $L^1(m) = L^1_w(m)$ if and only if $L^p(m)$ is reflexive for some/every $1 < p < \infty$.

The spaces $L^p(m)$ and $L^p_w(m)$

For $1 < p < q < \infty$,

$$\begin{array}{ccccccccc} & & L^q_w(m) & \subset & L^p_w(m) & \subset & L^1_w(m) & \subset L^0(m) & \\ & & \cup & & \cup & & \cup & & \\ L^\infty(m) & \subset & L^q(m) & \subset & L^p(m) & \subset & L^1(m) & & L^q_w(m) \subset L^p(m) \end{array}$$

Representation of Banach lattices.

Theorem.

- (G. Curbera, 1992) Every order continuous Banach lattice with weak unit is order isometric to a space $L^1(m)$.
- (A. Fernández, F. Mayoral, F. Naranjo, C. Sáez and E.A. Sánchez-Pérez, 2006) Every abstract p-convex Banach lattice with order continuous norm and a weak unit is Banach lattice isomorphic to a space $L^p(m)$.
- (G. Curbera and W. Ricker, 2007) Every abstract p-convex Banach lattice E with the σ-Fatou property and possessing a weak unit which belongs to $\{x \in E\colon |x| \geq u_n \downarrow 0 \text{ implies } \|u_n\| \downarrow 0\}$ is Banach lattice isomorphic to a space $L^p_w(m)$.

Optimal domains

For an order continuous Banach function space $X(\mu)$ (over a positive finite measure), a Banach space E and a continuos linear operator $T : X(\mu) \longrightarrow E$, define

$$m_T : A \in \Sigma \longrightarrow m_T(A) = T(\chi_A) \in E$$

Then:

- m_T is a ($\sigma-$additive) vector measure.
- $X(\mu) \hookrightarrow L^1(m_T)$ and

$$T(f\chi_A) = \int_A f \, dm_T, \; A \in \Sigma.$$

Optimal domains

The integration operator with respect to m_T extends T and in a natural sense $L^1(m_T)$ is the optimal domain for T.

- Convolution operators (G. Curbera,...)
- Kernel operators (G. Curbera, O. Delgado, W. Ricker,...)
- Hardy operator (O. Delgado, J. Soria)
- Fourier transform (G. Mockenhaupt, W. Ricker)
- ...

1. Spaces $L^p(m)$ and $L^p_w(m)$.
2. The interpolated vector measure.
   - The framework.
   - The interpolated measure.
   - The compatibility condition.
3. Complex interpolation of $L^p(m)-$spaces.

The motivation

If μ_0 and μ_1 are two scalar positive measures (over the same measurable space) then they are both absolutely continuous with respect to $\mu = \mu_0 + \mu_1$. The Radon–Nikodym theorem gives us $0 \leq f_0, f_1 \in L^1(\mu)$ such that

$$\mu_0(A) = \int_A f_0 \, d\mu \text{ and } \mu_1(A) = \int_A f_1 \, d\mu.$$

For each $0 < \alpha < 1$, Stein and Weiss consider the scalar positive measure defined by $\mu_\alpha(A) = \displaystyle\int_A f_0^{1-\alpha} f_1^\alpha \, d\mu$. Then the Stein-Weiss interpolation formula reads,

$$[L^{p_0}(\mu_0), L^{p_1}(\mu_1)]_{[\theta]} = L^{p(\theta)}(\mu_\alpha).$$

with

$$\frac{1}{p(\theta)} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \text{ and } \alpha = \frac{\theta p(\theta)}{p_1}.$$

The motivation

The measure μ_α can be defined by

$$\mu_\alpha(A) = \inf\left\{\sum_{B\in\pi}\mu_0(A\cap B)^{1-\alpha}\mu_1(A\cap B)^{\alpha} : \pi\in\Pi(\Omega)\right\}.$$

Here $\Pi(\Omega)$ is the family of finite measurable partitions of Ω.

The motivation

Example 1. The argument fails for positive vector measures.

Let $([0,1], \mathcal{M}, \lambda)$ be the Lebesgue measure space and consider the vector measures defined by

$$m_0(A) = (\lambda(A), 0) \in \mathbb{R}^2 \text{ and } m_1(A) = (0, \lambda(A)) \in \mathbb{R}^2.$$

- $L^1(m_0) = L^1(m_1) = L^1(m_0 + m_1) = L^1([0,1])$.
- $\displaystyle\int f \, d(m_0 + m_1) = \left(\int f \, d\lambda, \int f \, d\lambda \right)$.
- $m_0([0,1]) = (1, 0)$ and $m_1([0,1]) = (0, 1)$.
- There are no functions $f_0, f_1 \in L^1(m_0 + m_1)$ such that

$$\int_\Omega f_0 \, d(m_0 + m_1) = m_0(\Omega) \text{ and } \int_\Omega f_1 \, d(m_0 + m_1) = m_1(\Omega).$$

The framework

We consider vector measures with values in Köthe-Banach function spaces X on a complete $\sigma-$finite measure space (Θ, Λ, η).
That is, X is a Banach lattice consisting in classes, modulo equality $\eta - a.e.$, of locally integrable, real valued functions on Θ that satisfies

(a) If $f \in L^0(\eta), g \in X$ and $|f| \leq |g|$ $\eta-$a.e., then $f \in X$ and $\|f\|_X \leq \|g\|_X$.

(b) $\chi_A \in X$ for every $A \in \Lambda$ with finite measure.

If X has order-continuous norm, the dual X' of X coincides with the Köthe dual

$$X^\times := \left\{g \in L^0(\eta) : fg \in L^1(\eta) \text{ for every } f \in X\right\}.$$

The Calderón product

For a couple (X_0, X_1) of Köthe-Banach function spaces on the same measure space, and $0 < \alpha < 1$, the Calderón product $X(\alpha) := X_0^{1-\alpha} X_1^{\alpha}$ is the set of $x \in L^0(\eta)$ such that

$$|x| \leq x_0^{1-\alpha} x_1^{\alpha} \text{ for some } 0 \leq x_0 \in X_0, 0 \leq x_1 \in X_1.$$

- $X(\alpha)$ is a Köthe-Banach function space with the norm

$$\|x\|_{X(\alpha)} := \inf \left\{ \|x_0\|^{1-\alpha} \|x_1\|^{\alpha} : |x| \leq x_0^{1-\alpha} x_1^{\alpha}, 0 \leq x_i \in X_i \right\}$$

- For every $x_0 \in X_0, x_1 \in X_1$, $\| |x_0|^{1-\alpha} |x_1|^{\alpha} \|_{X(\alpha)} \leq \|x_0\|^{1-\alpha} \|x_1\|^{\alpha}$
- If X_0 or X_1 has order continuous norm then the norm of $X(\alpha)$ is order-continuous too and

$$[X_0, X_1]_{[\alpha]} = X(\alpha).$$

The interpolated measure

Let $0 < \alpha < 1$ and X_0 and X_1 be two Köthe-Banach function spaces such that

$$X(\alpha) := X_0^{1-\alpha} X_1^{\alpha}$$

is order-continuous. Now consider two positive vector measures on the same measurable space (Ω, Σ),

$$m_0 : \Sigma \longrightarrow X_0 \text{ and } m_1 : \Sigma \longrightarrow X_1.$$

For a measurable partition $\pi \in \Pi(\Omega)$ of Ω and a measurable subset $A \in \Sigma$, denote

$$C_\pi(A) := \sum_{B \in \pi} m_0(A \cap B)^{1-\alpha} m_1(A \cap B)^{\alpha} \in X(\alpha)$$

The interpolated measure

Definition

$$[m_0, m_1]_\alpha(A) := \lim_\pi C_\pi(A)(= \inf_\pi C_\pi(A))$$

- $[m_0, m_1]_\alpha(A) \in X(\alpha)$ is well-defined since $X(\alpha)$ is order-continuous.
- $0 \leq [m_0, m_1]_\alpha(A) \leq m_0(A)^{1-\alpha} m_1(A)^\alpha$ $(\mu - a.e.)$
- $\|[m_0, m_1]_\alpha(A)\|_{X(\alpha)} \leq \|m_0(A)\|_{X_0}^{1-\alpha} \|m_1(A)\|_{X_1}^\alpha$.

The interpolated measure

Lemma

Let m_0 and m_1 be two equivalent positive vector measures and $0 < \alpha < 1$.

- $[m_0, m_1]_\alpha : \Sigma \longrightarrow X(\alpha)$ is a (countably additive) positive vector measure.
- For every $A \in \Sigma$ and every $0 \leq x' \in X(\alpha)'$ such that $x' \leq (x_0')^{1-\alpha}(x_1')^\alpha,\ 0 \leq x_0' \in X_0', 0 \leq x_1' \in X_1'$, $\langle [m_0, m_1]_\alpha(A), x' \rangle \leq \langle m_0(A), x_0' \rangle^{1-\alpha} \langle m_1(A), x_1' \rangle^\alpha$
- In particular, $\langle [m_0, m_1]_\alpha, x' \rangle \leq [\langle m_0, x_0' \rangle, \langle m_1, x_1' \rangle]_\alpha$.
- $\|[m_0, m_1]_\alpha\|(A) \leq (\|m_0\|(A))^{1-\alpha}(\|m_1\|(A))^\alpha$.

The $L^1(m)-$space of the interpolated measure

Proposition

Let $m_0 : \Sigma \longrightarrow X_0$ and $m_1 : \Sigma \longrightarrow X_1$ two equivalent positive vector measures on (Ω, Σ). Then, for every $0 < \alpha < 1$,

$$\left(L^1(m_0)\right)^{1-\alpha} \left(L^1(m_1)\right)^{\alpha} \subseteq L^1\left([m_0, m_1]_\alpha\right)$$

is a continuous inclusion.

Remark. In general, this inclusion is non-injective. The interpolated measure $[m_0, m_1]_\alpha$ can be the null measure even if m_0 and m_1 are non-trivial. In this case, the inclusion is simply the zero map.

Examples

Example 1

Let $([0,1],\mathcal{M},\lambda)$ be the Lebesgue measure space and consider the vector measures defined by

$$m_0(A)=(\lambda(A),0)\in\mathbb{R}^2 \text{ and } m_1(A)=(0,\lambda(A))\in\mathbb{R}^2.$$

- $L^1(m_0)=L^1(m_1)=L^1([0,1])$.
- $[m_0,m_1]_\alpha=0$ for every $0<\alpha<1$.
- $\left(L^1(m_0)\right)^{1-\alpha}\left(L^1(m_1)\right)^{\alpha}=L^1([0,1])$.

Examples

Example 2

Let $([0,1],\mathcal{M},\lambda)$ be the Lebesgue measure space and consider $1 \leq s_1 \leq s_0 < \infty$ and a function $0 < g \in L^t(\lambda)$, where $\frac{1}{s_0}+\frac{1}{t}=\frac{1}{s_1}$. Consider the vector measures defined by

$$m_0 : A \in \mathcal{M} \longrightarrow m_0(A) = \chi_A \in L^{s_0}(\lambda),$$
$$m_1 : A \in \mathcal{M} \longrightarrow m_1(A) = g\chi_A \in L^{s_1}(\lambda).$$

- $[m_0, m_1]_\alpha(A) = g^\alpha \chi_A \in L^s(\lambda), \frac{1}{s} = \frac{1-\alpha}{s_0} + \frac{\alpha}{s_1}$.
- $g \in L^t(\lambda) \Longrightarrow g^\alpha \in L^s(\lambda)$ since $\alpha s < t$.
- $L^1(m_0) = L^{s_0}(\lambda),\ L^1(m_1) = \{f : fg \in L^{s_1}(\lambda)\} = L^{s_1}(g^{s_1}\lambda)$,
- $(L^1(m_0))^{1-\alpha}(L^1(m_1))^\alpha = L^s(g^{\alpha s}\lambda)$.
- $L^1([m_0, m_1]_\alpha) = \{f : fg^\alpha \in L^s(\lambda)\} = L^s(g^{\alpha s}\lambda)$.

Examples

Example 3

Let X_0 and X_1 be two Köthe-Banach function spaces over a $\sigma-$finite measure space. For a pair of positive unconditionally convergent series $\sum_n f_n$ in X_0 and $\sum_n g_n$ in X_1 consider the vector measures defined over $\mathcal{P}(\mathbb{N})$ by

$$m_0(A) = \sum_{n\in A} f_n \in X_0 \text{ and } m_1(A) = \sum_{n\in A} g_n \in X_1.$$

- $\sum_n f_n^{1-\alpha} g_n^\alpha$ is a positive unconditionally convergent series in $X_0^{1-\alpha} X_1^\alpha$.
- $[m_0, m_1]_\alpha(A) = \sum_{n\in A} f_n^{1-\alpha} g_n^\alpha$.
- $L^1(m_0)^{1-\alpha} L^1(m_1)^\alpha = L^1([m_0, m_1]_\alpha)$?

The compatibility condition

Definition. Compatibility

A pair of equivalent vector measures m_0 and m_1 are said to be $\alpha-$compatible, for $0 < \alpha < 1$ if

$$\left(L^1(m_0)\right)^{1-\alpha}\left(L^1(m_1)\right)^{\alpha} = L^1\left([m_0, m_1]_\alpha\right)$$

Equivalently, $\left[L^1(m_0), L^1(m_1)\right]_\alpha = L^1\left([m_0, m_1]_\alpha\right)$.

Remark. If $m_0, m_1 : \Sigma \longrightarrow X$ are two positive vector measures such that there exists a vector measure m and functions $0 < f_0, f_1 \in L^1(m)$ such that

$$m_0 = f_0 m \text{ and } m_1 = f_1 m \quad \left(f_i m(A) := \int_A f_i \, dm\right)$$

then m_0 and m_1 are $\alpha-$compatible for every $0 < \alpha < 1$.

Radon-Nikodym derivative with respect to a vector measure

Definitions

Let $m, n : \Sigma \longrightarrow X$ two vector measures with values in a Banach space. We say that

a) n is scalarly uniformly absolutely continuous with respect to m if $\forall \varepsilon > 0, \exists \delta > 0$ such that
$\forall x' \in X', A \in \Sigma : |\langle m, x' \rangle|(A) < \delta \Longrightarrow |\langle n, x' \rangle|(A) < \varepsilon$

b) n is scalarly dominated by m if there exists $M > 0$ such that
$|\langle n, x' \rangle|(A) \leq M |\langle m, x' \rangle|(A), \ \forall A \in \Sigma, x' \in X'.$

Radon-Nikodym derivative with respect to a vector measure

Theorem, Musial, 1993

The following conditions are equivalent:

1) n has a Radon-Nikodym derivative with respect to m. That is, there exists a (scalar) bounded measurable function f such that
$$n(A) = \int_A f\,dm \;\forall A \in \Sigma.$$
2) n is scalarly uniformly absolutely continuous with respect to m.
3) n is scalarly dominated by m.

1. Spaces $L^p(m)$ and $L^p_w(m)$.
2. The interpolated vector measure.
3. Complex interpolation of $L^p(m)-$spaces.
   - Interpolation of $L^1(m)-$spaces.
   - Interpolation of $L^p(m)-$spaces.
   - Interpolation of tensor products.

Interpolation of $L^1(m)-$spaces

Theorem

Let $m_0 : \Sigma \longrightarrow X_0$ and $m_1 : \Sigma \longrightarrow X_1$ be two $\alpha-$compatible vector measures and consider two functions $0 < f_0 \in L^1(m_0)$ and $0 < f_1 \in L^1(m_1)$. Then

$$\left(L^1(f_0 m_0)\right)^{1-\alpha} \left(L^1(f_1 m_1)\right)^{\alpha} = L^1\left(f_0^{1-\alpha} f_1^{\alpha}[m_0, m_1]_\alpha\right)$$

Interpolation of $L^p(m)-$spaces

Theorem

Let $m_0 : \Sigma \longrightarrow X_0$ and $m_1 : \Sigma \longrightarrow X_1$ be two $\alpha-$compatible vector measures and consider two functions $0 < f_0 \in L^1(m_0)$ and $0 < f_1 \in L^1(m_1)$. Then, for $0 < \theta < 1 \leq p_0, p_1 < \infty$,

$$[L^{p_0}(f_0 m_0), L^{p_1}(f_1 m_1)]_{[\theta]} = L^{p(\theta)}\left(f_0^{1-\alpha} f_1^{\alpha}[m_0, m_1]_\alpha\right)$$

with $\alpha = \dfrac{\theta p(\theta)}{p_1}$.

Corollary

Let m be a positive vector measure with values in a Köthe-Banach function space and consider two functions $0 < f_0, f_1 \in L^1(m)$. Then, for $0 < \theta < 1 \leq p_0, p_1 < \infty$,

$$[L^{p_0}(f_0 m), L^{p_1}(f_1 m)]_{[\theta]} = L^{p(\theta)}\left(f_0^{1-\alpha} f_1^{\alpha} m\right)$$

with $\alpha = \dfrac{\theta p(\theta)}{p_1}$.

Corollary

Let $0<\theta<1\leq p_0,p_1,q_0,q_1<\infty$ and $\alpha=\frac{\theta p(\theta)}{p_1}, \beta=\frac{\theta q(\theta)}{q_1}$ with $\frac{1}{p(\theta)}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}, \frac{1}{q(\theta)}=\frac{1-\theta}{q_0}+\frac{\theta}{q_1}$. Let (m_0,m_1) be a couple of $\alpha-$compatible vector measures and (n_0,n_1) be a couple of $\beta-$compatible vector measures. If

$$T: L^{p_0}(m_0)+L^{p_1}(m_1)\longrightarrow L^{q_0}(n_0)+L^{q_1}(n_1)$$

is a linear operator such that the restrictions $T_0: L^{p_0}(m_0)\longrightarrow L^{q_0}(n_0)$ and $T_1: L^{p_1}(m_1)\longrightarrow L^{q_1}(n_1)$ are well defined and continuous then

$$T: L^{p(\theta)}([m_0,m_1]_\alpha)\longrightarrow L^{q(\theta)}([n_0,n_1]_\beta)$$

is well defined and continuous

Interpolation of injective tensor products

Corollary

Let $0 < \theta < 1 \leq p_0, p_1, q_0, q_1 < \infty, \alpha, \beta, p(\theta), q(\theta)$, (m_0, m_1) and (n_0, n_1) be as before.

- If $L^{p_0}(m_0), L^{p_1}(m_1), L^{q_0}(n_0)$ and $L^{q_1}(n_1)$ are 2-concave Banach lattices, then

$$\left[L^{p_0}(m_0)\hat{\otimes}_\varepsilon L^{q_0}(n_0), L^{p_1}(m_1)\hat{\otimes}_\varepsilon L^{q_1}(n_1)\right]_{[\theta]} = \text{(Kouba,1991)}$$

$$= [L^{p_0}(m_0), L^{p_1}(m_1)]_{[\theta]}\hat{\otimes}_\varepsilon [L^{q_0}(n_0), L^{q_1}(n_1)]_{[\theta]}$$

$$= L^{p(\theta)}([m_0, m_1]_\alpha)\hat{\otimes}_\varepsilon L^{q(\theta)}([n_0, n_1]_\beta).$$

Interpolation of tensor products

Corollary

Let $0 < \theta < 1 \leq p_0, p_1, q_0, q_1 < \infty, \alpha, \beta, p(\theta), q(\theta)$, (m_0, m_1) and (n_0, n_1) be as before.

- If $p_0, p_1, q_0, q_1 \geq 2$ ($\Rightarrow$ the spaces are 2-convex Banach laticces), then

$$\left[L^{p_0}(m_0)\hat{\otimes}_\pi L^{q_0}(n_0), L^{p_1}(m_1)\hat{\otimes}_\pi L^{q_1}(n_1)\right]_{[\theta]} = \text{(Kouba,1991)}$$

$$= [L^{p_0}(m_0), L^{p_1}(m_1)]_{[\theta]}\hat{\otimes}_\pi[L^{q_0}(n_0), L^{q_1}(n_1)]_{[\theta]}$$

$$= L^{p(\theta)}([m_0, m_1]_\alpha)\hat{\otimes}_\pi L^{q(\theta)}([n_0, n_1]_\beta).$$

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That's all.
Thank you.