Complex interpolation of vector measures.
A generalization of the Stein-Weiss formula.

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Classical Results: Interpolation spaces of integrable function spaces with respect to a scalar positive measure.

- Complex interpolation (M. Riesz, 1926, G.O. Thorin, 1938, Calderón, 60’, Lions, 60’)

\[ [L^{p_0}(\mu), L^{p_1}(\mu)][\theta] = [L^{p_0}(\mu), L^{p_1}(\mu)][\theta] = L^{p(\theta)}(\mu). \]

- Real interpolation (J. Marcinkiewicz, 1939, Lions, Peetre, 60’)

\[ (L^{p_0}(\mu), L^{p_1}(\mu))_{\theta,q} = L^{p(\theta),q}(\mu). \]

For \( 1 \leq p_0 \neq p_1 \leq \infty \) and \( 0 < \theta < 1 \),

\[ \frac{1}{p(\theta)} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}. \]
Classical Results: Interpolation spaces with change of measure.

E.M. Stein and G. Weiss, 1958

Generalizations of the theorems of M. Riesz and J. Marcinkiewicz to operators defined on $L^p$ spaces with change of measure

- **Complex interpolation**
  \[
  [L^{p_0}(f_0\mu), L^{p_1}(f_1\mu)]_{\theta} = L^{p(\theta)}(f_0^{1-\alpha} f_1^\alpha \mu), \quad \alpha = \frac{\theta p(\theta)}{p_1}.
  \]

- **Real interpolation** (the diagonal case, Stein-Weiss, Lions, Peetre)
  \[
  (L^{p_0}(f_0\mu), L^{p_1}(f_1\mu))_{\theta,p(\theta)} = L^{p(\theta)}(f_0^{1-\alpha} f_1^\alpha \mu), \quad \alpha = \frac{\theta p(\theta)}{p_1}
  \]

- **Real interpolation** (the off-diagonal case, Lizorkin, 1976 and Freitag, 1978, Gilbert, Peetre,...)
Classical Results: Interpolation spaces with change of measure.

**Extensions**


$$
(L^{p_0}(A_0), L^{p_1}(A_1))_{\theta,q} = L^{p(\theta)}((A_0, A_1)_{\theta,q}), \quad q = p(\theta)
$$

has no natural extension for $q \neq p(\theta)$.

What’s about spaces of integrable functions with respect to a vector measure?

For a Banach space $X$, a measurable space $(\Omega, \Sigma)$ (where $\Sigma$ is a $\sigma-$algebra of subsets of $\Omega$) and a vector measure (a countably additive set function)

$$m : \Sigma \longrightarrow X$$

let us consider, for $1 \leq p < \infty$, the spaces

- $L^p_{w}(m)$ of scalar measurable functions $f$, on $(\Omega, \Sigma)$, such that $|f|^p$ is weakly integrable with respect to $m$.
- $L^p(m)$ of scalar measurable functions $f$, on $(\Omega, \Sigma)$, such that $|f|^p$ is integrable with respect to $m$. 
Interpolation of $L^p(m)$ and $L^p_w(m)$ spaces

  
  $[L^p_0(m), L^p_1(m)]_{[\theta]} = [L^p_w(m), L^p_{w_1}(m)]_{[\theta]} = L^p(\theta)(m)$

  $[L^p_0(m), L^p_1(m)]^{[\theta]} = [L^p_w(m), L^p_{w_1}(m)]^{[\theta]} = L^p_{p(\theta)}(m)$


  $(L^p_0(m), L^p_1(m))_{\theta,q} = (L^p_w(m), L^p_{w_1}(m))_{\theta,q} = L^{p(\theta),q}(\|m\|)$.


  $[L^p_0(m_0), L^p_1(m_1)]_{[\theta]} = ??$
1. Spaces $L^p(m)$ and $L^p_w(m)$.

2. The interpolated vector measure.

3. Complex interpolation of $L^p(m)$—spaces.
Spaces $L^p(m)$ and $L^p_w(m)$.
- Vector measures.
- The function spaces.
- Applications.

The interpolated vector measure.

Complex interpolation of $L^p(m)$—spaces.
Vector Measures

1. \((Ω, Σ)\) measurable space \((Σ\) is a \(σ\)—algebra over a set \(Ω)\).
2. \(m : Σ \rightarrow X\) (countably additive) vector measure in a Banach space \(X\) with dual \(X'\).
3. The semivariation of \(m\). For \(A ∈ Σ\),

\[
\|m\|(A) := \sup \left\{ |⟨m, x'⟩| (A) : x' \in X', \|x'\| ≤ 1 \right\}
\]

4. \(|⟨m, x'⟩|\) is the variation measure of the scalar measure \(<m, x'>\) defined by \(<m, x'> (A) := ⟨m(A), x'⟩\),

\[
Σ \xrightarrow{m} X \xrightarrow{x'} \mathbb{R}.
\]
The space \( L^p_w(m) \)

- \( L^p_w(m) \) is the space of all scalar measurable functions \( f \) on \( \Omega \) such that \(|f|^p\) is a \textit{weakly integrable function} with respect to \( m \). That is, \(|f|^p\) is integrable with respect to each \(|\langle m, x' \rangle|\), \( x' \in X' \).

- It is a Banach lattice with the natural order (a.e.) and the norm

\[
\|f\|_p := \sup \left\{ \int \Omega |f|^p d|\langle m, x' \rangle| : \|x'\| \leq 1 \right\}, \quad f \in L^p_w(m).
\]

- \( L^p_w(m) \) has the Fatou property.
The space $L^p(m)$

- $L^p(m)$ is the space of all scalar measurable functions $f$ on $\Omega$ such that $|f|^p$ is an integrable function with respect to $m$. That is, $f \in L^p_w(m)$ and for each $A \in \Sigma$ there exists $\int_A |f|^p dm \in X$ such that

$$\left\langle \int_A |f|^p dm, x' \right\rangle = \int_A |f|^p d\langle m, x' \rangle, \forall x' \in X'.$$

- $L^p(m)$ is an order-continuous closed ideal in $L^p_w(m)$.
- $L^p(m)$ is the closure, in $L^p_w(m)$, of the simple functions.
- $L^1(m) = L^1_w(m)$ if and only if $L^p(m)$ is reflexive for some/every $1 < p < \infty$. 

Vector measures.
The function spaces.
Applications.
The spaces \( L^p(m) \) and \( L^p_w(m) \)

For \( 1 < p < q < \infty \),

\[
L^q_w(m) \subset L^p(m) \subset L^1_w(m) \subset L^0(m)
\]

\[
L^\infty(m) \subset L^q(m) \subset L^p(m) \subset L^1(m) \]

\[
L^q_w(m) \subset L^p(m)
\]
Representation of Banach lattices.

**Theorem.**

- (G. Curbera, 1992) Every order continuous Banach lattice with weak unit is order isometric to a space $L^1(m)$.
- (A. Fernández, F. Mayoral, F. Naranjo, C. Sáez and E.A. Sánchez-Pérez, 2006) Every abstract $p$-convex Banach lattice with order continuous norm and a weak unit is Banach lattice isomorphic to a space $L^p(m)$.
- (G. Curbera and W. Ricker, 2007) Every abstract $p$-convex Banach lattice $E$ with the $\sigma$-Fatou property and possessing a weak unit which belongs to $\{x \in E : |x| \geq u_n \downarrow 0 \}$ implies $\|u_n\| \downarrow 0$ is Banach lattice isomorphic to a space $L^p_w(m)$. 
Optimal domains

For an order continuous Banach function space $X(\mu)$ (over a positive finite measure), a Banach space $E$ and a continuous linear operator $T : X(\mu) \rightarrow E$, define

$$m_T : A \in \Sigma \rightarrow m_T(A) = T(\chi_A) \in E$$

Then:

- $m_T$ is a ($\sigma-$additive) vector measure.
- $X(\mu) \hookrightarrow L^1(m_T)$ and

$$T(f\chi_A) = \int_A f \, dm_T, \ A \in \Sigma.$$
The integration operator with respect to $m_T$ extends $T$ and in a natural sense $L^1(m_T)$ is the optimal domain for $T$.

- Convolution operators (G. Curbera, ...)
- Kernel operators (G. Curbera, O. Delgado, W. Ricker, ...)
- Hardy operator (O. Delgado, J. Soria)
- Fourier transform (G. Mockenhaupt, W. Ricker)
- ...
1. Spaces $L^p(m)$ and $L^p_w(m)$.

2. The interpolated vector measure.
   - The framework.
   - The interpolated measure.
   - The compatibility condition.

3. Complex interpolation of $L^p(m)$–spaces.
The motivation

If $\mu_0$ and $\mu_1$ are two scalar positive measures (over the same measurable space) then they are both absolutely continuous with respect to $\mu = \mu_0 + \mu_1$. The Radon–Nikodym theorem gives us $0 \leq f_0, f_1 \in L^1(\mu)$ such that

$$\mu_0(A) = \int_A f_0 \, d\mu \quad \text{and} \quad \mu_1(A) = \int_A f_1 \, d\mu.$$ 

For each $0 < \alpha < 1$, Stein and Weiss consider the scalar positive measure defined by $\mu_\alpha(A) = \int_A f_0^{1-\alpha} f_1^\alpha \, d\mu$. Then the Stein-Weiss interpolation formula reads,

$$[L^{p_0}(\mu_0), L^{p_1}(\mu_1)][\theta] = L^{p(\theta)}(\mu_\alpha).$$

with

$$\frac{1}{p(\theta)} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \text{and} \quad \alpha = \frac{\theta p(\theta)}{p_1}.$$
The measure $\mu_\alpha$ can be defined by

$$
\mu_\alpha(A) = \inf \left\{ \sum_{B \in \pi} \mu_0(A \cap B)^{1-\alpha} \mu_1(A \cap B)^\alpha : \pi \in \Pi(\Omega) \right\}.
$$

Here $\Pi(\Omega)$ is the family of finite measurable partitions of $\Omega$. 
Example 1. The argument fails for positive vector measures.

Let ([0, 1], ℳ, λ) be the Lebesgue measure space and consider the vector measures defined by

\[ m_0(A) = (\lambda(A), 0) \in \mathbb{R}^2 \text{ and } m_1(A) = (0, \lambda(A)) \in \mathbb{R}^2. \]

- \( L^1(m_0) = L^1(m_1) = L^1(m_0 + m_1) = L^1([0, 1]). \)
- \( \int f \, d(m_0 + m_1) = \left( \int f \, d\lambda, \int f \, d\lambda \right). \)
- \( m_0([0, 1]) = (1, 0) \text{ and } m_1([0, 1]) = (0, 1). \)
- There are no functions \( f_0, f_1 \in L^1(m_0 + m_1) \) such that
  \[ \int_{\Omega} f_0 \, d(m_0 + m_1) = m_0(\Omega) \text{ and } \int_{\Omega} f_1 \, d(m_0 + m_1) = m_1(\Omega). \]
The framework

We consider vector measures with values in Köthe-Banach function spaces $X$ on a complete $\sigma$–finite measure space $(\Theta, \Lambda, \eta)$. That is, $X$ is a Banach lattice consisting in classes, modulo equality $\eta - a.e.$, of locally integrable, real valued functions on $\Theta$ that satisfies

(a) If $f \in L^0(\eta), g \in X$ and $|f| \leq |g|$ $\eta - a.e.$, then $f \in X$ and $\|f\|_X \leq \|g\|_X$.

(b) $\chi_A \in X$ for every $A \in \Lambda$ with finite measure.

If $X$ has order-continuous norm, the dual $X'$ of $X$ coincides with the Köthe dual

$$X^\times := \{ g \in L^0(\eta) : fg \in L^1(\eta) \text{ for every } f \in X \}.$$
The Calderón product

For a couple \((X_0, X_1)\) of Köthe-Banach function spaces on the same measure space, and \(0 < \alpha < 1\), the Calderón product \(X(\alpha) := X_0^{1-\alpha}X_1^\alpha\) is the set of \(x \in L^0(\eta)\) such that

\[ |x| \leq x_0^{1-\alpha}x_1^\alpha \text{ for some } 0 \leq x_0 \in X_0, 0 \leq x_1 \in X_1. \]

- \(X(\alpha)\) is a Köthe-Banach function space with the norm
  \[
  \|x\|_{X(\alpha)} := \inf \left\{ \|x_0\|^{1-\alpha}\|x_1\|^\alpha : |x| \leq x_0^{1-\alpha}x_1^\alpha, 0 \leq x_i \in X_i \right\}
  \]

- For every \(x_0 \in X_0, x_1 \in X_1\),
  \[
  \|x_0|^{1-\alpha}|x_1|^\alpha\|_{X(\alpha)} \leq \|x_0\|^{1-\alpha}\|x_1\|^\alpha
  \]

- If \(X_0\) or \(X_1\) has order continuous norm then the norm of \(X(\alpha)\) is order-continuous too and
  \[
  [X_0, X_1]_[\alpha] = X(\alpha).
  \]
The interpolated measure

Let $0 < \alpha < 1$ and $X_0$ and $X_1$ be two Köthe-Banach function spaces such that

$$X(\alpha) := X_0^{1-\alpha} X_1^\alpha$$

is order-continuous. Now consider two positive vector measures on the same measurable space $(\Omega, \Sigma)$,

$$m_0 : \Sigma \longrightarrow X_0 \quad \text{and} \quad m_1 : \Sigma \longrightarrow X_1.$$ 

For a measurable partition $\pi \in \Pi(\Omega)$ of $\Omega$ and a measurable subset $A \in \Sigma$, denote

$$C_\pi(A) := \sum_{B \in \pi} m_0(A \cap B)^{1-\alpha} m_1(A \cap B)^\alpha \in X(\alpha)$$
The interpolated measure

Definition

\[ [m_0, m_1]_\alpha(A) := \lim_{\pi} C_\pi(A)(= \inf_{\pi} C_\pi(A)) \]

- \([m_0, m_1]_\alpha(A) \in X(\alpha)\) is well-defined since \(X(\alpha)\) is order-continuous.
- \(0 \leq [m_0, m_1]_\alpha(A) \leq m_0(A)^{1-\alpha} m_1(A)^\alpha (\mu - a.e.)\)
- \(\|[m_0, m_1]_\alpha(A)\|_{X(\alpha)} \leq \|m_0(A)\|_{X_0}^{1-\alpha} \|m_1(A)\|_{X_1}^\alpha.\)
The interpolated measure

Lemma

Let $m_0$ and $m_1$ be two equivalent positive vector measures and $0 < \alpha < 1$.

1. $[m_0, m_1]_\alpha : \Sigma \rightarrow X(\alpha)$ is a (countably additive) positive vector measure.

2. For every $A \in \Sigma$ and every $0 \leq x' \in X(\alpha)'$ such that $x' \leq (x_0')^{1-\alpha}(x_1')^\alpha$, $0 \leq x_0' \in X_0'$, $0 \leq x_1' \in X_1'$,
   \[ \langle [m_0, m_1]_\alpha(A), x' \rangle \leq \langle m_0(A), x_0' \rangle^{1-\alpha} \langle m_1(A), x_1' \rangle^\alpha \]

3. In particular, $\langle [m_0, m_1]_\alpha, x' \rangle \leq \langle [m_0, x_0'], [m_1, x_1'] \rangle_\alpha$.

4. $\|[m_0, m_1]_\alpha\|(A) \leq (\|m_0\|(A))^{1-\alpha} (\|m_1\|(A))^\alpha$. 

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Complex interpolation of vector measures
The $L^1(m)$—space of the interpolated measure

**Proposition**

Let $m_0 : \Sigma \rightarrow X_0$ and $m_1 : \Sigma \rightarrow X_1$ two equivalent positive vector measures on $(\Omega, \Sigma)$. Then, for every $0 < \alpha < 1$,

$$(L^1(m_0))^{1-\alpha} (L^1(m_1))^\alpha \subseteq L^1([m_0, m_1]_\alpha)$$

is a continuous inclusion.

**Remark.** In general, this inclusion is non-injective. The interpolated measure $[m_0, m_1]_\alpha$ can be the null measure even if $m_0$ and $m_1$ are non-trivial. In this case, the inclusion is simply the zero map.
Examples

Example 1

Let \(([0, 1], \mathcal{M}, \lambda)\) be the Lebesgue measure space and consider the vector measures defined by

\[ m_0(A) = (\lambda(A), 0) \in \mathbb{R}^2 \text{ and } m_1(A) = (0, \lambda(A)) \in \mathbb{R}^2. \]

- \(L^1(m_0) = L^1(m_1) = L^1([0, 1]).\)
- \([m_0, m_1]_\alpha = 0\) for every \(0 < \alpha < 1.\)
- \((L^1(m_0))^{1-\alpha} (L^1(m_1))^\alpha = L^1([0, 1]).\)
Example 2

Let \([0, 1], \mathcal{M}, \lambda\) be the Lebesgue measure space and consider \(1 \leq s_1 \leq s_0 < \infty\) and a function \(0 < g \in L^t(\lambda)\), where \(\frac{1}{s_0} + \frac{1}{t} = \frac{1}{s_1}\). Consider the vector measures defined by

\[
m_0 : A \in \mathcal{M} \mapsto m_0(A) = \chi_A \in L^{s_0}(\lambda),
\]
\[
m_1 : A \in \mathcal{M} \mapsto m_1(A) = g\chi_A \in L^{s_1}(\lambda).
\]

- \([m_0, m_1]_\alpha(A) = g^\alpha \chi_A \in L^s(\lambda)\), \(\frac{1}{s} = \frac{1-\alpha}{s_0} + \frac{\alpha}{s_1}\).
- \(g \in L^t(\lambda) \implies g^\alpha \in L^s(\lambda)\) since \(\alpha s < t\).
- \(L^1(m_0) = L^{s_0}(\lambda), \quad L^1(m_1) = \{f : fg \in L^{s_1}(\lambda)\} = L^{s_1}(g^{s_1} \lambda)\),
- \((L^1(m_0))^{1-\alpha}(L^1(m_1))^\alpha = L^s(g^{\alpha s} \lambda)\).
- \(L^1([m_0, m_1]_\alpha) = \{f : fg^\alpha \in L^s(\lambda)\} = L^s(g^{\alpha s} \lambda)\).
Example 3

Let $X_0$ and $X_1$ be two Köthe-Banach function spaces over a $\sigma$—finite measure space. For a pair of positive unconditionally convergent series $\sum_n f_n$ in $X_0$ and $\sum_n g_n$ in $X_1$ consider the vector measures defined over $\mathcal{P}(\mathbb{N})$ by

$$m_0(A) = \sum_{n \in A} f_n \in X_0 \text{ and } m_1(A) = \sum_{n \in A} g_n \in X_1.$$ 

- $\sum_n f_n^{1-\alpha} g_n^\alpha$ is a positive unconditionally convergent series in $X_0^{1-\alpha} X_1^\alpha$.
- $[m_0, m_1]_\alpha(A) = \sum_{n \in A} f_n^{1-\alpha} g_n^\alpha$.
- $L^1(m_0)^{1-\alpha} L^1(m_1)^\alpha = L^1([m_0, m_1]_\alpha)$?
The compatibility condition

**Definition. Compatibility**

A pair of equivalent vector measures $m_0$ and $m_1$ are said to be $\alpha$–compatible, for $0 < \alpha < 1$ if

$$\left(L^1(m_0)\right)^{1-\alpha} \left(L^1(m_1)\right)^{\alpha} = L^1([m_0, m_1]_{\alpha})$$

Equivalently, $[L^1(m_0), L^1(m_1)]_{\alpha} = L^1([m_0, m_1]_{\alpha})$.

**Remark.** If $m_0, m_1 : \Sigma \longrightarrow X$ are two positive vector measures such that there exists a vector measure $m$ and functions $0 < f_0, f_1 \in L^1(m)$ such that

$$m_0 = f_0 m \text{ and } m_1 = f_1 m \quad \left( f_i m(A) := \int_A f_i \, dm \right)$$

then $m_0$ and $m_1$ are $\alpha$–compatible for every $0 < \alpha < 1$. 
Radon-Nikodym derivative with respect to a vector measure

Definitions

Let $m, n : \Sigma \rightarrow X$ two vector measures with values in a Banach space. We say that

a) $n$ is **scalarly uniformly absolutely continuous** with respect to $m$ if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that

$\forall x' \in X', A \in \Sigma : |\langle m, x' \rangle|(A) < \delta \implies |\langle n, x' \rangle|(A) < \varepsilon$

b) $n$ is **scalarly dominated** by $m$ if there exists $M > 0$ such that

$|\langle n, x' \rangle|(A) \leq M |\langle m, x' \rangle|(A)$, $\forall A \in \Sigma, x' \in X'$. 
Radon-Nikodym derivative with respect to a vector measure

Theorem, Musial, 1993

The following conditions are equivalent:

1) \( n \) has a Radon-Nikodym derivative with respect to \( m \). That is, there exists a (scalar) bounded measurable function \( f \) such that

\[
  n(A) = \int_A f \, dm \quad \forall A \in \Sigma.
\]

2) \( n \) is scalarly uniformly absolutely continuous with respect to \( m \).

3) \( n \) is scalarly dominated by \( m \).
1. Spaces $L^p(m)$ and $L^p_w(m)$.

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3. Complex interpolation of $L^p(m)$—spaces.
   - Interpolation of $L^1(m)$—spaces.
   - Interpolation of $L^p(m)$—spaces.
   - Interpolation of tensor products.
Interpolation of $L^1(m)$–spaces

Theorem

Let $m_0 : \Sigma \rightarrow X_0$ and $m_1 : \Sigma \rightarrow X_1$ be two $\alpha$–compatible vector measures and consider two functions $0 < f_0 \in L^1(m_0)$ and $0 < f_1 \in L^1(m_1)$. Then

$$(L^1(f_0 m_0))^{1-\alpha} (L^1(f_1 m_1))^\alpha = L^1(f_0^{1-\alpha} f_1^\alpha [m_0, m_1]_\alpha)$$
Interpolation of $L^p(m)$—spaces

**Theorem**

Let $m_0 : \Sigma \rightarrow X_0$ and $m_1 : \Sigma \rightarrow X_1$ be two $\alpha$—compatible vector measures and consider two functions $0 < f_0 \in L^1(m_0)$ and $0 < f_1 \in L^1(m_1)$. Then, for $0 < \theta < 1 \leq p_0, p_1 < \infty$,

$$[L^{p_0}(f_0 m_0), L^{p_1}(f_1 m_1)][\theta] = L^{p(\theta)} \left( f_0^{1-\alpha} f_1^\alpha [m_0, m_1]_\alpha \right)$$

with $\alpha = \frac{\theta p(\theta)}{p_1}$. 

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Complex interpolation of vector measures
Corollary

Let $m$ be a positive vector measure with values in a Köthe-Banach function space and consider two functions $0 < f_0, f_1 \in L^1(m)$. Then, for $0 < \theta < 1 \leq p_0, p_1 < \infty$,

$$\left[ L^{p_0}(f_0 m), L^{p_1}(f_1 m) \right]_{[\theta]} = L^{p(\theta)} \left( f_0^{1-\alpha} f_1^\alpha m \right)$$

with $\alpha = \frac{\theta p(\theta)}{p_1}$.
Corollary

Let $0 < \theta < 1 \leq p_0, p_1, q_0, q_1 < \infty$ and $\alpha = \frac{\theta p(\theta)}{p_1}, \beta = \frac{\theta q(\theta)}{q_1}$ with

$$\frac{1}{p(\theta)} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q(\theta)} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$  

Let $(m_0, m_1)$ be a couple of $\alpha$–compatible vector measures and $(n_0, n_1)$ be a couple of $\beta$–compatible vector measures. If

$$T : L^{p_0}(m_0) + L^{p_1}(m_1) \longrightarrow L^{q_0}(n_0) + L^{q_1}(n_1)$$

is a linear operator such that the restrictions

$T_0 : L^{p_0}(m_0) \longrightarrow L^{q_0}(n_0)$ and $T_1 : L^{p_1}(m_1) \longrightarrow L^{q_1}(n_1)$ are well defined and continuous then

$$T : L^{p(\theta)}([m_0, m_1]_\alpha) \longrightarrow L^{q(\theta)}([n_0, n_1]_\beta)$$

is well defined and continuous.
Interpolation of injective tensor products

Corollary

Let $0 < \theta < 1 \leq p_0, p_1, q_0, q_1 < \infty, \alpha, \beta, p(\theta), q(\theta), (m_0, m_1)$ and $(n_0, n_1)$ be as before.

- If $L^{p_0}(m_0), L^{p_1}(m_1), L^{q_0}(n_0)$ and $L^{q_1}(n_1)$ are 2-concave Banach lattices, then

$$[L^{p_0}(m_0) \hat{\otimes} \varepsilon L^{q_0}(n_0), L^{p_1}(m_1) \hat{\otimes} \varepsilon L^{q_1}(n_1)]_{[\theta]} = (\text{Kouba, 1991})$$

$$= [L^{p_0}(m_0), L^{p_1}(m_1)]_{[\theta]} \hat{\otimes} \varepsilon [L^{q_0}(n_0), L^{q_1}(n_1)]_{[\theta]}$$

$$= L^{p(\theta)}([m_0, m_1]_{\alpha}) \hat{\otimes} \varepsilon L^{q(\theta)}([n_0, n_1]_{\beta}).$$
Corollary

Let $0 < \theta < 1 \leq p_0, p_1, q_0, q_1 < \infty$, $\alpha, \beta, p(\theta), q(\theta)$, $(m_0, m_1)$ and $(n_0, n_1)$ be as before.

- If $p_0, p_1, q_0, q_1 \geq 2$ ($\Rightarrow$ the spaces are 2-convex Banach laticces), then

\[
\left[ L^{p_0}(m_0) \hat{\otimes}_\pi L^{q_0}(n_0), L^{p_1}(m_1) \hat{\otimes}_\pi L^{q_1}(n_1) \right][\theta] = (\text{Kouba, 1991})
\]

\[
= \left[ L^{p_0}(m_0), L^{p_1}(m_1) \right][\theta] \hat{\otimes}_\pi \left[ L^{q_0}(n_0), L^{q_1}(n_1) \right][\theta]
\]

\[
= L^{p(\theta)}([m_0, m_1]_\alpha) \hat{\otimes}_\pi L^{q(\theta)}([n_0, n_1]_\beta).
\]
Introduction
Spaces $L^p(m)$ and $L^p_w(m)$.
The interpolated vector measure.
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References


That’s all.
Thank you.